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<td>Ōshima, Hideaki</td>
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Osaka University
ON THE HOMOTOPY GROUP $\pi_{2n+9}(U(n))$ FOR $n \geq 6$

HIDEAKI OSHIMA

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The homotopy groups $\pi_{2n+i}(U(n))$ of the unitary group $U(n)$ for $0 \leq i \leq 8$, $i=10$ and 12 were determined by Borel and Hirzebruch [2], Bott [3], Kervaire [7], Toda [22, 23], Matsunaga [8–12], Mimura and Toda [13], Mosher [14, 15], and Imanishi [6]. For $n \geq 5$ and $i=9$, 11 or 13 the odd components were determined by [12] and [6], but the 2-component had not been completely determined. Indeed Mosher [15] has not determined some group extensions which appear in case of $i=9$ only if $n \equiv 2, 4$ or $6 \mod (8)$ and $n \geq 6$. In this note we shall determine these group extensions for $i=9$. $\pi_{2n+9}(U(n))$ for $n \leq 5$ was determined by [6], [13], [15] and [23]. Therefore we shall complete the computation of $\pi_{2n+9}(U(n))$. While the group $\pi_{2n+9}(U(n))$ has been computed by Vastersavendts [24] for $n \equiv 0 \mod (4), 6 \mod (8)$ or $2 \mod (16)$, her results contradict Mosher's [15] and ours for $n \equiv 0 \mod (16)$ and $n \equiv 6 \mod (8)$ respectively.

We shall prove

**Theorem.** The 2-component of $\pi_{2n+9}(U(n))$ for $n \equiv 2, 4$ or $6 \mod (8)$ and $n \geq 6$ is given by the following table:

<table>
<thead>
<tr>
<th>$n \mod (_)$</th>
<th>$\pi_{2n+9}(U(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(16)</td>
<td>$Z_2 \oplus Z_4 \oplus Z_2$</td>
</tr>
<tr>
<td>10(32)</td>
<td>$Z_2 \oplus Z_4 \oplus Z_4$</td>
</tr>
<tr>
<td>26(64)</td>
<td>$Z_2 \oplus Z_4 \oplus Z_8$</td>
</tr>
<tr>
<td>58(64)</td>
<td>$Z_2 \oplus Z_4 \oplus Z_{16}$</td>
</tr>
<tr>
<td>4(8)</td>
<td>$Z_2 \oplus Z_2 \oplus Z_8$</td>
</tr>
<tr>
<td>6(8)</td>
<td>$Z_2 \oplus Z_4$</td>
</tr>
</tbody>
</table>

where $Z_m = Z/mZ$ is the cyclic group of order $m$.

We shall use the notations and terminologies defined in [20] or the book of Toda [23] without any reference.

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1. Method of computation

By Theorem 4.3 of Toda [22] we know that \( \pi_{2n+9}(U(n)) \) is isomorphic to the stable homotopy group \( \pi_{2n+9}^{s}(P_{n+6,6}) \) of the stunted complex projective space \( P_{n+6,6} = P_{n+6}/P_n \) if \( n \geq 5 \). We shall compute \( \pi_{2n+9}(P_{n+6,6}) \).

Consider the canonical cofibration

\[
S^{2(n+k)-3} \overset{p_{n+k-1,k-1}}{\longrightarrow} P_{n+k-1,k-1} \overset{i_1}{\longrightarrow} P_{n+k,k} \overset{q_k}{\longrightarrow} S^{2(n+k)-2}
\]

and the associated exact sequence

\[(S)_k: \quad \cdots \to G_{i-2k+2} \overset{p_{\ast}}{\longrightarrow} \pi_{2n-1+i}(P_{n+k-1,k-1}) \overset{i_{1\ast}}{\longrightarrow} \pi_{2n-1+i}(P_{n+k,k}) \overset{q_{\ast}}{\longrightarrow} G_{i-2k+1} \overset{p_{\ast}}{\longrightarrow} \cdots
\]

We set the two steps of computation:

1. determine the \( G_{\ast}-\)module structure of \( \pi_{2n}^{s}(P_{n+k-1,k-1}) \),
2. describe \( p_{n+k-1,k-1} \in \pi_{2n+9}^{s}(P_{n+k-1,k-1}) \) explicitly.

If these two are possible, we know \( \pi_{2n+9}^{s}(P_{n+k,k}) \) up to group extension

\[
0 \to \text{Cokernel of } p_{\ast} \to \pi_{2n-1+i}(P_{n+k,k}) \to \text{Kernel of } p_{\ast} \to 0.
\]

To determine this group extension, we prepare a lemma.

**Lemma 1** (cf. Theorem 2.1 of [13]). Let \( A \overset{f}{\longrightarrow} X \overset{i}{\longrightarrow} C(f) \) be a cofibration and

\[
\cdots \to \pi_{s}^{s}(X) \overset{i_{\ast}}{\longrightarrow} \pi_{s}^{s}(C(f)) \overset{\Delta}{\longrightarrow} \pi_{s-1}^{s}(A) \overset{f_{\ast}}{\longrightarrow} \pi_{s-1}^{s}(X) \to \cdots
\]

an associated stable exact sequence. Assume that \( \alpha \in \pi_{s-1}^{s}(A) \) satisfies \( f_{\ast}(\alpha) = 0 \), and the order of \( \alpha \) is \( k \). For an arbitrary element \( \beta \) of \( \langle f, \alpha, k \rangle \subseteq \pi_{s}^{s}(X) \), there exists an element \( [\alpha] \in \pi_{s}^{s}(C(f)) \) such that

\[
\Delta([\alpha]) = \alpha \quad \text{and} \quad i_{\ast}(\beta) = -k[\alpha].
\]

Proof. By definition of Toda bracket, there exists a commutative stable diagram with \( \beta = a \circ b \):

\[
\begin{array}{ccc}
S_{n}^{s} & \overset{b}{\longrightarrow} & A \\
\downarrow k_{\ast} & & \downarrow a \\
S_{n-1}^{s} & \overset{\alpha}{\longrightarrow} & C(\alpha) \\
\downarrow f & & \downarrow a' \\
A & \overset{i}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
& \overset{\alpha'}{\longrightarrow} & C(f) \\
& \downarrow & \\
& \overset{-\alpha}{\longrightarrow} & EA \\
& \downarrow & \\
& \overset{=}{\longrightarrow} & EA
\end{array}
\]

Then we may put \( [\alpha] = -a' \).
For the above (2), we consider \((S)_k\) for \(i = 2k - 2\):

\[
\pi^2_{2n-2} (P_{n+k,k}) \xrightarrow{q_*} G_0 \xrightarrow{p_*} \pi^2_{2n-3} (P_{n+k-1,k-1})
\]

The exactness of this shows that

\[
\#p_{n+k-1,k-1} = \#(\text{Cokernel of } q_*)
\]

On the other hand by (4.5) of [20] we know that

\[
\#(\text{Cokernel of } q_*) = Q^j \{n+k, k\}
\]

\[
= C \{j M_k(C) - n - k, k\} \quad \text{for large } j
\]

and this number was determined for \(k \leq 8\) in (3.1) of [20]. We shall need the 2-component of this number for \(k = 5\) and 6. Let \(v_2(m)\) be the exponent of 2 in the factorization of an integer \(m\) into the prime powers.

**Lemma 2** ((3.1) of [20]). \(v_2(\#P_{n+4,4})\) and \(v_2(\#P_{n+5,5})\) are given by the following table:

<table>
<thead>
<tr>
<th>(v_2(#P_{n+4,4}))</th>
<th>(n \mod ( ))</th>
<th>(v_2(#P_{n+5,5}))</th>
<th>(n \mod ( ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4, 6(8)</td>
<td>4</td>
<td>4, 6(8)</td>
</tr>
<tr>
<td>3</td>
<td>0(8), 2(16)</td>
<td>3</td>
<td>0, 2(16)</td>
</tr>
<tr>
<td>2</td>
<td>10(16)</td>
<td>2</td>
<td>8(16), 10(32)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>26(64)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>58(64)</td>
</tr>
</tbody>
</table>

Considering the above (1) and (2), we shall compute inductively \(\pi^2_{2n-1+i}(P_{n+k,k})\) for \(k \leq 6\) and some \(i \leq 10\). Since the suspension \(EP_{n+k,k}\) is 2n-connected and the pair \((W_{n+k,k}, EP_{n+k,k})\) is \((4n+3)\)-connected, it follows that \(\pi^2_{2n-1+i}(P_{n+k,k})\) is isomorphic to \(\pi^2_{2n+i}(W_{n+k,k})\) for \(i \leq 2n\), where \(W_{n+k,k} = U(n+k)/U(n)\) is the complex Stiefel manifold. Nomura and Furukawa [16] have computed \(\pi^2_{2n+i}(W_{n+k,k})\) for \(k = 2, 3\) and \(i \leq 21, 19\) respectively. Therefore we already know \(\pi^2_{2n-1+i}(P_{n+k,k})\) for \(2 \leq k \leq 3\) and \(i \leq 10\). But informations for (1) from [16] are not sufficient for our purpose. So we shall recompute some \(\pi^2_{2n-1+i}(P_{n+k,k})\) for \(k \leq 3\).

**2. Computation**

From now on, \(n\) means always an even integer \(\geq 6\), \(\pi^*_k(\ )\) and \(G_\ast\) often denote only the 2-primary component of itself. We work in the stable category of pointed spaces and stable maps between them.
Since $p_{n+1,1}n_1=0$, it follows that $P_{n+2,2}=S^{2n+1} \vee S^{2n+2}$. Let $s: S^{2n+2} \to P_{n+2,2}$ be an inclusion map which is a right inverse of $q_1$. Then

\begin{equation}
(2.1) \quad i_1 \ast + s_1: G_{i-1} \ast G_{i-3} \to \pi^{i+1}_{2n-1+i}(P_{n+2,2}) \text{ is an isomorphism.}
\end{equation}

By the proof of (1.11), (i) of (1.13) and (1.14) of [20], we have

\[ p_{n+2,2} = (n/2)i_1(n+\alpha) + s_1: S^{2n+3} \to P_{n+2,2} = S^{2n} \vee S^{2n+2}. \]

Put

\[ e_n = \begin{cases} 1 & \text{if } n \equiv 0 \mod (4) \\ 2 & \text{if } n \equiv 2 \mod (4). \end{cases} \]

Then by (2.1) and (S) for $i=8$, we have a short exact sequence

\[ 0 \to Z_{10} \{i_{2n} \sigma\} \to \pi^{i+1}_{2n+1}(P_{n+3,3}) \to Z_{\eta/e_n} \{e_n \nu\} \to 0. \]

We have

\[ \langle p_{n+2,2}, e_n \nu, (8/e_n) \xi \rangle = \langle (n/2)i_{1n} \nu, e_n \nu, (8/e_n) \xi \rangle + \langle s_1 \eta, e_n \nu, (8/e_n) \xi \rangle \]
\[ \supseteq i_1 \langle (n/2) \nu, e_n \nu, (8/e_n) \xi \rangle + s_1 \langle \eta, e_n \nu, (8/e_n) \xi \rangle \]
\[ \supseteq i_1 \langle (n/e_n/4) \langle 2/e_n \nu, e_n \nu, (8/e_n) \xi \rangle \rangle \]
\[ \supseteq 0 \]

since $\langle \eta, e_n \nu, (8/e_n) \xi \rangle \subseteq G_5 = 0$ and $\langle (2/e_n) \nu, e_n \nu, (8/e_n) \xi \rangle \supseteq 0$ (see e.g. [16]). Therefore by Lemma 1 the above short exact sequence splits, that is, there exists $[e_n \nu] \in \pi^{i+1}_{2n+1}(P_{n+3,3})$ with $q_2[e_n \nu] = e_n \nu$ and

\begin{equation}
(2.2) \quad \pi^{i+1}_{2n+1}(P_{n+3,3}) = Z_{10} \{i_{2n} \sigma\} \oplus Z_{\eta/e_n} \{[e_n \nu]\}. \end{equation}

It follows from (S) for $i=9$ that $i_1: \pi^{i+1}_{2n+3}(P_{n+2,2}) \to \pi^{i+1}_{2n+1}(P_{n+3,3})$ is an isomorphism. Hence by (2.1) we have

\begin{equation}
(2.3) \quad \pi^{i+1}_{2n+3}(P_{n+3,3}) = Z_{10} \{i_{2n} \sigma\} \oplus Z_2 \{i_{2n} \nu\} \oplus Z_2 \{i_{2n} \nu\} \oplus Z_{\eta/e_n} \{i_{2n} \nu^3\}. \end{equation}

From (2.1) and (S) for $i=10$ it follows that

\begin{equation}
(2.4) \quad \pi^{i+1}_{2n+4}(P_{n+3,3}) = Z_{10} \{i_{2n} \sigma\} \oplus Z_2 \{i_{2n} \mu\} \oplus Z_2 \{i_{2n} \eta \nu\} \oplus Z_{\eta/e_n} \{i_{2n} \nu^3\}. \end{equation}

Analysing $p_{n+k,i}$ for $k=3, 4$ and 5, we consider the followings. Put

\[ L_{m,k} = \begin{cases} 1 & \text{if } m+k \equiv 1 \mod (2) \\ 2 & \text{if } m+k \equiv 0 \mod (2). \end{cases} \]

Then, since $L_{m,k}(m+k-1) \equiv 0 \mod (2)$, $q_{l-1}(L_{m,k}P_{m+k,i}) = L_{m,i}(m+k-1) \eta = 0$ and hence $i_{1n}(L_{m,k}P_{m+k,i})$ is not empty for $1 < l < m+k$, and
HOMOTOPY GROUP \( \pi_{2n+4}(U(n)) \)

\((T)_k\)

\[ i_{1*}^{-1}(L_{m,k}p_{m+k}) = i_{1*}^{-1}(L_{m,k}q_{m-1*}p_{m+k}) \cup q_{m-1*}i_{1*}^{-1}(L_{m,k}p_{m+k}) \]

and by (1.15) of [20]

\((T)_0^t\)

\[ q_{m-2*}^{-1}(L_{m,k}p_{m+k}) = \]

\[ = \{ (m+k-2)(v+\alpha_1) \quad \text{if } m+k \equiv 0 \mod (2) \}

\[ \{ (1/2)(m+k+1)(v+\alpha_1), (1/2)(m+k+1)(v+\alpha_1)+4v \} \]

\[ \text{if } m+k \equiv 1 \mod (2). \]

Now \( q_{1*}=s_{1*}: \pi_{2n+5}(P_{n+3,3}) \rightarrow \pi_{2n+5}(S^{2n+3}) = G_3 \) by (2.1), since \( q_{1*}s_{1*}=1 \).

Then by \((T)_0^t\)

\[ q_{n-1*}^{-1}(p_{n*,3}) \ni ((n+4)/2)s_{1*}(v+\alpha_1) \]

and by \((T)_3\)

\[ p_{n+3,3} = ((n+4)/2)i_{1*}s_{1*}(v+\alpha_1) \]

so that \( p_{n+3,3} \circ \eta = 0 \) and

\[ \langle p_{n+3,3}, \eta, 2\alpha \rangle \cup i_{1*}s_{1*} \langle ((n+4)/2)v, \eta, 2\alpha \rangle = 0 \]

and by Lemma 1 there exists \([\eta] \in \pi_{2n+7}(P_{n+4})\) with \( q_{3*} [\eta] = \eta \) and

\[ \pi_{2n+7}(P_{n+4}) = Z_{10} \{ i_{2*}e \} \oplus Z_{10} \{ i_{1*}e \} \oplus Z_{10} \{ i_{1*}e \} \oplus Z_{10} \{ i_{1*}e \} \oplus Z_{2} \{ [\eta] \eta \} . \]

We have also the following from (2.3) and (5) for \( i = 9 \)

\[ \pi_{2n+8}(P_{n+4,4}) = Z_{10} \{ i_{2*}e \} \oplus Z_{2} \{ i_{2*}e \} \oplus Z_{10} \{ i_{2*}e \} \oplus Z_{10} \{ i_{2*}e \} \oplus Z_{2} \{ [\eta] \eta \} . \]

By the same argument as the proof of (2.2) we know that there exists \([e_{e*}] \in \pi_{2n+9}(P_{n+4,4})\) with \( q_{3*} [e_{e*}] = e_{e*} \) and

\[ \pi_{2n+9}(P_{n+4,4}) = Z_{10} \{ i_{2*}e \} \oplus Z_{2} \{ i_{2*}e \} \oplus Z_{10} \{ i_{2*}e \} \oplus Z_{10} \{ [e_{e*}] \} . \]

To compute \( \pi_{2n+9}(P_{n+5,5}) \) we shall prepare four lemmas.

Remember that in [20] we used the notations: \( HP_{m+k,h} = HP_{m+k}/HP_m \), the stunted quaternionic projective space; \( \pi: P_{2n+2,2k} \rightarrow HP_{m+k,h} \), the canonical quotient map;

\[ S^{2n+4h-1} \rightarrow HP_{(n/2)+k} \rightarrow HP_{(n/2)+k+1} \]

the canonical cofibration.

**Lemma 3.** We have
Proof. Considering the stable homotopy exact sequence associated with (2.8) for \( k=1 \), we obtain (ii) and (iii) immediately since \( G_4=G_5=0 \) and \( p^H_{(n/2)+1,1}=(n/2)(\nu+\alpha_1) \) and by Lemma 1 we have a split exact sequence:

\[
0 \to Z_6\{i^H_0\sigma\} \to \pi^H_{2n+7}(HP_{(n/2)+2,2}) \to Z_{n/2}\{e_\nu\} \to 0 .
\]

Then the following commutative diagram induces (i):

\[
\begin{array}{ccccccccc}
S^{2n} & \xrightarrow{i_1} & P_{n+2,2} & \xrightarrow{i_1} & P_{n+3,3} & \xrightarrow{q_2} & S^{2n+4} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{2n} & = & HP_{(n/2)+1,1} & \xrightarrow{i^H_1} & HP_{(n/2)+2,2} & \xrightarrow{q^H_1} & S^{2n+4} \\
\end{array}
\]

Since \( q^H_0 \circ p^H_{(n/2)+2,2} \circ \eta = ((n/2)+1)(\nu+\alpha_1)\eta = 0 \), there exists a map \( f: S^{2n+8} \to S^{2n} \) with \( i^H_0 \circ f = p^H_{(n/2)+2,2} \circ \eta \). It is easily seen that \( i^H_1: \{HP_{(n/2)+2,2}, S^{2n-1}\} \to \{S^{2n}, S^{2n-1}\} = Z_2\{\eta\} \) is an isomorphism. Let \( h \equiv \{HP_{(n/2)+2,2}, S^{2n-1}\} \) be the element with \( h \circ i^H_1 = \eta \). It follows from (2.7) of [21] that

\[
h \circ p^H_{(n/2)+2,2} = \begin{cases} 
\varepsilon & \text{if } n \equiv 2 \text{ mod (8)} \\
\nu & \text{if } n \equiv 4 \text{ mod (8)} \\
\varepsilon + \nu & \text{if } n \equiv 6 \text{ mod (8)} \\
0 & \text{if } n \equiv 0 \text{ mod (8)} 
\end{cases}
\]

Since \( \eta \circ f = h \circ p^H_{(n/2)+2,2} \circ \eta \) and \( \eta: G_8 \to G_9 \) is a monomorphism, we obtain (iv). This completes the proof of Lemma 3.

**Lemma 4.** For suitably chosen \([e_\nu\nu]\) it holds that \([\eta] \gamma^2 = (4|e_\nu\nu|)[e_\nu\nu]\).

**Proof.** By Proposition 1.4 of Toda [23]

(2.9) \[ [\eta] \gamma^2 = [\eta] \circ [2\eta, 2\nu] = [\gamma], 2\nu, \eta \circ 2\nu . \]
Let \( \text{Indet} \langle \alpha, \beta, \gamma \rangle \) be an indeterminacy of a Toda bracket \( \langle \alpha, \beta, \gamma \rangle \). Then

\[
\pi_{2n+3}(P_{n+4,i}) \supset \text{Indet} \langle [\gamma], 2\iota, \eta \rangle = [\gamma] \circ \{ S_{2n+9}, S_{2n+1} \} + \pi_{2n+8}(P_{n+4,i}) \circ \eta
\]

\[
= Z_2 \{ [\gamma] \gamma^3 \} + Z_2 \{ i_{3*} \eta \} + Z_2 \{ i_{3*} \nu^2 \}
\]

and

\[
q_{3*} \text{Indet} \langle [\gamma], 2\iota, \eta \rangle = Z_2 \{ \gamma \} = Z_2 \{ 4\nu \} = \text{Indet} \langle \eta, 2\iota, \eta \rangle
\]

and, since \( q_{3*} \langle [\gamma], 2\iota, \eta \rangle \subset q_{3*} \langle [\gamma], 2\iota, \eta \rangle = \langle \eta, 2\iota, \eta \rangle \), we have

\[
q_{3*} \langle [\gamma], 2\iota, \eta \rangle = \langle \eta, 2\iota, \eta \rangle = \{ 2\nu, 6\nu \}.
\]

Hence there exists an element in \( \langle [\gamma], 2\iota, \eta \rangle \) which is mapped to \( 2\nu \) by \( q_{3*} \). By (2.7) this element has a form as \( (2/[e_n])[[e_n^\nu]] + i_{3*} x \) for some \( x \in \pi_{2n+9}(P_{n+3,3}) \), and from (2.9) it follows that \( 4i_{3*} x = 0 \). Then by (2.7) \( 2i_{3*} x \) is divisible by \( 8 \), that is, \( 2i_{3*} x = 8i_{3*} y \) for some \( y \in \pi_{2n+9}(P_{n+3,3}) \). Then

\[
[\gamma] \gamma^2 = 2 \{ (2/[e_n])[[e_n^\nu]] + i_{3*} x \}
\]

\[
= (4/[e_n])[[e_n^\nu]] + 2e_{3*} i_{3*} y.
\]

Since \( q_{3*}([[e_n^\nu]] + 2e_{3*} i_{3*} y) = e_n \nu \) and the order of \( [[e_n^\nu]] + 2e_{3*} i_{3*} y \) is \( 8/e_n \), we may change \( [[e_n^\nu]] \) for \( [[e_n^\nu]] + 2e_{3*} i_{3*} y \). So the conclusion follows.

**Appointment:** From now on we assume that \( [[e_n^\nu]] \) satisfies \( [\gamma] \gamma^2 = (4/[e_n])[[e_n^\nu]] \).

Since \( q_{3*} f_{n+4,i} = (n+3)[\gamma] = \eta \), by (2.5) we can put

\[
p_{n+4,i} = a_i i_{3*} \sigma + b_i i_{3*} [e_n^\nu] + [\gamma] + \text{odd torsion}
\]

for some integers \( a_i \) and \( b_i \). By Lemma 2 and (2.5) we have

\[
a_i \equiv \begin{cases} 
1 \mod (2) & \text{if } n \equiv 4 \text{ or } 6 \mod (8) \\
0 \mod (2) & \text{if } n \equiv 0 \text{ or } 2 \mod (8). 
\end{cases}
\]

By \( (T)_6 \) and \( (T)_4 \), for any \( p' \in q_{n-18} f_{n+1} \subset \pi_{2n+7}(P_{n+3,3}) \) we have

\[
i_{3*} p' = 2p_{n+4,i} \quad \text{and} \quad q_{3*} p' = (n+2)(\nu + \alpha_i).
\]

Then \( p' = 2a_i i_{3*} \sigma + 2b_i [e_n^\nu] + \text{odd torsion} \). Applying \( q_{3*} \) to this equation we know that \( 2b_i e_{n+2} \equiv n+2 \mod (8) \), and

\[
b_i \equiv \begin{cases} 
1 \mod (2) & \text{if } n \equiv 0 \mod (4) \text{ or } 2 \mod (8) \\
0 \mod (2) & \text{if } n \equiv 6 \mod (8).
\end{cases}
\]

**Lemma 5.** We have
Proof. By Lemma 1 we can easily construct a commutative diagram:

\[
\begin{array}{cccccc}
S^{2n+8} & \xrightarrow{e_\nu} & S^{2n+4} & \xrightarrow{C(e_\nu)} & S^{2n+7} & \xrightarrow{e_\nu} & S^{2n+4} \\
\eta \downarrow & & \eta \downarrow & & \eta \downarrow \\
S^{2n+3} & \xrightarrow{\sigma} & P_{n+2,2} & \xrightarrow{a} & P_{n+3,3} & \xrightarrow{i_1} & S^{2n+4}
\end{array}
\]

Then \( a \circ b \equiv \langle \sigma_{n+2,2}, e_\nu, \eta \rangle \) and this Toda bracket is a coset of

\[
\pi_{2n+6}(P_{n+2,2})/\pi_{2n+7}(P_{n+2,2}) \circ \eta = [Z_2\{i_{18}\} \oplus Z_2\{i_{18}\} \{0, i_{18}(\varepsilon + \varphi)\}] \oplus Z_2\{s_8\varphi^2\}.
\]

We have

\[
\langle \sigma_{n+2,2}, e_\nu, \eta \rangle = \langle (n/2)i_{18}\nu, e_\nu, \eta \rangle + \langle s_8\eta, e_\nu, \eta \rangle
\]

\[
\equiv \langle \sigma_{n/4}i_{18}\varepsilon + e_\nu s_8\varphi^2 \rangle
\]

since \( \langle 2/e_\nu, e_\nu, \eta \rangle = \varepsilon + G; \sigma \eta \) and \( \langle \eta, e_\nu, \eta \rangle = e_\nu^2 \) by Toda [23]. Hence

(2.12) \[ e_\nu \eta = i_{18}(a \circ b) \]

\[
= (ne_\nu/4)i_{28}\varepsilon + e_\sigma i_{18}s_8\varphi^2 \text{ or } ((ne_\nu/4) + 1)i_{28}\varepsilon + i_{28}\varphi + e_\sigma i_{18}s_8\varphi^2.
\]

Thus Lemma 5 follows if \( n \equiv 6 \pmod{8} \). By Lemma 4

(2.13) \[ p_{n+4,4} \circ \eta^2 = a_{n}i_{18}(\varepsilon + \varphi^3) + b_{n}i_{18}[e_\nu \eta^2] + (4/e_\nu)[[e_\nu \eta^2]] \]

and by (iii) of Lemma 3, the fact \( 4/e_\nu \equiv 0 \pmod{2} \) and the commutativity of the diagram in the proof of Lemma 3 it follows that

\[
p_{(n+2)/2+2,2} \circ \eta^2 = \pi \circ p_{n+4,4} \circ \eta^2
\]

\[
= a_{n}i_{18}(\varepsilon + \varphi^3) + b_{n}i_{18}[e_\nu \eta^2].
\]

Then the conclusions for \( n \equiv 6 \pmod{8} \) follow from (iii) and (iv) of Lemma 3, (2.10), (2.11) and (2.12). This completes the proof of Lemma 5.

Lemma 6. We have
Homotopy Group $\pi_{2n+9}(U(n))$

\[ p_{n+4,4}\circ \eta^2 = \begin{cases} 
  i_3 \gamma \varepsilon + 2[2\nu] & \text{if } n \equiv 2 \mod (4) \\
  \frac{n}{4} i_3 \delta \nu^3 + 4[[\nu]] & \text{if } n \equiv 0 \mod (4) .
\end{cases} \]

Proof. The conclusion follows from (2.7), (2.10), (2.11), Lemma 5 and (2.13).

Now we compute $\pi_{2n+9}(P_{n+5,5})$. Since $p_{n+4,4}\circ \eta = [\gamma]\gamma + \text{(other term)}$ is non-zero, it follows from (2.7), Lemma 6 and $(S)_5$ for $i = 10$ that

\[ \pi_{2n+9}(P_{n+5,5}) = Z_{16} \{ i_3 \sigma \} \oplus Z_2 \{ i_4 \mu \} \oplus H_n \]

where

\[ H_n = \begin{cases} 
  Z_4 \{ i_1 [[2\nu]] \} \text{ with the relations } i_4 \gamma \varepsilon = 2 i_1 [[2\nu]] \text{ and } i_4 \delta \nu^3 = 0 & \text{if } n \equiv 2 \mod (4) \\
  Z_2 \{ i_4 \gamma \varepsilon \} \oplus Z_2 \{ i_4 \delta \nu^3 \} \oplus Z_4 \{ i_1 [[\nu]] \} & \text{if } n \equiv 0 \mod (8) \\
  Z_2 \{ i_4 \gamma \varepsilon \} \oplus Z_8 \{ i_1 [[\nu]] \} \text{ with the relation } i_4 \delta \nu^3 = 4 i_1 [[\nu]] & \text{if } n \equiv 4 \mod (8) .
\end{cases} \]

By $(T)_5$

\[ q_{3\circ q_{n-1,4,1}}(p_{n+5}) = \{ ((n+6)/2)(\nu + \alpha_i), ((n+6)/2)(\nu + \alpha_i) + 4\nu \} \]

and hence we can choose a map $\tilde{p} \in q_{n-1,4,1}(p_{n+5}) \subseteq \pi_{2n+9}(P_{n+4,4})$ with

\[ q_{3\circ \tilde{p}} = \begin{cases} 
  ((n+6)/2)(\nu + \alpha_i) + 4\nu & \text{if } n \equiv 2 \mod (16) \\
  ((n+6)/2)(\nu + \alpha_i) & \text{otherwise}
\end{cases} \]

and then by $(T)_5$

\[ i_3 \circ \tilde{p} = p_{n+5,5} . \]

By (2.7) we can put

\[ \tilde{p} = a_n i_3 \sigma + b_n i_3 \mu + c_n i_3 \gamma \varepsilon + d_n i_3 \delta \nu^3 + d_n [[\nu]] + \text{odd torsion} \]

for some integers $a_n$, $b_n$, $c_n$, $d_n$ and $d_n$. Remark that $i_3 \delta \nu^3 = 0$ if $n \equiv 2 \mod (4)$. We have

\[ d_n e_n \nu + \text{odd torsion} = q_{3\circ \tilde{p}} \]

\[ = \begin{cases} 
  (((n+6)/2)+4)\nu + \text{odd torsion} & \text{if } n \equiv 2 \mod (16) \\
  ((n+6)/2)\nu + \text{odd torsion} & \text{otherwise}
\end{cases} \]

and

\[ d_n \equiv \begin{cases} 
  1 \mod (2) & \text{if } n \equiv 0 \mod (4) \text{ or } 6 \mod (8) \\
  0 \mod (4) & \text{if } n \equiv 2 \mod (8) .
\end{cases} \]
Put $p_{n+5,5} = a_{n} i_{5,n} s_{a} \sigma + b_{n} i_{1} \mu + \hat{p}$. Then the 2-primary part of $\hat{p}$ is contained in $H_{n}$. Hence by Lemma 2 and (2.14) we have

$$a_{n} \equiv 1 \mod (2) \quad \text{if} \ n \equiv 4 \text{ or } 6 \mod (8).$$

Then by (2.14), (2.16) and $(S)_{5}$ for $i=10$ we have

$$(2.17) \quad \pi_{2n+6}^{2n+6}(P_{n+6,6}) = Z_{g/\mu} \{i_{s_{a} s_{a} \sigma}\} \oplus Z_{2} \{i_{s_{a} \mu}\} \oplus Z_{2/\eta} \text{ if } n \equiv 4 \text{ or } 6 \mod (8)$$

where if $n \equiv 4 \mod (8)$, $Z_{2/\eta} = Z_{2}$ is generated by $i_{s_{a} \eta \sigma}$.

Next suppose that $n \equiv 2 \mod (8)$. Let $l$ be the odd component of the order of $\hat{p}$. Of course $l$ is an odd integer. Put $\hat{p} = l a_{n} s_{a} \sigma + b_{n} i_{1} \mu + c_{n} i_{1} \eta \epsilon$. Then by (2.15) and (2.16)

$$l \hat{p} = i_{2n} \hat{p}$$

and we have a commutative diagram in which the each horizontal sequences are cofibrations and $l$ denotes a multiplication by $l$:

We calculate the Adams' $e_{c}$ and $e_{g}$ invariants of $\pi \circ \hat{p} \in G_{9}$.

**Lemma 7.** We have

(i) $e_{c}(\pi \circ \hat{p}) = 0$ and $b_{n} \equiv 0 \mod (2)$,

(ii) $e_{g}(\pi \circ \hat{p}) = \begin{cases} 1 & \text{if } n \equiv 2 \mod (16) \\ 0 & \text{if } n \equiv 10 \mod (16) \end{cases}$ and $c_{n} \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 2 \mod (16) \\ 0 \mod (2) & \text{if } n \equiv 10 \mod (16). \end{cases}$

**Proof.** Applying $\tilde{K}$ to the above diagram, we can show the first part of (i) by the similar method as the proof of (1.12) of [20]. Then the second part of (i) follows, since $\pi \circ s = \eta^{2}$ or $0$, $e_{c}(\eta^{2} \sigma) = e_{c}(\eta \epsilon) = 0$ and $e_{c}(\mu) \neq 0$ by [1].
Put \(n = 8m + 2\). Applying \(\widetilde{KO}^{-4}\) to the above diagram, we have the following commutative diagram in which the horizontal sequences are exact:

![Diagram (2.18)]

By Theorem 2 of Fujii [4] it is easily seen that

\[
\widetilde{KO}^{-4}(P_{8m+7,5}) = Z\{z_2, z_2 z_0^{4m}, z_2 z_0^{4m+1}, z_2 z_0^{4m+2}\} + Z\{z_0, z_0 z_0^{4m+1}\}
\]

Also note that a generator \(d\) of \(\widetilde{KO}^{-4}(S^{16m+4}) = Z\) satisfies

\[
\pi^* d = z_2 z_0^{4m} + x z_2 z_0^{4m+1}
\]

for some integer \(x\). We shall not need the explicit value of \(x\). Here we regard \(\widetilde{KO}^{-4}(X/A)\) as a subgroup of \(\widetilde{KO}^{-4}(X)\) if the quotient map \(X \to X/A\) induces a monomorphism. Similar remarks shall hold in the forthcoming proof of (A). By chasing diagram, we know that there exist elements \([z_2 z_0^{4m}]\) and \([z_2 z_0^{4m+1}]\) in \(\widetilde{KO}^{-4}(C(l\tilde{p}))\) such that

\[
\pi^* [z_2 z_0^{4m}] = l_{i_1}^*[z_2 z_0^{4m}] \quad \text{and} \quad \pi^* [z_2 z_0^{4m+1}] = l_{i_2}^*[z_2 z_0^{4m+1}].
\]

Put \(a' = [z_2 z_0^{4m}] + x'[z_2 z_0^{4m+1}]\). Then there exists an element \(a \in \widetilde{KO}^{-4}(C(\pi \circ \tilde{p}))\) such that

\[
\pi^* a = i_{i_2}^* a' \quad \text{and} \quad j^* a = d.
\]

Let \(b \in \widetilde{KO}^{-4}(C(\pi \circ \tilde{p}))\) and \(b' \in \widetilde{KO}^{-4}(C(l\tilde{p}))\) be the images of the generator of \(\widetilde{KO}^{-4}(S^{16m+14}) = Z_2\).

Now we assume the followings which shall be proved later:
(A) \( i^*_2 [x_2 z_0^4 m + 2] = e_{2m} i^* b' \),
(B) the order of \( i^*_2 [x_2 z_0^4 m + 1] \) is 2.

Remark that \( e_{2m} = 1 \) if \( m \equiv 0 \mod (2) \), or 2 if \( m \equiv 1 \mod (2) \), and \( i^* b' \) is the generator of the 2-torsion of \( KO^{-4}(C(\bar{p})) \). We have
\[
\psi^3 a = 3^{2m+4} a + \lambda b
\]
for some \( \lambda \in \mathbb{Z}_2 \), and
\[
e_2 (\pi \circ \bar{p}) = \lambda
\]
and
\[
\pi^* \psi^3 a = \pi^* (3^{2m+4} a + \lambda b) = i^*_2 (3^{2m+4} a' + \lambda b')
\]
On the other hand
\[
\pi^* \psi^3 a = \psi^3 \pi^* a = \psi^3 i^* a' = i^*_2 \psi^3 a'
\]
and
(2.19)
\[
i^*_2 (3^{2m+4} a' + \lambda b') = i^*_2 \psi^3 a'.
\]
Since the order of \( i^*_2 [x_2 z_0^4 m + 3] \) is 2 and \( i^*_2 [x_2 z_0^4 m + 2] = e_{2m} i^* b' \) by (A), we have
\[
i^* \psi^3 a' = \psi^3 i^* a'
\]
\[
= \psi^3 \{ i^*_2 (x_2 z_0^4 m + x_2 z_0^4 m + 1) \}
\]
\[
= i^*_2 \psi^3 (x_2 z_0^4 m + x_2 z_0^4 m + 1)
\]
\[
= i^*_2 \{ 3^{2m+4} x_2 z_0^4 m + ((8m+2)3^{2m+3} + x3^{2m+5}) x_2 z_0^4 m + 1
\]
\[
+ ((4m+1)(8m+1)3^{2m+2} + x(8m+4)3^{2m+5}) x_2 z_0^4 m + 2
\]
\[
+ ((4m+1)(8m+1)8m3^{2m} + x(4m+2)(8m+3)3^{2m+3}) x_2 z_0^4 m + 3 \}
\]
\[
i^*_2 \{ 3^{2m+4} [x_2 z_0^4 m] + ((8m+2)3^{2m+3} + x3^{2m+5}) [x_2 z_0^4 m + 1] + e_{2m} b' \}
\]
Then, since \( i^* \) is a monomorphism,
\[
\psi^3 a' = 3^{2m+4} [x_2 z_0^4 m] + ((8m+2)3^{2m+3} + x3^{2m+5}) [x_2 z_0^4 m + 1] + e_{2m} b'
\]
and by (B)
\[
i^*_2 \psi^3 a' = 3^{2m+4} i^*_2 [x_2 z_0^4 m] + x i^*_2 [x_2 z_0^4 m + 1] + e_{2m} i^*_2 b'
\]
also by (2.19) this equals to
\[
i^*_2 (3^{2m+4} a' + \lambda b') = i^*_2 \{ 3^{2m+4} (x [x_2 z_0^4 m] + x [x_2 z_0^4 m + 1]) + \lambda b' \}
\]
\[
= 3^{2m+4} i^*_2 [x_2 z_0^4 m] + x i^*_2 [x_2 z_0^4 m + 1] + \lambda i^*_2 b'
\]
and, since \( i^*_2 b' \) is non-zero,
\[
\lambda = e_{2m} \quad \text{in} \mathbb{Z}_2
\]
and this implies the first part of (ii). Since $\pi \circ s = 0$ or $\eta$, and $\alpha_s \equiv 0 \mod (2)$ by Lemma 2, it follows that by the second part of (i) we have

$$\pi \circ \tilde{p} = c_n \eta \xi$$

and the above proof of the first part of (ii) shows that

$$c_n \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 2 \mod (16) \\ 0 \mod (2) & \text{if } n \equiv 10 \mod (16) \end{cases}.$$ 

This implies the second part of (ii).

We shall give the proofs of (A) and (B).

The proof of (A): We have the following commutative diagram:

\[
\begin{array}{cccccc}
S^{16m+13} & \xrightarrow{i_{16}} & P_{8m+7,5} & \xrightarrow{i_1} & P_{8m+8,6} & \xrightarrow{i_1} S^{16m+14} \\
S^{16m+13} & \xrightarrow{q_3} & P_{8m+6,4} & \xrightarrow{i_1} & C(\tilde{p}) & \xrightarrow{q_2} S^{16m+14} \\
S^{16m+13} & \xrightarrow{q_3} & P_{8m+7,2} & \xrightarrow{i_1} & P_{8m+8,3} & \xrightarrow{q_2} S^{16m+14} \\
S^{16m+13} & \xrightarrow{q_3} & P_{8m+6,1} & \xrightarrow{i_1} & C(q_3 \circ \tilde{p}) & \xrightarrow{q_2} S^{16m+14}
\end{array}
\]

We have

$$\widetilde{KO}^{-4}(P_{8m+8,3}) = Z \{x_{2^1}z_{2}^{4m+2}\} \oplus Z_2 \{x_{2^1}z_{2}^{4m+3}\},$$

$$q_3^*x_{2^1}z_{2}^{4m+2} = x_{2^1}z_{2}^{4m+2}.$$ 

It suffices for our purpose to compute $i_{16}^*x_{2^1}z_{2}^{4m+2}$, since $i_{16}^*x_{2^1}z_{2}^{4m+2}$ is contained in the image of $\widetilde{KO}^{-4}(S^{16m+14})$, and $q_3^*$ induces an isomorphism between the images of $\widetilde{KO}^{-4}(S^{16m+14})$. We have chosen $\tilde{p}$ such that $q_3 \circ \tilde{p} = (m+1)\alpha_1$. Let $u_m$ be the order of $q_3 \circ \tilde{p}$. Then $u_m = 1$ or 3. Applying $\pi_{16m+14}(\ )$ to the above diagram, we know easily that there exists uniquely an element $u \in \pi_{16m+14}(P_{8m+8,3})$ such that $q_3 \circ u = u_m$, moreover there exists $\tilde{u} \in \pi_{16m+11}(C(q_3 \circ \tilde{p}))$ such that $q_2 \circ \tilde{u} = u_m$ and $u = \tilde{u} \circ \tilde{u}$, where $\iota$ is the identity map of $S^{16m+14}$. Since $\tilde{q}_2^*$ is the identity homomorphism of $\widetilde{KO}^{-4}(S^{16m+14}) = Z_2$, it follows that $u_2^* : \widetilde{KO}^{-4}(C(q_3 \circ \tilde{p})) \rightarrow \widetilde{KO}(S^{16m+14})$ is the inverse of $q_2^*$. Thus

$$i_{16}^*x_{2^1}z_{2}^{4m+2} = q_3^*u^*x_{2^1}z_{2}^{4m+2} = q_2^*u^*x_{2^1}z_{2}^{4m+2}.$$ 

Next we determine $u^*x_{2}z_{2}^{4m+2}$. Consider the commutative diagram:
Recall that \( \mathcal{K}(P_{8m+8,3}) = \mathbb{Z}\{z_{8m+5}, z_{8m+6}, z_{8m+7}\} \) and the real restriction homomorphism \( r \) in the right hand side is an epimorphism. We can prove the followings:

\[
\begin{align*}
(2.21) & \quad \left\{ \begin{array}{l}
\tilde{r}(g^2z_{8m+5}) = z_{8m+5}^4 + (8m+5)z_{8m+6}^4 + 2z_{8m+7}^4, \\
\tilde{r}(g^2z_{8m+6}) = z_{8m+5}^4 + 2z_{8m+6}^4 + 2z_{8m+7}^4, \\
\tilde{r}(g^2z_{8m+7}) = z_{8m+5}^4 + 2z_{8m+6}^4 + 3
\end{array} \right. \\
(2.22) & \quad \tilde{r}(g^2z_{8m+5}z_{8m+6}z_{8m+7}) = z_{8m+5}^4 + 2z_{8m+6}^4 + 2z_{8m+7}^4, \\
(2.23) & \quad \left\{ \begin{array}{l}
u^*_mz_{8m+5} = (1/3)(8m+5)(3m+2)u_m\beta, \\
\nu^*_mz_{8m+6} = (4m+3)u_m\beta, \\
\nu^*_mz_{8m+7} = u_m\beta
\end{array} \right.
\]

where \( \beta \in \mathcal{K}(S^{16m+14}) = \mathbb{Z} \) is the generator such that \( q^*\beta = z_{8m+7} \). (2.23) follows from the relation \( \psi^2u^* = u^*\psi^2 \). For (2.21) we consider the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{K}^{-i}(P_{8m+8,3}) & \xrightarrow{q^*} & \tilde{K}^{-i}(P_{8m+9}) \\
\xrightarrow{r} & & \xleftarrow{i^*_1} \\
\tilde{KO}^{-i}(P_{8m+8,3}) & \xrightarrow{q^*} & \tilde{KO}^{-i}(P_{8m+9})
\end{array}
\]

Since \( \tilde{KO}^{-i}(P_{8m+9}) = \mathbb{Z}\{z_2, z_2z_0, \ldots, z_2z_0^4 + 3\} \) is torsion free (see [4]), by the aid of the complexification homomorphism we can describe \( r \) in the right hand side explicitly. In particular we have

\[
\begin{align*}
r(g^2z_{8m+5}) &= (1/3)(8m+5)(8m^2 + 10m + 3)z_2z_0^{4m+3} + (8m+5)z_2z_0^{4m+2}, \\
r(g^2z_{8m+6}) &= (4m+3)z_2z_0^{4m+3} + 2z_2z_0^{4m+2}, \\
r(g^2z_{8m+7}) &= (8m+7)z_2z_0^{4m+3}.
\end{align*}
\]

Hence \( r \) in the left hand side satisfies (2.21). Then (2.22) follows from (2.21). By (2.22) and (2.23)

\[
\begin{align*}
u^*z_2z_0^{4m+2} = r(g^2u^*(z_{8m+5}(4m+2)z_{8m+6}+z_{8m+7})) = v_m\tilde{r}(g^2\beta)
\end{align*}
\]

where \( v_m = ((1/3)(8m+5)(3m+2) - (4m+2)(4m+3) + 1)u_m \).
Now

\[ \pi_1^* x_2^* x_0^{4m+2} = \pi_1^* q_1^* x_2^* x_0^{4m+2} \]

\[ = q_1^* \pi_1^* x_2^* x_0^{4m+2} \]

\[ = q_1^* q_2^* u^* x_0^{4m+2} \]

\[ = v_m q_1^* q_2^* r(q_2^* \beta) \]

\[ = v_m \tilde{t}^* b' \]

where the third equality follows from (2.20). Therefore (A) follows since \( v_m \equiv e_{2m} \mod (2) \).

The proof of (B): It suffices to show that the second short exact sequence from the bottom on the diagram (2.18) splits. Naturally we have a commutative diagram in which the horizontal sequences are exact:

\[
\begin{array}{c}
0 \to \widetilde{KO}^{-4}(P_{6m+4,2}) \to \widetilde{KO}^{-4}(C(p)) \to \widetilde{KO}^{-4}(S^{16m+14}) \to 0
\end{array}
\]

It is easily seen that \( q_1^* \) is a monomorphism. By Propositions 3.3 and 7.1 of Adams [1] we have a homomorphism

\[ e: G_7 = \pi_{16m+13}(S^{16m+6}) \to \text{Ext}^1(\widetilde{KO}^{-4}(S^{16m+6}), \widetilde{KO}^{-4}(S^{16m+14})) = \mathbb{Z}_2. \]

Since \( q_1 \circ \tilde{p} = la' \sigma \), and \( a_4' \equiv 0 \mod (2) \) by Lemma 2, it follows that \( q_1 \circ \tilde{p} \) is divisible by 2, and \( e(q_1 \circ \tilde{p}) = 0 \). This implies that the above lower sequence splits (see [1]), and also the upper one does. Then (B) follows and the proof of Lemma 7 is completed.

Now we proceed the computation of \( \pi_{2n+9}(P_{n+6}) \) for \( n \equiv 2 \mod (8) \). By (2.15), (2.16) and Lemma 7

\[ p_{n+5,5} = i_1 \circ \tilde{p} = a_n^* l_3 s_{*} \sigma + c_{n} i_1 e + \text{odd torsion}. \]

Then we obtain the following table by Lemma 2:

<table>
<thead>
<tr>
<th>( n \mod (\ ) )</th>
<th>( v_2(P_{n+5,5}) )</th>
<th>( a_4^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(16)</td>
<td>3</td>
<td>2(4)</td>
</tr>
<tr>
<td>10(32)</td>
<td>2</td>
<td>4(8)</td>
</tr>
<tr>
<td>26(64)</td>
<td>1</td>
<td>8(16)</td>
</tr>
<tr>
<td>58(64)</td>
<td>0</td>
<td>0(16)</td>
</tr>
</tbody>
</table>
Put

\[ e''_n = \begin{cases} 
2 & \text{if } n \equiv 2 \text{ mod } (16) \\
2^2 & \text{if } n \equiv 10 \text{ mod } (32) \\
2^3 & \text{if } n \equiv 26 \text{ mod } (64) \\
2^4 & \text{if } n \equiv 58 \text{ mod } (64).
\end{cases} \]

Then from (2.14) and (S) for \( i = 10 \) it follows

\[ (2.24) \quad \pi_{2n+i}(P_{n+i,0}) = Z_2 \oplus Z_2 \{ i_{2n} \mathfrak{u} \} \oplus Z_2 \{ i_{2n} [[2\nu]] \} \quad \text{if } n \equiv 2 \text{ mod } (8) \]

where \( Z_2 \mathfrak{u} \) is generated by \( i_{2n} \mathfrak{u} \) if \( n \equiv 10 \text{ mod } (16) \), or \( i_{2n} \mathfrak{u} + i_{2n} [[2\nu]] \) if \( n \equiv 2 \text{ mod } (16) \).

(2.17) and (2.24) give the proof of Theorem.

Added in proof. Professor Y. Furukawa has pointed out to the author that in [5], [17], [18] and [19] the stable homotopy groups \( \pi_{2n+i}(W_{n+t, k}) \) have been calculated for \( k \leq 4 \) and \( i \leq 36 \), and K. Oguchi [19] partly treated them for \( k = 5 \).

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