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ON THE HOMOTOPY GROUP $\pi_{2n+9}(U(n))$ FOR $n\geqq 6$

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The homotopy groups $\pi_{2n+i}(U(n))$ of the unitary group $U(n)$ for $0\leqq i\leqq 8$, $i=10$ and 12 were determined by Borel and Hirzebruch [2], Bott [3], Kervaire [7], Toda [22, 23], Matsunaga [8–12], Mimura and Toda [13], Mosher [14, 15], and Imanishi [6]. For $n\geqq 5$ and $i=9$, 11 or 13 the odd components were determined by [12] and [6], but the 2-component had not been completely determined. Indeed Mosher [15] has not determined some group extensions which appear in case of $i=9$ only if $n\equiv 2$, 4 or 6 mod (8) and $n\geqq 6$. In this note we shall determine these group extensions for $i=9$. $\pi_{2n+9}(U(n))$ for $n\leqq 5$ was determined by [6], [13], [15] and [23]. Therefore we shall complete the computation of $\pi_{2n+9}(U(n))$. While the group $\pi_{2n+9}(U(n))$ has been computed by Vastersavendts [24] for $n\equiv 0$ mod (4), 6 mod (8) or 2 mod (16), her results contradict Mosher's [15] and ours for $n\equiv 0$ mod (16) and $n\equiv 6$ mod (8) respectively.

We shall prove

Theorem. *The 2-component cf $\pi_{2n+9}(U(n))$ foı $n\equiv 2, 4$ or 6 mod (8) and $n\geqq 6$ is given by the following table:*

n mod ( )	$\pi_{2n+9}(U(n))$
2(16)	$Z_2\oplus Z_4\oplus Z_2$
10(32)	$Z_2\oplus Z_4\oplus Z_4$
26(64)	$Z_2\oplus Z_4\oplus Z_8$
58(64)	$Z_2\oplus Z_4\oplus Z_{16}$
4(8)	$Z_2\oplus Z_2\oplus Z_8$
6(8)	$Z_2\oplus Z_4$

where $Z_m=Z/mZ$ is the cyclic group of order m.

We shall use the notations and terminologies defined in [20] or the book of Toda [23] without any reference.

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1. Method of computation

By Theorem 4.3 of Toda [22] we know that $\pi_{2n+9}(U(n))$ is isomorphic to the stable homotopy group $\pi^s_{2n+9}(P_{n+6,6})$ of the stunted complex projective space $P_{n+6,6}=P_{n+6}/P_n$ if $n\geqq 5$. We shall compute $\pi^s_{2n+9}(P_{n+6,6})$.

Consider the canonical cofibration

$$S^{2(n+k)-3} \xrightarrow{p_{n+k-1,k-1}} P_{n+k-1,k-1} \xrightarrow{i_1} P_{n+k,k} \xrightarrow{q_{k-1}} S^{2(n+k)-2}$$

and the associated exact sequence

$$(S)_k: \qquad \cdots \to G_{i-2k+2} \xrightarrow{p_*} \pi^s_{2n-1+i}(P_{n+k-1,k-1}) \xrightarrow{i_{1*}} \pi^s_{2n-1+i}(P_{n+k,k}) \xrightarrow{q_*} G_{i-2k+1} \xrightarrow{p_*} \cdots .$$

We set the two steps of computation:

(1) *determine the G_*-module structure of $\pi^s_*(P_{n+k-1,k-1})$,*

(2) *describe $p_{n+k-1,k-1}\in\pi^s_{2(n+k)-3}(P_{n+k-1,k-1})$ explicitly.*

If these two are possible, we know $\pi^s_{2n-1+i}(P_{n+k,k})$ up to group extension

$$0 \to \text{Cokernel of } p_* \to \pi^s_{2n-1+i}(P_{n+k,k}) \to \text{Kernel of } p_* \to 0 .$$

To determine this group extension, we prepare a lemma.

Lemma 1 (cf. Theorem 2.1 of [13]). *Let $A \xrightarrow{f} X \xrightarrow{i} C(f)$ be a cofibration and*

$$\cdots \to \pi^s_n(X) \xrightarrow{i_*} \pi^s_n(C(f)) \xrightarrow{\Delta} \pi^s_{n-1}(A) \xrightarrow{f_*} \pi^s_{n-1}(X) \to \cdots$$

an associated stable exact sequence. Assume that $\alpha\in\pi^s_{n-1}(A)$ satisfies $f_*(\alpha)=0$, and the order of α is k. For an arbitrary element β of $\langle f, \alpha, k\iota\rangle\subset\pi^s_n(X)$, there exists an element $[\alpha]\in\pi^s_n(C(f))$ such that

$$\Delta([\alpha]) = \alpha \quad \textit{and} \quad i_*(\beta) = -k[\alpha] .$$

Proof. By definition of Toda bracket, there exists a commutative stable diagram with $\beta=a\circ b$:

$$\begin{tikzcd}
 & & S^n \arrow[d, "b"'] \arrow[dr, "k\iota"] & & \\
S^{n-1} \arrow[r, "\alpha"] & A \arrow[r] \arrow[d, "="'] & C(\alpha) \arrow[r] \arrow[d, "a"'] & S^n \arrow[r, "-\alpha"] \arrow[d, "a'"'] & EA \arrow[d, "="'] \\
 & A \arrow[r, "f"] & X \arrow[r, "i"] & C(f) \arrow[r] & EA
\end{tikzcd}$$

Then we may put $[\alpha]=-a'$.

For the above (2), we consider $(S)_k$ for $i=2k-2$:

$$\pi^s_{2(n+k)-2}(P_{n+k,k}) \xrightarrow{q_*} G_0 \xrightarrow{p_*} \pi^s_{2(n+k)-3}(P_{n+k-1,k-1}) .$$

The exactness of this shows that

$$\# p_{n+k-1,k-1} = \#(\textit{Cokernel of } q_*) .$$

On the other hand by (4.5) of [20] we know that

$$\begin{aligned} \#(\textit{Cokernel of } q_*) &= Q^s\{n+k, k\} \\ &= C\{jM_k(C)-n-k, k\} \qquad \textit{for large } j \end{aligned}$$

and this number was determined for $k\leqq 8$ in (3.1) of [20]. We shall need the 2-component of this number for $k=5$ and 6. Let $\nu_2(m)$ be the exponent of 2 in the factorization of an integer m into the prime powers.

Lemma 2 ((3.1) of [20]). *$\nu_2(\# p_{n+4,4})$ and $\nu_2(\# p_{n+5,5})$ are given by the following table*:

$\nu_2(\# p_{n+4,4})$	n mod ( )	$\nu_2(\# p_{n+5,5})$	n mod ( )
4	4, 6(8)	4	4, 6(8)
3	0(8), 2(16)	3	0, 2(16)
2	10(16)	2	8(16), 10(32)
		1	26(64)
		0	58(64)

Considering the above (1) and (2), we shall compute inductively $\pi^s_{2n-1+i}(P_{n+k,k})$ for $k\leqq 6$ and some $i\leqq 10$. Since the suspension $EP_{n+k,k}$ is $2n$-connected and the pair $(W_{n+k,k}, EP_{n+k,k})$ is $(4n+3)$-connected, it follows that $\pi^s_{2n-1+i}(P_{n+k,k})$ is isomorphic to $\pi_{2n+i}(W_{n+k,k})$ for $i\leqq 2n$, where $W_{n+k,k}=U(n+k)/U(n)$ is the complex Stiefel manifold. Nomura and Furukawa [16] have computed $\pi_{2n+i}(W_{n+k,k})$ for $k=2, 3$ and $i\leqq 21$, 19 respectively. Therefore we already know $\pi^s_{2n-1+i}(P_{n+k,k})$ $2\leqq k\leqq 3$ and $i\leqq 10$. But informations for (1) from [16] are not sufficient for our purpose. So we shall recompute some $\pi^s_{2n-1+i}(P_{n+k,k})$ for $k\leqq 3$.

2. Computation

From now on, n means always an even integer $\geqq 6$, $\pi^s_*(\ )$ and G_* often denote only the 2-primary component of itself. We work in the stable category of pointed spaces and stable maps between them.

Since $p_{n+1,1}=n\eta=0$, it follows that $P_{n+2,2}=S^{2n}\vee S^{2n+2}$. Let s: $S^{2n+2}\to p_{n+2,2}$ be an inclusion map which is a right inverse of q_1. Then

(2.1) $\quad i_{1*}+s_*\colon G_{i-1}\oplus G_{i-3}\to \pi^s_{2n-1+i}(P_{n+2,2})$ is an isomorphism.

By the proof of (1.11), (i) of (1.13) and (1.14) of [20], we have

$$p_{n+2,2}=(n/2)i_{1*}(\nu+\alpha_1)+s_*\eta\colon S^{2n+3}\to P_{n+2,2}=S^{2n}\vee S^{2n+2}.$$

Put

$$e_n=\begin{cases}1 & \textit{if } n\equiv 0 \bmod (4)\\ 2 & \textit{if } n\equiv 2 \bmod (4).\end{cases}$$

Then by (2.1) and $(S)_3$ for $i=8$, we have a short exact sequence

$$0\to Z_{16}\{i_{2*}\sigma\}\to \pi^s_{2n+7}(P_{n+3,3})\to Z_{8/e_n}\{e_n\nu\}\to 0.$$

We have

$$\begin{aligned}\langle p_{n+2,2}, e_n\nu, (8/e_n)\iota\rangle &= \langle (n/2)i_{1*}\nu, e_n\nu, (8/e_n)\iota\rangle+\langle s_*\eta, e_n\nu, (8/e_n)\iota\rangle\\ &\supset i_{1*}\langle (n/2)\nu, e_n\nu, (8/e_n)\iota\rangle+s_*\langle \eta, e_n\nu, (8/e_n)\iota\rangle\\ &\supset i_{1*}\{(ne_n/4)\langle (2/e_n)\nu, e_n\nu, (8/e_n)\iota\rangle\}\\ &\ni 0\end{aligned}$$

since $\langle \eta, e_n\nu, (8/e_n)\iota\rangle\subset G_5=0$ and $\langle (2/e_n)\nu, e_n\nu, (8/e_n)\iota\rangle\ni 0$ (see e.g. [16]). Therefore by Lemma 1 the above short exact sequence splits, that is, there exists $[e_n\nu]\in\pi^s_{2n+7}(P_{n+3,3})$ with $q_{2*}[e_n\nu]=e_n\nu$ and

(2.2) $$\pi^s_{2n+7}(P_{n+3,3})=Z_{16}\{i_{2*}\sigma\}\oplus Z_{8/e_n}\{[e_n\nu]\}.$$

It follows from $(S)_3$ for $i=9$ that $i_{1*}\colon \pi^s_{2n+8}(P_{n+2,2})\to\pi^s_{2n+8}(P_{n+3,3})$ is an isomorphism. Hence by (2.1) we have

(2.3) $$\pi^s_{2n+8}(P_{n+3,3})=Z_2\{i_{2*}\varepsilon\}\oplus Z_2\{i_{2*}\bar{\nu}\}\oplus Z_2\{i_{1*}s_*\nu^2\}.$$

From (2.1) and $(S)_3$ for $i=10$ it follows that

(2.4) $$\pi^s_{2n+9}(P_{n+3,3})=Z_{16}\{i_{1*}s_*\sigma\}\oplus Z_2\{i_{2*}\mu\}\oplus Z_2\{i_{2*}\eta\varepsilon\}\oplus Z_{2/e_n}\{i_{2*}\nu^3\}.$$

Analysing $p_{n+k,k}$ for $k=3$, 4 and 5, we consider the followings. Put

$$L_{m,k}=\begin{cases}1 & \textit{if } m+k\equiv 1 \bmod (2)\\ 2 & \textit{if } m+k\equiv 0 \bmod (2).\end{cases}$$

Then, since $L_{m,k}(m+k-1)\equiv 0 \bmod (2)$, $q_{l-1*}(L_{m,k}p_{m+k,l})=L_{m,k}(m+k-1)\eta=0$ and hence $i^{-1}_{1*}(L_{m,k}p_{m+k,l})$ is not empty for $1<l<m+k$, and

$(T)_k$ $\qquad i_{1*}^{-1}(L_{m,k}p_{m+k,k}) = i_{1*}^{-1}(L_{m,k}q_{m-1*}p_{m+k}) \supset q_{m-1*}i_{1*}^{-1}(L_{m,k}p_{m+k})$

and by (1.15) of [20]

$$(T)'_k \qquad \begin{aligned} & q_{k-2*}q_{m-1*}i_{1*}^{-1}(L_{m,k}p_{m+k}) \\ & = q_{m+k-3*}i_{1*}^{-1}(L_{m,k}p_{m+k}) \\ & = \begin{cases} (m+k-2)(\nu+\alpha_1) & \text{if } m+k\equiv 0 \bmod (2) \\ \{(1/2)(m+k+1)(\nu+\alpha_1),\ (1/2)(m+k+1)(\nu+\alpha_1)+4\nu\} & \\ & \text{if } m+k\equiv 1 \bmod (2)\,. \end{cases} \end{aligned}$$

Now $q_{1*}=s_*^{-1}$: $\pi_{2n+5}^s(P_{n+2,2}) \xrightarrow[\cong]{} \pi_{2n+5}^s(S^{2n+2}) = G_3$ by (2.1), since $q_1 \circ s=1$. Then by $(T)'_3$

$$q_{n-1*}i_{1*}^{-1}(p_{n+3}) \ni ((n+4)/2)s_*(\nu+\alpha_1)$$

and by $(T)_3$

$$p_{n+3,3} = ((n+4)/2)i_{1*}s_*(\nu+\alpha_1)$$

so that $p_{n+3,3}\circ\eta=0$ and

$$\langle p_{n+3,3},\ \eta,\ 2\iota\rangle \supset i_{1*}s_*\langle((n+4)/2)\nu,\ \eta,\ 2\iota\rangle = 0$$

and by Lemma 1 there exists $[\eta]\in\pi_{2n+7}^s(P_{n+4,4})$ with $q_{3*}[\eta]=\eta$ and

$$(2.5) \qquad \pi_{2n+7}^s(P_{n+4,4}) = Z_{16}\{i_{3*}\sigma\}\oplus Z_{8/e_n}\{i_{1*}[e_n\nu]\}\oplus Z_2\{[\eta]\}\,.$$

We have also the following from (2.3) and $(S)_4$ for $i=9$

$$(2.6) \qquad \pi_{2n+8}^s(P_{n+4,4}) = Z_2\{i_{3*}\varepsilon\}\oplus Z_2\{i_{3*}\bar{\nu}\}\oplus Z_{2/e_n}\{i_{2*}s_*\nu^2\}\oplus Z_2\{[\eta]\eta\}\,.$$

By the same argument as the proof of (2.2) we know that there exists $[[e_n\nu]]\in\pi_{2n+9}^s(P_{n+4,4})$ with $q_{3*}[[e_n\nu]]=e_n\nu$ and

$$(2.7) \qquad \begin{aligned} \pi_{2n+9}^s(P_{n+4,4}) = {} & Z_{16}\{i_{2*}s_*\sigma\}\oplus Z_2\{i_{3*}\mu\}\oplus Z_2\{i_{3*}\eta\varepsilon\} \\ & \oplus Z_{2/e_n}\{i_{3*}\nu^3\}\oplus Z_{8/e_n}\{[[e_n\nu]]\}\,. \end{aligned}$$

To compute $\pi_{2n+9}^s(P_{n+5,5})$ we shall prepare four lemmas.

Remember that in [20] we used the notations: $HP_{m+k,k}=HP_{m+k}/HP_m$, the stunted quaternionic projective space; π: $P_{2m+2k,2k}\to HP_{m+k,k}$, the canonical quotient map;

$$(2.8) \qquad S^{2n+4k-1} \xrightarrow{p^H_{(n/2)+k,k}} HP_{(n/2)+k,k} \xrightarrow{i_1^H} HP_{(n/2)+k+1,k+1}$$

the canonical cofibration.

Lemma 3. *We have*

(i) $\pi^s_{2n+7}(HP_{(n/2)+2,2})=Z_{16}\{i^H_{1*}\sigma\}\oplus Z_{8/e_n}\{\pi_* i_{1*}[e_n\nu]\}$,
(ii) $\pi^s_{2n+8}(HP_{(n/2)+2,2})=Z_2\{i^H_{1*}\varepsilon\}\oplus Z_2\{i^H_{1*}\bar{\nu}\}$,
(iii) $\pi^s_{2n+9}(HP_{(n/2)+2,2})=Z_2\{i^H_{1*}\mu\}\oplus Z_2\{i^H_{1*}\eta\varepsilon\}\oplus Z_{2/e_n}\{i^H_{1*}\nu^3\}$,

(iv) $$p^H_{(n/2)+2,2}\circ\eta=\begin{cases} i^H_{1*}\varepsilon & \textit{if } n\equiv 2 \bmod (8) \\ i^H_{1*}\bar{\nu} & \textit{if } n\equiv 4 \bmod (8) \\ i^H_{1*}(\varepsilon+\bar{\nu}) & \textit{if } n\equiv 6 \bmod (8) \\ 0 & \textit{if } n\equiv 0 \bmod (8). \end{cases}$$

Proof. Considering the stable homotopy exact sequence associated with (2.8) for $k=1$, we obtain (ii) and (iii) immediately since $G_4=G_5=0$ and $p^H_{(n/2)+1,1}=(n/2)(\nu+\alpha_1)$ and by Lemma 1 we have a split exact sequence:

$$0\to Z_{16}\{i^H_{1*}\sigma\}\to \pi^s_{2n+7}(HP_{(n/2)+2,2})\to Z_{8/e_n}\{e_n\nu\}\to 0\,.$$

Then the following commutative diagram induces (i):

$$\begin{tikzcd}
 & & S^{2n+7} \arrow[r, "e_n\nu"] \arrow[d, "{[e_n\nu]}"] & S^{2n+4} \arrow[d, "="] \\
S^{2n} \arrow[r, "i_1"] \arrow[ddd, "="] & P_{n+2,2} \arrow[r, "i_1"] \arrow[ddd, "\pi"] & P_{n+3,3} \arrow[r, "q_2"] \arrow[d, "i_1"] & S^{2n+4} \arrow[ddd, "="] \\
 & & P_{n+4,4} \arrow[dd, "\pi"] & \\
 & & & \\
S^{2n} \arrow[r, phantom, "="] & HP_{(n/2)+1,1} \arrow[r, "i^H_1"] & HP_{(n/2)+2,2} \arrow[r, "q^H_1"] & S^{2n+4}
\end{tikzcd}$$

Since $q^H_1\circ p^H_{(n/2)+2,2}\circ\eta=((n/2)+1)(\nu+\alpha_1)\eta=0$, there exists a map $f\colon S^{2n+8}\to S^{2n}$ with $i^H_1\circ f=p^H_{(n/2)+2,2}\circ\eta$. It is easily seen that $i^{H*}_1\colon \{HP_{(n/2)+2,2}, S^{2n-1}\}\to\{S^{2n}, S^{2n-1}\}=Z_2\{\eta\}$ is an isomorphism. Let $h\in\{HP_{(n/2)+2,2}, S^{2n-1}\}$ be the element with $h\circ i^H_1=\eta$. It follows from (2.7) of [21] that

$$h\circ p^H_{(n/2)+2,2}=\begin{cases} \varepsilon & \textit{if } n\equiv 2 \bmod (8) \\ \bar{\nu} & \textit{if } n\equiv 4 \bmod (8) \\ \varepsilon+\bar{\nu} & \textit{if } n\equiv 6 \bmod (8) \\ 0 & \textit{if } n\equiv 0 \bmod (8)\,. \end{cases}$$

Since $\eta\circ f=h\circ p^H_{(n/2)+2,2}\circ\eta$ and $\eta\circ\colon G_8\to G_9$ is a monomorphism, we obtain (iv). This completes the proof of Lemma 3.

Lemma 4. *For suitably chosen* $[[e_n\nu]]$ *it holds that* $[\eta]\eta^2=(4/e_n)[[e_n\nu]]$.

Proof. By Proposition 1.4 of Toda [23]

$$[\eta]\eta^2=[\eta]\circ\langle 2\iota, \eta, 2\iota\rangle=\langle[\eta], 2\iota, \eta\rangle\circ 2\iota\,. \tag{2.9}$$

Let Indet $\langle\alpha, \beta, \gamma\rangle$ be an indeterminacy of a Toda bracket $\langle\alpha, \beta, \gamma\rangle$. Then

$$\begin{aligned}\pi^s_{2n+9}(P_{n+4,4})\supset \text{Indet}\langle[\eta], 2\iota, \eta\rangle &= [\eta]\circ\{S^{2n+9}, S^{2n+7}\}+\pi^s_{2n+8}(P_{n+4,4})\circ\eta \\ &= Z_2\{[\eta]\eta^2\}+Z_2\{i_{3*}\eta\varepsilon\}+Z_{2/e_n}\{i_{3*}\nu^3\}\end{aligned}$$

and

$$q_{3*}\ \text{Indet}\langle[\eta], 2\iota, \eta\rangle = Z_2\{\eta^3\} = Z_2\{4\nu\} = \text{Indet}\langle\eta, 2\iota, \eta\rangle$$

and, since $q_{3*}\langle[\eta], 2\iota, \eta\rangle\subset\langle q_{3*}[\eta], 2\iota, \eta\rangle=\langle\eta, 2\iota, \eta\rangle$, we have

$$q_{3*}\langle[\eta], 2\iota, \eta\rangle = \langle\eta, 2\iota, \eta\rangle = \{2\nu, 6\nu\}\ .$$

Hence there exists an element in $\langle[\eta], 2\iota, \eta\rangle$ which is mapped to 2ν by q_{3*}. By (2.7) this element has a form as $(2/e_n)[[e_n\nu]]+i_{1*}x$ for some $x\in\pi^s_{2n+9}(P_{n+3,3})$, and from (2.9) it follows that $4i_{1*}x=0$. Then by (2.7) $2i_{1*}x$ is divisible by 8, that is, $2i_{1*}x=8i_{1*}y$ for some $y\in\pi^s_{2n+9}(P_{n+3,3})$. Then

$$\begin{aligned}[\eta]\eta^2 &= 2\{(2/e_n)[[e_n\nu]]+i_{1*}x\} \\ &= (4/e_n)([[e_n\nu]]+2e_n i_{1*}y)\ .\end{aligned}$$

Since $q_{3*}([[e_n\nu]]+2e_n i_{1*}y)=e_n\nu$ and the order of $[[e_n\nu]]+2e_n i_{1*}y$ is $8/e_n$, we may change $[[e_n\nu]]$ for $[[e_n\nu]]+2e_n i_{1*}y$. So the conclusion follows.

Appointment: From now on we assume that $[[e_n\nu]]$ satisfies $[\eta]\eta^2=(4/e_n)[[e_n\nu]]$.

Since $q_3\circ p_{n+4,4}=(n+3)\eta=\eta$, by (2.5) we can put

$$p_{n+4,4} = a_n i_{3*}\sigma+b_n i_{1*}[e_n\nu]+[\eta]+\textit{odd torsion}$$

for some integers a_n and b_n. By Lemma 2 and (2.5) we have

$$(2.10)\qquad a_n\equiv\begin{cases} 1 \bmod (2) & \textit{if } n\equiv 4 \textit{ or } 6 \bmod (8) \\ 0 \bmod (2) & \textit{if } n\equiv 0 \textit{ or } 2 \bmod (8).\end{cases}$$

By $(T)_4$, and $(T)'_4$, for any $p'\in q_{n-1*}i^{-1}_{1*}(2p_{n+4})\subset\pi^s_{2n+7}(P_{n+3,3})$ we have

$$i_1\circ p' = 2p_{n+4,4} \quad \textit{and} \quad q_2\circ p' = (n+2)(\nu+\alpha_1)\ .$$

Then $p'=2a_n i_{2*}\sigma+2b_n[e_n\nu]+\textit{odd torsion}$. Applying q_{2*} to this equation we know that $2b_n e_n\equiv n+2 \bmod (8)$, and

$$(2.11)\qquad b_n\equiv\begin{cases} 1 \bmod (2) & \textit{if } n\equiv 0 \bmod (4) \textit{ or } 2 \bmod (8) \\ 0 \bmod (2) & \textit{if } n\equiv 6 \bmod (8)\ .\end{cases}$$

Lemma 5. *We have*

$$[2\nu]\eta = \begin{cases} i_{2*}\varepsilon & \text{if } n\equiv 2 \bmod (8) \\ i_{2*}\varepsilon \text{ or } i_{2*}\bar{\nu} & \text{if } n\equiv 6 \bmod (8) \end{cases}, \text{ and}$$

$$[\nu]\eta = (n/4)i_{2*}\varepsilon + i_{1*}s_*\nu^2 \quad \text{if } n\equiv 0 \bmod (4).$$

Proof. By Lemma 1 we can easily construct a commutative diagram:

$$\begin{tikzcd}
 & & S^{2n+8} \arrow[d, "b"] \arrow[dr, "\eta"] & & \\
S^{2n+6} \arrow[r, "e_n\nu"] & S^{2n+3} \arrow[r] \arrow[d, "="] & C(e_n\nu) \arrow[r] \arrow[d, "a"] & S^{2n+7} \arrow[r, "-e_n\nu"] \arrow[d, "-{[e_n\nu]}"] & S^{2n+4} \arrow[d, "="] \\
 & S^{2n+3} \arrow[r, "p_{n+2,2}"'] & P_{n+2,2} \arrow[r, "i_1"'] & P_{n+3,3} \arrow[r] & S^{2n+4}
\end{tikzcd}$$

Then $a\circ b \in \langle p_{n+2,2}, e_n\nu, \eta\rangle$ and this Toda backet is a coset of

$$\begin{aligned} &\pi^s_{2n+8}(P_{n+2,2})/\pi^s_{2n+7}(P_{n+2,2})\circ\eta \\ &\quad = [Z_2\{i_{1*}\varepsilon\}\oplus Z_2\{i_{1*}\bar{\nu}\}/\{0, i_{1*}(\varepsilon+\bar{\nu})\}]\oplus Z_2\{s_*\nu^2\}. \end{aligned}$$

We have

$$\begin{aligned} \langle p_{n+2,2}, e_n\nu, \eta\rangle &= \langle (n/2)i_{1*}\nu, e_n\nu, \eta\rangle + \langle s_*\eta, e_n\nu, \eta\rangle \\ &\supset i_{1*}\{(ne_n/4)\langle (2/e_n)\nu, e_n\nu, \eta\rangle\} + s_*\langle \eta, e_n\nu, \eta\rangle \\ &\ni (ne_n/4)i_{1*}\varepsilon + e_n s_*\nu^2 \end{aligned}$$

since $\langle (2/e_n)\nu, e_n\nu, \eta\rangle = \varepsilon + G_7\circ\eta$ and $\langle \eta, e_n\nu, \eta\rangle = e_n\nu^2$ by Toda [23]. Hence

$$\begin{aligned} (2.12)\qquad [e_n\nu]\eta &= i_{1*}(a\circ b) \\ &= (ne_n/4)i_{2*}\varepsilon + e_n i_{1*}s_*\nu^2 \ \text{ or } \ ((ne_n/4)+1)i_{2*}\varepsilon + i_{2*}\bar{\nu} + e_n i_{1*}s_*\nu^2. \end{aligned}$$

Thus Lemma 5 follows if $n\equiv 6 \bmod (8)$. By Lemma 4

$$(2.13)\qquad p_{n+4,4}\circ\eta^2 = a_n i_{3*}(\eta\varepsilon+\nu^3) + b_n i_{1*}[e_n\nu]\eta^2 + (4/e_n)[[e_n\nu]]$$

and by (iii) of Lemma 3, the fact $4/e_n\equiv 0 \bmod (2)$ and the commutativity of the diagram in the proof of Lemma 3 it follows that

$$\begin{aligned} p^H_{(n/2)+2,2}\circ\eta^2 &= \pi\circ p_{n+4,4}\circ\eta^2 \\ &= a_n i^H_{1*}(\eta\varepsilon+\nu^3) + b_n\pi_* i_{1*}[e_n\nu]\eta^2. \end{aligned}$$

Then the conclusions for $n\not\equiv 6 \bmod (8)$ follow from (iii) and (iv) of Lemma 3, (2.10), (2.11) and (2.12). This completes the proof of Lemma 5.

Lemma 6. *We have*

$$p_{n+4,4}\circ\eta^2=\begin{cases} i_{3*}\eta\varepsilon+2[[2\nu]] & \text{if } n\equiv 2 \bmod (4)\\ (n/4)i_{3*}\nu^3+4[[\nu]] & \text{if } n\equiv 0 \bmod (4)\,. \end{cases}$$

Proof. The conclusion follows from (2.7), (2.10), (2.11), Lemma 5 and (2.13).

Now we compute $\pi^s_{2n+9}(P_{n+5,5})$. Since $p_{n+4,4}\circ\eta=[\eta]\eta+(\textit{other term})$ is non-zero, it follows from (2.7), Lemma 6 and $(S)_5$ for $i=10$ that

$$\pi^s_{2n+9}(P_{n+5,5})=Z_{16}\{i_{3*}s_*\sigma\}\oplus Z_2\{i_{4*}\mu\}\oplus H_n \tag{2.14}$$

where

$$H_n=\begin{cases} Z_4\{i_{1*}[[2\nu]]\} \textit{ with the relations } i_{4*}\eta\varepsilon=2i_{1*}[[2\nu]] \textit{ and } i_{4*}\nu^3=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n\equiv 2 \bmod (4)\\ Z_2\{i_{4*}\eta\varepsilon\}\oplus Z_2\{i_{4*}\nu^3\}\oplus Z_4\{i_{1*}[[\nu]]\} \qquad \text{if } n\equiv 0 \bmod (8)\\ Z_2\{i_{4*}\eta\varepsilon\}\oplus Z_8\{i_{1*}[[\nu]]\} \textit{ with the relation } i_{4*}\nu^3=4i_{1*}[[\nu]]\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n\equiv 4 \bmod (8)\,. \end{cases}$$

By $(T)'_5$

$$q_{3*}\,q_{n-1*}i_{1*}^{-1}(p_{n+5})=\{((n+6)/2)(\nu+\alpha_1),\ ((n+6)/2)(\nu+\alpha_1)+4\nu\}$$

and hence we can choose a map $\tilde{p}\in q_{n-1*}i_{1*}^{-1}(p_{n+5})\subset\pi^s_{2n+9}(P_{n+4,4})$ with

$$q_3\circ\tilde{p}=\begin{cases} ((n+6)/2)(\nu+\alpha_1)+4\nu & \text{if } n\equiv 2 \bmod (16)\\ ((n+6)/2)(\nu+\alpha_1) & \textit{otherwise} \end{cases}$$

and then by $(T)_5$

$$i_1\circ\tilde{p}=p_{n+5,5}\,.$$

By (2.7) we can put

$$\tilde{p}=a'_n i_{2*}s_*\sigma+b'_n i_{3*}\mu+c_n i_{3*}\eta\varepsilon+d'_n i_{3*}\nu^3+d_n[[e_n\nu]]+\textit{odd torsion} \tag{2.15}$$

for some integers a'_n, b'_n, c_n, d'_n and d_n. Remark that $i_{3*}\nu^3=0$ if $n\equiv 2 \bmod (4)$. We have

$$\begin{aligned} d_n e_n\nu+\textit{odd torsion} &= q_3\circ\tilde{p}\\ &=\begin{cases} (((n+6)/2)+4)\nu+\textit{odd torsion} & \text{if } n\equiv 2 \bmod (16)\\ ((n+6)/2)\nu+\textit{odd torsion} & \textit{otherwise} \end{cases} \end{aligned}$$

and

$$d_n\equiv\begin{cases} 1 \bmod (2) & \text{if } n\equiv 0 \bmod (4) \textit{ or } 6 \bmod (8)\\ 0 \bmod (4) & \text{if } n\equiv 2 \bmod (8)\,. \end{cases} \tag{2.16}$$

Put $p_{n+5,5}=a'_n i_{3*}s_*\sigma+b'_n i_{4*}\mu+\check{p}$. Then the 2-primary part of $\check{p}$ is contained in H_n. Hence by Lemma 2 and (2.14) we have

$$a'_n\equiv 1 \bmod (2) \qquad \textit{if } n\equiv 4 \textit{ or } 6 \bmod (8)\,.$$

Then by (2.14), (2.16) and $(S)_6$ for $i=10$ we have

$$\text{(2.17)} \quad \pi^s_{2n+9}(P_{n+6,6}) = Z_{8/e_n}\{i_{4*}s_*\sigma\}\oplus Z_2\{i_{5*}\mu\}\oplus Z_{2/e_n} \qquad \textit{if } n\equiv 4 \textit{ or } 6 \bmod (8)$$

where if $n\equiv 4 \bmod (8)$, $Z_{2/e_n}=Z_2$ is generated by $i_{5*}\eta\varepsilon$.

Next suppose that $n\equiv 2 \bmod (8)$. Let l be the odd component of the order of $\tilde{p}$. Of course l is an odd integer. Put $\hat{p}=la'_n s_*\sigma+b'_n i_{1*}\mu+c_n i_{1*}\eta\varepsilon$. Then by (2.15) and (2.16)

$$l\tilde{p} = i_{2*}\hat{p}$$

and we have a commutative diagram in which the each horizontal sequences are cofibrations and l denotes a multiplication by l:

$$\begin{tikzcd}
S^{2n+9} \arrow[r, "p_{n+5,5}"] & P_{n+5,5} \arrow[r] & P_{n+6,6} \\
S^{2n+9} \arrow[u, "="] \arrow[d, "="'] \arrow[r, "\tilde{p}"] & P_{n+4,4} \arrow[u, "i_1"] \arrow[d, "l"] \arrow[r] & C(\tilde{p}) \arrow[u, "\bar{i}_1"] \arrow[d, "\bar{l}"] \\
S^{2n+9} \arrow[r, "l\tilde{p}"] & P_{n+4,4} \arrow[r] & C(l\tilde{p}) \\
S^{2n+9} \arrow[u, "="] \arrow[d, "="'] \arrow[r, "\hat{p}"] & P_{n+2,2} \arrow[u, "i_2"] \arrow[d, "\pi"] \arrow[r] & C(\hat{p}) \arrow[u, "\bar{i}_2"] \arrow[d, "\bar{\pi}"] \\
S^{2n+9} \arrow[r, "\pi\circ\hat{p}"] & HP_{(n/2)+1,1} \arrow[r, "j"] & C(\pi\circ\hat{p})
\end{tikzcd}$$

We calculate the Adams' e_C and e_R invariants of $\pi\circ\hat{p}\in G_9$.

Lemma 7. *We have*

(i) $e_C(\pi\circ\hat{p})=0$ *and* $b'_n\equiv 0 \bmod (2)$,

(ii) $e_R(\pi\circ\hat{p})=\begin{cases} 1 & \textit{if } n\equiv 2 \bmod (16) \\ 0 & \textit{if } n\equiv 10 \bmod (16) \end{cases}$ *and*

$$c_n\equiv\begin{cases} 1 \bmod (2) & \textit{if } n\equiv 2 \bmod (16) \\ 0 \bmod (2) & \textit{if } n\equiv 10 \bmod (16)\,. \end{cases}$$

Proof. Applying $\tilde{K}$ to the above diagram, we can show the first part of (i) by the similar method as the proof of (1.12) of [20]. Then the second part of (i) follows, since $\pi\circ s=\eta^2$ or 0, $e_C(\eta^2\sigma)=e_C(\eta\varepsilon)=0$ and $e_C(\mu)\neq 0$ by [1].

Put $n=8m+2$. Applying $\widetilde{KO}^{-4}$ to the above diagram, we have the following commutative diagram in which the horizontal sequences are exact:

diagram (2.18)

$$\begin{tikzcd}
0 & \widetilde{KO}^{-4}(P_{8m+7,5}) \arrow[l] \arrow[d, "i_1^*"] & \widetilde{KO}^{-4}(P_{8m+8,6}) \arrow[l] \arrow[d, "\bar{i}_1^*"] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] \arrow[d, "="] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-4}(P_{8m+6,4}) \arrow[l] & \widetilde{KO}^{-4}(C(\bar{p})) \arrow[l] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-4}(P_{8m+6,4}) \arrow[l] \arrow[u, "l"] \arrow[d, "i_2^*"] & \widetilde{KO}^{-4}(C(l\bar{p})) \arrow[l] \arrow[u, "\bar{l}^*"] \arrow[d, "\bar{i}_2^*"] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] \arrow[u, "="] \arrow[d, "="] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-4}(P_{8m+4,2}) \arrow[l] & \widetilde{KO}^{-4}(C(\hat{p})) \arrow[l] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-4}(S^{16m+4}) \arrow[l] \arrow[u, "\pi^*"] & \widetilde{KO}^{-4}(C(\pi\circ\hat{p})) \arrow[l, "j^*"'] \arrow[u, "\bar{\pi}^*"] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] \arrow[u, "="] & 0 \arrow[l]
\end{tikzcd}$$

By Theorem 2 of Fujii [4] it is easily seen that

$$\begin{aligned}
\widetilde{KO}^{-4}(P_{8m+8,6}) &= Z\{z_2z_0^{4m},\ z_2z_0^{4m+1},\ z_2z_0^{4m+2}\}\oplus Z_2\{z_2z_0^{4m+3}\}\\
\widetilde{KO}^{-4}(P_{8m+7,5}) &= Z\{z_2z_0^{4m},\ z_2z_0^{4m+1},\ z_2z_0^{4m+2}\}\\
\widetilde{KO}^{-4}(P_{8m+6,4}) &= Z\{z_2z_0^{4m},\ z_2z_0^{4m+1}\}\\
\widetilde{KO}^{-4}(P_{8m+4,2}) &= Z\{z_2z_0^{4m}\}\oplus Z_2\{z_2z_0^{4m+1}\}\ .
\end{aligned}$$

Also note that a generator d of $\widetilde{KO}^{-4}(S^{16m+4})=Z$ satisfies

$$\pi^*d = z_2z_0^{4m}+xz_2z_0^{4m+1}$$

for some integer x. We shall not need the explicit value of x. Here we regard $\widetilde{KO}^{-4}(X/A)$ as a subgroup of $\widetilde{KO}^{-4}(X)$ if the quotient map $X\to X/A$ induces a monomorphism. Similar remarks shall hold in the forthcoming proof of (A). By chasing diagram, we know that there exist elements $[z_2z_0^{4m}]$ and $[z_2z_0^{4m+1}]$ in $\widetilde{KO}^{-4}(C(l\bar{p}))$ such that

$$\bar{l}^*[z_2z_0^{4m}] = l\bar{i}_1^*z_2z_0^{4m} \quad \textit{and} \quad \bar{l}^*[z_2z_0^{4m+1}] = l\bar{i}_1^*z_2z_0^{4m+1}\,.$$

Put $a'=[z_2z_0^{4m}]+x[z_2z_0^{4m+1}]$. Then there exists an element $a\in\widetilde{KO}^{-4}(C(\pi\circ\hat{p}))$ such that

$$\bar{\pi}^*a = \bar{i}_2^*a' \quad \textit{and} \quad j^*a = d\,.$$

Let $b\in\widetilde{KO}^{-4}(C(\pi\circ\hat{p}))$ and $b'\in\widetilde{KO}^{-4}(C(l\bar{p}))$ be the images of the generator of $\widetilde{KO}^{-4}(S^{16m+14})=Z_2$.

Now we assume the followings which shall be proved later:

(A) $\bar{i}_1^* z_2 z_0^{4m+2} = e_{2m} \bar{l}^* b'$,

(B) *the order of* $\bar{i}_2^*[z_2 z_0^{4m+1}]$ *is* 2.

Remark that $e_{2m}=1$ if $m\equiv 0 \bmod (2)$, or 2 if $m\equiv 1 \bmod (2)$, and $\bar{l}^* b'$ is the generator of the 2-torsion of $\widetilde{KO}^{-4}(C(\tilde{p}))$. We have

$$\psi^3 a = 3^{8m+4} a + \lambda b$$

for some $\lambda \in Z_2$, and

$$e_R(\pi \circ \hat{p}) = \lambda$$

and

$$\bar{\pi}^* \psi^3 a = \bar{\pi}^*(3^{8m+4} a + \lambda b) = \bar{i}_2^*(3^{8m+4} a' + \lambda b') .$$

On the other hand

$$\bar{\pi}^* \psi^3 a = \psi^3 \bar{\pi}^* a = \psi^3 \bar{i}_2^* a' = \bar{i}_2^* \psi^3 a'$$

and

$$\bar{i}_2^*(3^{8m+4} a' + \lambda b') = \bar{i}_2^* \psi^3 a'. \tag{2.19}$$

Since the order of $\bar{i}_1^* z_2 z_0^{4m+3}$ is 2 and $\bar{i}_1^* z_2 z_0^{4m+2} = e_{2m} \bar{l}^* b'$ by (A), we have

$$\begin{aligned}
\bar{l}^* \psi^3 a' &= \psi^3 \bar{l}^* a' \\
&= \psi^3 \{ l \bar{i}_1^* (z_2 z_0^{4m} + x z_2 z_0^{4m+1}) \} \\
&= l \bar{i}_1^* \psi^3 (z_2 z_0^{4m} + x z_2 z_0^{4m+1}) \\
&= l \bar{i}_1^* \{ 3^{8m+4} z_2 z_0^{4m} + ((8m+2) 3^{8m+3} + x 3^{8m+6}) z_2 z_0^{4m+1} \\
&\quad + ((4m+1)(8m+1) 3^{8m+2} + x(8m+4) 3^{8m+5}) z_2 z_0^{4m+2} \\
&\quad + ((4m+1)(8m+1) 8m 3^{8m} + x(4m+2)(8m+3) 3^{8m+4}) z_2 z_0^{4m+3} \} \\
&= \bar{l}^* \{ 3^{8m+4} [z_2 z_0^{4m}] + ((8m+2) 3^{8m+3} + x 3^{8m+6}) [z_2 z_0^{4m+1}] + e_{2m} b' \} .
\end{aligned}$$

Then, since $\bar{l}^*$ is a monomorphism,

$$\psi^3 a' = 3^{8m+4} [z_2 z_0^{4m}] + ((8m+2) 3^{8m+3} + x 3^{8m+6}) [z_2 z_0^{4m+1}] + e_{2m} b'$$

and by (B)

$$\bar{i}_2^* \psi^3 a' = 3^{8m+4} \bar{i}_2^* [z_2 z_0^{4m}] + x \bar{i}_2^* [z_2 z^{4m+1}] + e_{2m} \bar{i}_2^* b'$$

also by (2.19) this equals to

$$\begin{aligned}
\bar{i}_2^*(3^{8m+4} a' + \lambda b') &= \bar{i}_2^* \{ 3^{8m+4} ([z_2 z_0^{4m}] + x [z_2 z_0^{4m+1}]) + \lambda b' \} \\
&= 3^{8m+4} \bar{i}_2^* [z_2 z_0^{4m}] + x \bar{i}_2^* [z_2 z_0^{4m+1}] + \lambda \bar{i}_2^* b'
\end{aligned}$$

and, since $\bar{i}_2^* b'$ is non-zero,

$$\lambda = e_{2m} \qquad in\ Z_2$$

and this implies the first part of (ii). Since $\pi\circ s=0$ or η^2, and $a'_n\equiv 0 \bmod (2)$ by Lemma 2, it follows that by the second part of (i) we have

$$\pi\circ\hat{p} = c_n\eta\varepsilon$$

and the above proof of the first part of (ii) shows that

$$c_n \equiv \begin{cases} 1 \bmod (2) & \textit{if } n\equiv 2 \bmod (16) \\ 0 \bmod (2) & \textit{if } n\equiv 10 \bmod (16)\,. \end{cases}$$

This implies the second part of (ii).

We shall give the proofs of (A) and (B).

The proof of (A): We have the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & S^{16m+13} \arrow[rr] \arrow[dd, "="] & & P_{8m+7,5} \arrow[rr] \arrow[dd] & & P_{8m+8,6} \arrow[rr] \arrow[dd, "q_3"] & & S^{16m+14} \arrow[dd, "="] \\
S^{16m+13} \arrow[ur, "="] \arrow[rr, "\tilde{p}"] \arrow[dd, "="] & & P_{8m+6,4} \arrow[ur] \arrow[rr] \arrow[dd, "q_3"] & & C(\tilde{p}) \arrow[ur, "\bar{i}_1"] \arrow[rr] \arrow[dd, "\bar{q}_3"] & & S^{16m+14} \arrow[ur, "="] \arrow[dd, "="] & \\
 & S^{16m+13} \arrow[rr] & & P_{8m+7,2} \arrow[rr] & & P_{8m+8,3} \arrow[rr, "q_2"] & & S^{16m+14} \\
S^{16m+13} \arrow[ur, "="] \arrow[rr] & & P_{8m+6,1} \arrow[ur] \arrow[rr] & & C(q_3\circ\tilde{p}) \arrow[ur, "\hat{i}_1"] \arrow[rr, "\hat{q}_2"] & & S^{16m+14} \arrow[ur, "="] &
\end{tikzcd}$$

We have

$$\widetilde{KO}^{-4}(P_{8m+8,3}) = Z\{z_2z_0^{4m+2}\}\oplus Z_2\{z_2z_0^{4m+3}\}\,,$$
$$q_3^* z_2z_0^{4m+2} = z_2z_0^{4m+2}\,.$$

It suffices for our purpose to compute $\hat{i}_1^* z_2z_0^{4m+2}$, since $\bar{i}_1^* z_2z_0^{4m+2}$ is contained in the image of $\widetilde{KO}^{-4}(S^{16m+14})$, and $\bar{q}_3^*$ induces an isomorphism between the images of $\widetilde{KO}^{-4}(S^{16m+14})$. We have chosen $\tilde{p}$ such that $q_3\circ\tilde{p}=(m+1)\alpha_1$. Let u_m be the order of $q_3\circ\tilde{p}$. Then $u_m=1$ or 3. Applying $\pi^s_{16m+14}(\ )$ to the above diagram, we know easily that there exists uniquely an element $u\in\pi^s_{16m+14}(P_{8m+8,3})$ such that $q_2\circ u=u_m\iota$, moreover there exists $\hat{u}\in\pi^s_{16m+14}(C(q_3\circ\tilde{p}))$ such that $\hat{q}_2\circ\hat{u}=u_m\iota$ and $u=\hat{i}_1\circ\hat{u}$, where ι is the identity map of S^{16m+14}. Since $\hat{q}_2^*$: $\widetilde{KO}^{-4}(S^{16m+14})\to\widetilde{KO}^{-4}(C(q_3\circ\tilde{p}))$ is an isomorphism, and $\hat{u}^*\hat{q}_2^*$ is the multiplication by u_m which is the identity homomorphism of $\widetilde{KO}^{-4}(S^{16m+14})=Z_2$, it follows that $\hat{u}^*$: $\widetilde{KO}^{-4}(C(q_3\circ\tilde{p}))\to\widetilde{KO}(S^{16m+14})$ is the inverse of $\hat{q}_2^*$. Thus

$$\hat{i}_1^* z_2z_0^{4m+2} = \hat{q}_2^*\hat{u}^*\hat{i}_1^* z_2z_0^{4m+2} = \hat{q}_2^* u^* z_2z_0^{4m+2}\,. \tag{2.20}$$

Next we determine $u^*z_2z_0^{4m+2}$. Consider the commutative diagram:

$$\begin{tikzcd}
\tilde{K}(P_{8m+8,3}) \arrow[r, "u^*"] \arrow[d, "\cong"] & \tilde{K}(S^{16m+14}) \arrow[d, "\cong"] \\
\tilde{K}^{-4}(P_{8m+8,3}) \arrow[r, "u^*"] \arrow[d, "r"] & \tilde{K}^{-4}(S^{16m+14}) \arrow[d, "r"] \\
\widetilde{KO}^{-4}(P_{8m+8,3}) \arrow[r, "u^*"] & \widetilde{KO}^{-4}(S^{16m+14})
\end{tikzcd}$$

Recall that $\tilde{K}(P_{8m+8,3})=Z\{z^{8m+5}, z^{8m+6}, z^{8m+7}\}$ and the real restriction homomorphism r in the right hand side is an epimorphism. We can prove the followings:

$$\text{(2.21)}\qquad \begin{cases} r(g_C^2 z^{8m+5}) = z_2 z_0^{4m+3} + (8m+5) z_2 z_0^{4m+2} \\ r(g_C^2 z^{8m+6}) = z_2 z_0^{4m+3} + 2 z_2 z_0^{4m+2} \\ r(g_C^2 z^{8m+7}) = z_2 z_0^{4m+3}, \end{cases}$$

$$\text{(2.22)}\qquad r(g_C^2(z^{8m+5}-(4m+2)z^{8m+6}+z^{8m+7})) = z_2 z_0^{4m+2},$$

$$\text{(2.23)}\qquad \begin{cases} u^* z^{8m+5} = (1/3)(8m+5)(3m+2) u_m \beta \\ u^* z^{8m+6} = (4m+3) u_m \beta \\ u^* z^{8m+7} = u_m \beta \end{cases}$$

where $\beta \in \tilde{K}(S^{16m+14})=Z$ is the generator such that $q_2^* \beta = z^{8m+7}$. (2.23) follows from the relation $\psi^2 u^* = u^* \psi^2$. For (2.21) we consider the following commutative diagram:

$$\begin{tikzcd}
\tilde{K}^{-4}(P_{8m+8,3}) \arrow[r, phantom, "\overset{q^*}{\subset}"] \arrow[d, "r"] & \tilde{K}^{-4}(P_{8m+8}) \arrow[d, "r"] & \tilde{K}^{-4}(P_{8m+9}) \arrow[l, "i_1^*"'] \arrow[d, "r"] \\
\widetilde{KO}^{-4}(P_{8m+8,3}) \arrow[r, phantom, "\overset{q^*}{\subset}"] & \widetilde{KO}^{-4}(P_{8m+8}) & \widetilde{KO}^{-4}(P_{8m+9}) \arrow[l, "i_1^*"']
\end{tikzcd}$$

Since $\widetilde{KO}^{-4}(P_{8m+9})=Z\{z_2, z_2 z_0, \cdots, z_2 z_0^{4m+3}\}$ is torsion free (see [4]), by the aid of the complexification homomorphism we can describe r in the right hand side explicitly. In particular we have

$$\begin{aligned} r(g_C^2 z^{8m+5}) &= (1/3)(8m+5)(8m^2+10m+3) z_2 z_0^{4m+3} + (8m+5) z_2 z_0^{4m+2}, \\ r(g_C^2 z^{8m+6}) &= (4m+3)^2 z_2 z_0^{4m+3} + 2 z_2 z_0^{4m+2}, \\ r(g_C^2 z^{8m+7}) &= (8m+7) z_2 z_0^{4m+3}. \end{aligned}$$

Hence r in the left hand side satisfies (2.21). Then (2.22) follows from (2.21). By (2.22) and (2.23)

$$\begin{aligned} u^* z_2 z_0^{4m+2} &= r(g_C^2 u^*(z^{8m+5}-(4m+2)z^{8m+6}+z^{8m+7})) \\ &= v_m r(g_C^2 \beta) \end{aligned}$$

where $v_m=((1/3)(8m+5)(3m+2)-(4m+2)(4m+3)+1)u_m$.

Now

$$
\begin{aligned}
\bar{i}_1^* z_2 z_0^{4m+2} &= \bar{i}_1^* q_3^* z_2 z_0^{4m+2} \\
&= \bar{q}_3^* \hat{i}_1^* z_2 z_0^{4m+2} \\
&= \bar{q}_3^* \hat{q}_2^* u^* z_2 z_0^{4m+2} \\
&= v_m \bar{q}_3^* \hat{q}_2^* r(g_C^2 \beta) \\
&= v_m \bar{l}^* b'
\end{aligned}
$$

where the third equality follows from (2.20). Therefore (A) follows since $v_m \equiv e_{2m} \bmod (2)$.

The proof of (B): It suffices to show that the second short exact sequence from the bottom on the diagram (2.18) splits. Naturally we have a commutative diagram in which the horizontal sequences are exact:

$$
\begin{tikzcd}
0 & \widetilde{KO}^{-4}(P_{8m+4,2}) \arrow[l] & \widetilde{KO}^{-4}(C(p)) \arrow[l] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-4}(P_{8m+4,1}) \arrow[l] \arrow[u, "q_1^*"] & \widetilde{KO}^{-4}(C(q_1 \circ p)) \arrow[l] \arrow[u] & \widetilde{KO}^{-4}(S^{16m+14}) \arrow[l] \arrow[u, "="] & 0 \arrow[l]
\end{tikzcd}
$$

It is easily seen that q_1^* is a monomorphism. By Propositions 3.3 and 7.1 of Adams [1] we have a homomorphism

$$
e\colon\ G_7 = \pi_{16m+13}(S^{16m+6}) \to \mathrm{Ext}^1(\widetilde{KO}^{-4}(S^{16m+6}), \widetilde{KO}^{-4}(S^{16m+14})) = Z_2 .
$$

Since $q_1 \circ p = l a_n' \sigma$, and $a_n' \equiv 0 \bmod (2)$ by Lemma 2, it follows that $q_1 \circ p$ is divisible by 2, and $e(q_1 \circ p) = 0$. This implies that the above lower sequence splits (see [1]), and also the upper one does. Then (B) follows and the proof of Lemma 7 is completed.

Now we proceed the computation of $\pi_{2n+9}^s(P_{n+6,6})$ for $n \equiv 2 \bmod (8)$. By (2.15), (2.16) and Lemma 7

$$
p_{n+5,5} = i_1 \circ \tilde{p} = a_n' i_{3*} s_* \sigma + c_n i_{4*} \eta \varepsilon + \textit{odd torsion}.
$$

Then we obtain the following table by Lemma 2

n mod ( )	$\nu_2(\# p_{n+5,5})$	a_n'
2(16)	3	2(4)
10(32)	2	4(8)
26(64)	1	8(16)
58(64)	0	0(16)

Put

$$e'_n = \begin{cases} 2 & \text{if } n \equiv 2 \bmod (16) \\ 2^2 & \text{if } n \equiv 10 \bmod (32) \\ 2^3 & \text{if } n \equiv 26 \bmod (64) \\ 2^4 & \text{if } n \equiv 58 \bmod (64)\,. \end{cases}$$

Then from (2.14) and $(S)_6$ for $i=10$ it follows

(2.24) $\pi^s_{2n+9}(P_{n+6,6}) = Z_{e'_n} \oplus Z_2\{i_{5*}\mu\} \oplus Z_4\{i_{2*}[[2\nu]]\}$ *if* $n \equiv 2 \bmod (8)$

where $Z_{e'_n}$ is generated by $i_{4*}s_*\sigma$ if $n \equiv 10 \bmod (16)$, or $i_{4*}s_*\sigma + i_{2*}[[2\nu]]$ if $n \equiv 2 \bmod (16)$.

(2.17) and (2.24) give the proof of Theorem.

Added in proof. Professor Y. Furukawa has pointed out to the author that in [5], [17], [18] and [19] the stable homotopy groups $\pi_{2n+i}(W_{n+k,k})$ have been calculated for $k \leqq 4$ and $i \leqq 36$, and K. Oguchi [19] partly treated them for $k=5$.

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