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ON THE HOMOTOPY GROUP $\pi_{2n+9}(U(n))$ FOR $n \ge 6$

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The homotopy groups $\pi_{2n+i}(U(n))$ of the unitary group U(n) for $0 \le i \le 8$, i=10 and 12 were determined by Borel and Hirzebruch [2], Bott [3], Kervaire [7], Toda [22, 23], Matsunaga [8–12], Mimura and Toda [13], Mosher [14, 15], and Imanishi [6]. For $n \ge 5$ and i=9, 11 or 13 the odd components were determined by [12] and [6], but the 2-component had not been completely determined. Indeed Mosher [15] has not determined some group extensions which appear in case of i=9 only if n=2, 4 or 6 mod (8) and $n\ge 6$. In this note we shall determine these group extensions for i=9. $\pi_{2n+9}(U(n))$ for $n\le 5$ was determined by [6], [13], [15] and [23]. Therefore we shall complete the computation of $\pi_{2n+9}(U(n))$. While the group $\pi_{2n+9}(U(n))$ has been computed by Vastersavendts [24] for $n\equiv 0 \mod (4)$, 6 mod (8) or 2 mod (16), her results contradict Mosher's [15] and ours for $n\equiv 0 \mod (16)$ and $n\equiv 6 \mod (8)$ respectively.

We shall prove

Theorem. The 2-component of $\pi_{2n+9}(U(n))$ for $n \equiv 2, 4$ or $6 \mod (8)$ and $n \ge 6$ is given by the following table:

<i>n</i> mod ()	$\pi_{2n+9}(U(n))$
2(16)	$Z_2 \oplus Z_4 \oplus Z_2$
10(32)	$Z_2 \oplus Z_4 \oplus Z_4$
26(64)	$Z_2 \oplus Z_4 \oplus Z_8$
58(64)	$Z_2 \oplus Z_4 \oplus Z_{16}$
4(8)	$Z_2 \oplus Z_2 \oplus Z_8$
6(8)	$Z_2 \oplus Z_4$

where $Z_m = Z/mZ$ is the cyclic group of order m.

We shall use the notations and terminologies defined in [20] or the book of Toda [23] without any reference.

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1. Method of computation

By Theorem 4.3 of Toda [22] we know that $\pi_{2n+9}(U(n))$ is isomorphic to the stable homotopy group $\pi_{2n+9}^s(P_{n+6,6})$ of the stunted complex projective space $P_{n+6,6} = P_{n+6}/P_n$ if $n \ge 5$. We shall compute $\pi_{2n+9}^s(P_{n+6,6})$.

Consider the canonical cofibration

$$S^{2(n+k)-3} \xrightarrow{p_{n+k-1,k-1}} P_{n+k-1,k-1} \xrightarrow{i_1} P_{n+k,k} \xrightarrow{q_{k-1}} S^{2(n+k)-2}$$

and the associated exact sequence

$$(S)_{k}: \qquad \cdots \to G_{i-2k+2} \xrightarrow{p_{*}} \pi_{2n-1+i}^{s} (P_{n+k-1,k-1}) \xrightarrow{i_{1*}} \\ \pi_{2n-1+i}^{s} (P_{n+k,k}) \xrightarrow{q_{*}} G_{i-2k+1} \xrightarrow{p_{*}} \cdots .$$

We set the two steps of computation:

- (1) determine the G_* -module structure of $\pi_*^s(P_{n+k-1,k-1})$,
- (2) describe $p_{n+k-1,k-1} \in \pi_{2(n+k)-3}^{s}(P_{n+k-1,k-1})$ explicitly.

If these two are possible, we know $\pi^{s}_{2n-1+i}(P_{n+k,k})$ up to group extension

 $0 \rightarrow \text{Cokernel of } p_* \rightarrow \pi^s_{2n-1+i}(P_{n+k,k}) \rightarrow \text{Kernel of } p_* \rightarrow 0$.

To determine this group extension, we prepare a lemma.

Lemma 1 (cf. Theorem 2.1 of [13]). Let $A \xrightarrow{f} X \xrightarrow{i} C(f)$ be a cofibration and

$$\cdots \to \pi_n^s(X) \xrightarrow{i_*} \pi_n^s(C(f)) \xrightarrow{\Delta} \pi_{n-1}^s(A) \xrightarrow{f_*} \pi_{n-1}^s(X) \to \cdots$$

an associated stable exact sequence. Assume that $\alpha \in \pi_{n-1}^s(A)$ satisfies $f_*(\alpha) = 0$, and the order of α is k. For an arbitrary element β of $\langle f, \alpha, k\iota \rangle \subset \pi_n^s(X)$, there exists an element $[\alpha] \in \pi_n^s(C(f))$ such that

$$\Delta([\alpha]) = \alpha \quad and \quad i_*(\beta) = -k[\alpha].$$

Proof. By definition of Toda bracket, there exists a commutative stable diagram with $\beta = a \circ b$:



Then we may put $[\alpha] = -a'$.

For the above (2), we consider $(S)_k$ for i=2k-2:

$$\pi_{2(n+k)-2}^{s}(P_{n+k,k}) \xrightarrow{q_{*}} G_{0} \xrightarrow{p_{*}} \pi_{2(n+k)-3}^{s}(P_{n+k-1,k-1}).$$

The exactness of this shows that

$$\#p_{n+k-1,k-1} = \#(Cokernel \ of \ q_*).$$

On the other hand by (4.5) of [20] we know that

$$\label{eq:cohernel of q_*} = Q^s \{n+k, k\}$$

= $C \{jM_k(C) - n - k, k\}$ for large j

and this number was determined for $k \le 8$ in (3.1) of [20]. We shall need the 2-component of this number for k=5 and 6. Let $\nu_2(m)$ be the exponent of 2 in the factorization of an integer *m* into the prime powers.

Lemma 2 ((3.1) of [20]). $\nu_2(\#p_{n+4,4})$ and $\nu_2(\#p_{n+5,5})$ are given by the following table:

$\nu_2(\#p_{n+4,4})$	<i>n</i> mod ()	$\nu_2(\#p_{n+5,5})$	<i>n</i> mod ()
4	4, 6(8)	4	4, 6(8)
3	0(8), 2(16)	3	0, 2(16)
2	10(16)	2	8(16), 10(32)
		1	26(64)
		0	58(64)

Considering the above (1) and (2), we shall compute inductively $\pi_{2n-1+i}^{s}(P_{n+k,k})$ for $k \leq 6$ and some $i \leq 10$. Since the suspension $EP_{n+k,k}$ is 2*n*-connected and the pair $(W_{n+k,k}, EP_{n+k,k})$ is (4n+3)-connected, it follows that $\pi_{2n-1+i}^{s}(P_{n+k,k})$ is isomorphic to $\pi_{2n+i}(W_{n+k,k})$ for $i \leq 2n$, where $W_{n+k,k} = U(n+k)/U(n)$ is the complex Stiefel manifold. Nomura and Furukawa [16] have computed $\pi_{2n+i}(W_{n+k,k})$ for k=2, 3 and $i \leq 21, 19$ respectively. Therefore we already know $\pi_{2n-1+i}^{s}(P_{n+k,k}) \ 2 \leq k \leq 3$ and $i \leq 10$. But informations for (1) from [16] are not sufficient for our purpose. So we shall recompute some $\pi_{2n-1+i}^{s}(P_{n+k,k})$ for $k \leq 3$.

2. Computation

From now on, *n* means always an even integer ≥ 6 , $\pi_*^{s}()$ and G_* often denote only the 2-primary component of itself. We work in the stable category of pointed spaces and stable maps between them.

Since $p_{n+1,1}=n\eta=0$, it follows that $P_{n+2,2}=S^{2n}\vee S^{2n+2}$. Let $s: S^{2n+2} \rightarrow p_{n+2,2}$ be an inclusion map which is a right inverse of q_1 . Then

(2.1) $i_{1*}+s_*: G_{i-1}\oplus G_{i-3} \to \pi^s_{2n-1+i}(P_{n+2,2})$ is an isomorphism.

By the proof of (1.11), (i) of (1.13) and (1.14) of [20], we have

$$p_{n+2,2} = (n/2)i_{1*}(\nu+\alpha_1) + s_*\eta \colon S^{2n+3} \to P_{n+2,2} = S^{2n} \vee S^{2n+2}$$

Put

$$e_n = \begin{cases} 1 & \text{if } n \equiv 0 \mod (4) \\ 2 & \text{if } n \equiv 2 \mod (4) \end{cases}$$

Then by (2.1) and $(S)_3$ for i=8, we have a short exact sequence

$$0 \to Z_{16}\{i_{2*}\sigma\} \to \pi^{s}_{2n+7}(P_{n+3,3}) \to Z_{8/e_{n}}\{e_{n}\nu\} \to 0$$

We have

$$\langle p_{n+2,2}, e_n \nu, (8/e_n) \iota \rangle = \langle (n/2)i_{1*}\nu, e_n \nu, (8/e_n)\iota \rangle + \langle s_*\eta, e_n \nu, (8/e_n)\iota \rangle \supset i_{1*} \langle (n/2)\nu, e_n \nu, (8/e_n)\iota \rangle + s_* \langle \eta, e_n \nu, (8/e_n)\iota \rangle \supset i_{1*} \{ (ne_n/4) \langle (2/e_n)\nu, e_n \nu, (8/e_n)\iota \rangle \} \supseteq 0$$

since $\langle \eta, e_n \nu, (8/e_n) \nu \rangle \subset G_5 = 0$ and $\langle (2/e_n) \nu, e_n \nu, (8/e_n) \nu \rangle \ni 0$ (see e.g. [16]). Therefore by Lemma 1 the above short exact sequence splits, that is, there exists $[e_n \nu] \in \pi_{2n+7}^s(P_{n+3,3})$ with $q_{2*}[e_n \nu] = e_n \nu$ and

(2.2)
$$\pi_{2n+7}^{s}(P_{n+3,3}) = Z_{16}\{i_{2*}\sigma\} \oplus Z_{8/e_{n}}\{[e_{n}\nu]\}$$

It follows from $(S)_3$ for i=9 that $i_{1*}: \pi^s_{2n+8}(P_{n+2,2}) \rightarrow \pi^s_{2n+8}(P_{n+3,3})$ is an isomorphism. Hence by (2.1) we have

(2.3)
$$\pi_{2n+8}^{s}(P_{n+3,3}) = Z_{2}\{i_{2*}\varepsilon\} \oplus Z_{2}\{i_{2*}\bar{\nu}\} \oplus Z_{2}\{i_{1*}s_{*}\nu^{2}\} .$$

From (2.1) and $(S)_3$ for i=10 it follows that

$$(2.4) \qquad \pi^{s}_{2n+9}(P_{n+3,3}) = Z_{16}\{i_{1*}s_{*}\sigma\} \oplus Z_{2}\{i_{2*}\mu\} \oplus Z_{2}\{i_{2*}\eta\mathcal{E}\} \oplus Z_{2/e_{n}}\{i_{2*}\nu^{3}\}.$$

Analysing $p_{n+k,k}$ for k=3, 4 and 5, we consider the followings. Put

$$L_{m,k} = \begin{cases} 1 & \text{if } m + k \equiv 1 \mod (2) \\ 2 & \text{if } m + k \equiv 0 \mod (2) \end{cases}$$

Then, since $L_{m,k}(m+k-1)\equiv 0 \mod (2)$, $q_{l-1*}(L_{m,k}p_{m+k,l})=L_{m,k}(m+k-1)\eta=0$ and hence $i_{1*}^{-1}(L_{m,k}p_{m+k,l})$ is not empty for 1 < l < m+k, and

$$(T)_{k} \qquad i_{1*}^{-1}(L_{m,k}p_{m+k,k}) = i_{1*}^{-1}(L_{m,k}q_{m-1*}p_{m+k}) \supset q_{m-1*}i_{1*}^{-1}(L_{m,k}p_{m+k})$$

and by (1.15) of [20]

$$(T)'_{k} \qquad q_{k-2*} q_{m-1*} i_{1*}^{-1} (L_{m,k} p_{m+k}) \\ = q_{m+k-3*} i_{1*}^{-1} (L_{m,k} p_{m+k}) \\ = \begin{cases} (m+k-2)(\nu+\alpha_{1}) & \text{if } m+k \equiv 0 \mod (2) \\ \{(1/2)(m+k+1)(\nu+\alpha_{1}), (1/2)(m+k+1)(\nu+\alpha_{1})+4\nu\} \\ & \text{if } m+k \equiv 1 \mod (2) . \end{cases}$$

Now $q_{1*} = s_*^{-1}$: $\pi_{2n+5}^s(P_{n+2,2}) \xrightarrow{\simeq} \pi_{2n+5}^s(S^{2n+2}) = G_3$ by (2.1), since $q_1 \circ s = 1$. Then by $(T)'_3$

$$q_{n-1*}i_{1*}^{-1}(p_{n+3}) \ni ((n+4)/2)s_*(\nu+\alpha_1)$$

and by $(T)_3$

$$p_{n+3,3} = ((n+4)/2)i_{1*}s_*(\nu+\alpha_1)$$

so that $p_{n+3,3} \circ \eta = 0$ and

$$\langle p_{n+3,3}, \eta, 2\iota
angle \supset i_{1*}s_* \langle ((n+4)/2)\nu, \eta, 2\iota
angle = 0$$

and by Lemma 1 there exists $[\eta] \in \pi^s_{2n+7}(P_{n+4,4})$ with $q_{3*}[\eta] = \eta$ and

(2.5)
$$\pi_{2n+7}^{s}(P_{n+4,4}) = Z_{16}\{i_{3*}\sigma\} \oplus Z_{8/e_{n}}\{i_{1*}[e_{n}\nu]\} \oplus Z_{2}\{[\eta]\}.$$

We have also the following from (2.3) and (S)₄ for i=9

(2.6)
$$\pi_{2n+8}^{s}(P_{n+4,4}) = Z_{2}\{i_{3*}\varepsilon\} \oplus Z_{2}\{i_{3*}\bar{\nu}\} \oplus Z_{2/e_{n}}\{i_{2*}s_{*}\nu^{2}\} \oplus Z_{2}\{[\eta]\eta\} .$$

By the same argument as the proof of (2.2) we know that there exists $[[e_n\nu]] \in \pi^s_{2n+9}(P_{n+4,4})$ with $q_{3*}[[e_n\nu]] = e_n\nu$ and

(2.7)
$$\pi_{2n+9}^{s}(P_{n+4,4}) = Z_{16}\{i_{2*}s_{*}\sigma\} \oplus Z_{2}\{i_{3*}\mu\} \oplus Z_{2}\{i_{3*}\eta\varepsilon\} \oplus Z_{2/e_{n}}\{i_{3*}\nu^{3}\} \oplus Z_{8/e_{n}}\{[[e_{n}\nu]]\},$$

To compute $\pi_{2n+9}^{s}(P_{n+5,5})$ we shall prepare four lemmas.

Remember that in [20] we used the notations: $HP_{m+k,k} = HP_{m+k}/HP_m$, the stunted quaternionic projective space; $\pi: P_{2m+2k,2k} \to HP_{m+k,k}$, the canonical quotient map;

(2.8)
$$S^{2n+4k-1} \xrightarrow{p_{(n/2)+k,k}^{H}} HP_{(n/2)+k,k} \xrightarrow{i_1^{H}} HP_{(n/2)+k+1,k+1}$$

the canonical cofibration.

Lemma 3. We have

(i)
$$\pi_{2n+7}^{s}(HP_{(n/2)+2,2}) = Z_{16}\{i_{1*}^{H}\sigma\} \oplus Z_{8/e_{n}}\{\pi_{*}i_{1*}[e_{n}\nu]\},$$

(ii) $\pi_{2n+8}^{s}(HP_{(n/2)+2,2}) = Z_{2}\{i_{1*}^{H}\mu\} \oplus Z_{2}\{i_{1*}^{H}\nu\},$
(iii) $\pi_{2n+9}^{s}(HP_{(n/2)+2,2}) = Z_{2}\{i_{1*}^{H}\mu\} \oplus Z_{2}\{i_{1*}^{H}\eta\xi\} \oplus Z_{2/e_{n}}\{i_{1*}^{H}\nu^{3}\},$
(iv) $p_{(n/2)+2,2}^{H}\circ\eta = \begin{cases} i_{1*}^{H}\varepsilon & \text{if } n\equiv 2 \mod (8) \\ i_{1*}^{H}\overline{\nu} & \text{if } n\equiv 4 \mod (8) \\ i_{1*}^{H}(\varepsilon+\overline{\nu}) & \text{if } n\equiv 6 \mod (8) \\ 0 & \text{if } n\equiv 0 \mod (8). \end{cases}$

Proof. Considering the stable homotopy exact sequence associated with (2.8) for k=1, we obtain (ii) and (iii) immediately since $G_4=G_5=0$ and $p_{(n/2)+1,1}^H=(n/2)(\nu+\alpha_1)$ and by Lemma 1 we have a split exact sequence:

$$0 \to Z_{16}\{i_{1*}^{H}\sigma\} \to \pi_{2n+7}^{s}(HP_{(n/2)+2,2}) \to Z_{8/e_{n}}\{e_{n}\nu\} \to 0 \; .$$

Then the following commutative diagram induces (i):



Since $q_1^H \circ p_{(n/2)+2,2}^H \circ \eta = ((n/2)+1)(\nu+\alpha_1)\eta = 0$, there exists a map $f: S^{2n+8} \to S^{2n}$ with $i_1^H \circ f = p_{(n/2)+2,2}^H \circ \eta$. It is easily seen that $i_1^{H*}: \{HP_{(n/2)+2,2}, S^{2n-1}\} \to \{S^{2n}, S^{2n-1}\} = \mathbb{Z}_2\{\eta\}$ is an isomorphism. Let $h \in \{HP_{(n/2)+2,2}, S^{2n-1}\}$ be the element with $h \circ i_1^H = \eta$. It follows from (2.7) of [21] that

$$h \circ p^{H}_{(n/2)+2,2} = \begin{cases} \varepsilon & \text{if } n \equiv 2 \mod (8) \\ \overline{\nu} & \text{if } n \equiv 4 \mod (8) \\ \varepsilon + \overline{\nu} & \text{if } n \equiv 6 \mod (8) \\ 0 & \text{if } n \equiv 0 \mod (8) \end{cases}$$

Since $\eta \circ f = h \circ p_{(\pi/2)+2,2}^{H} \circ \eta$ and $\eta \circ : G_8 \to G_9$ is a monomorphism, we obtain (iv). This completes the proof of Lemma 3.

Lemma 4. For suitably chosen $[[e_n\nu]]$ it holds that $[\eta]\eta^2 = (4/e_n)[[e_n\nu]]$.

Proof. By Proposition 1.4 of Toda [23]

(2.9)
$$[\eta]\eta^2 = [\eta] \circ \langle 2\iota, \eta, 2\iota \rangle = \langle [\eta], 2\iota, \eta \rangle \circ 2\iota .$$

Let Indet $\langle \alpha, \beta, \gamma \rangle$ be an indeterminacy of a Toda bracket $\langle \alpha, \beta, \gamma \rangle$. Then

$$\pi_{2n+9}^{s}(P_{n+4,4}) \supset \operatorname{Indet} \langle [\eta], 2\iota, \eta \rangle = [\eta] \circ \{S^{2n+9}, S^{2n+7}\} + \pi_{2n+8}^{s}(P_{n+4,4}) \circ \eta$$
$$= Z_2\{[\eta]\eta^2\} + Z_2\{i_{3*}\eta\mathcal{E}\} + Z_{2/e_n}\{i_{3*}\nu^3\}$$

and

$$q_{3*} \operatorname{Indet} \langle [\eta], 2\iota, \eta \rangle = Z_2 \{\eta^3\} = Z_2 \{4\nu\} = \operatorname{Indet} \langle \eta, 2\iota, \eta \rangle$$

and, since $q_{3*}\langle [\eta], 2\iota, \eta \rangle \subset \langle q_{3*}[\eta], 2\iota, \eta \rangle = \langle \eta, 2\iota, \eta \rangle$, we have

$$q_{3*}\langle [\eta], 2\iota, \eta \rangle = \langle \eta, 2\iota, \eta \rangle = \{2\nu, 6\nu\}$$

Hence there exists an element in $\langle [\eta], 2\iota, \eta \rangle$ which is mapped to 2ν by q_{3*} . By (2.7) this element has a form as $(2/e_n)[[e_n\nu]] + i_{1*}x$ for some $x \in \pi_{2n+9}^s(P_{n+3,3})$, and from (2.9) it follows that $4i_{1*}x=0$. Then by (2.7) $2i_{1*}x$ is divisible by 8, that is, $2i_{1*}x=8i_{1*}y$ for some $y \in \pi_{2n+9}^s(P_{n+3,3})$. Then

$$\begin{aligned} [\eta]\eta^2 &= 2\{(2/e_n)[[e_n\nu]] + i_{1*}x\} \\ &= (4/e_n)([[e_n\nu]] + 2e_ni_{1*}y) \,. \end{aligned}$$

Since $q_{3*}([[e_n\nu]]+2e_ni_{1*}y)=e_n\nu$ and the order of $[[e_n\nu]]+2e_ni_{1*}y$ is $8/e_n$, we may change $[[e_n\nu]]$ for $[[e_n\nu]]+2e_ni_{1*}y$. So the conclusion follows.

Appointment: From now on we assume that $[[e_n\nu]]$ satisfies $[\eta]\eta^2 = (4/e_n)[[e_n\nu]]$.

Since $q_3 \circ f_{n+4,4} = (n+3)\eta = \eta$, by (2.5) we can put

$$p_{n+4,4} = a_n i_{3*} \sigma + b_n i_{1*} [e_n \nu] + [\eta] + odd \ torsion$$

for some integers a_n and b_n . By Lemma 2 and (2.5) we have

(2.10)
$$a_n \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 4 \text{ or } 6 \mod (8) \\ 0 \mod (2) & \text{if } n \equiv 0 \text{ or } 2 \mod (8). \end{cases}$$

By $(T)_4$, and $(T)'_4$, for any $p' \in q_{n-1*}i_{1*}^{-1}(2p_{n+4}) \subset \pi_{2n+7}^s(P_{n+3,3})$ we have

$$i_1 \circ p' = 2p_{n+4,4}$$
 and $q_2 \circ p' = (n+2)(\nu+\alpha_1)$.

Then $p'=2a_ni_{2*}\sigma+2b_n[e_n\nu]+odd$ torsion. Applying q_{2*} to this equation we know that $2b_ne_n\equiv n+2 \mod (8)$, and

(2.11)
$$b_n \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 0 \mod (4) \text{ or } 2 \mod (8) \\ 0 \mod (2) & \text{if } n \equiv 6 \mod (8) . \end{cases}$$

Lemma 5. We have

$$[2\nu]\eta = \begin{cases} i_{2*}\varepsilon & \text{if } n \equiv 2 \mod (8) \\ i_{2*}\varepsilon \text{ or } i_{2*}\overline{\nu} & \text{if } n \equiv 6 \mod (8) \end{cases}, \text{ and}$$
$$[\nu]\eta = (n/4)i_{2*}\varepsilon + i_{1*}s_*\nu^2 & \text{if } n \equiv 0 \mod (4).$$

Proof. By Lemma 1 we can easily construct a commutative diagram:

Then $a \circ b \in \langle p_{n+2,2}, e_n \nu, \eta \rangle$ and this Toda backet is a coset of

$$\pi^{s}_{2n+8}(P_{n+2,2})/\pi^{s}_{2n+7}(P_{n+2,2})\circ\eta = [Z_{2}\{i_{1*}arepsilon\}\oplus Z_{2}\{i_{1*}arepsilon\}/\{0, i_{1*}(arepsilon+arpsilon)\}]\oplus Z_{2}\{s_{*}
u^{2}\} .$$

We have

$$\langle p_{n+2,2}, e_n \nu, \eta \rangle = \langle (n/2)i_{1*}\nu, e_n \nu, \eta \rangle + \langle s_*\eta, e_n \nu, \eta \rangle$$

 $\supset i_{1*} \{ (ne_n/4) \langle (2/e_n)\nu, e_n \nu, \eta \rangle \} + s_* \langle \eta, e_n \nu, \eta \rangle$
 $\supseteq (ne_n/4)i_{1*} \mathcal{E} + e_n s_* \nu^2$

since $\langle (2/e_n)\nu, e_n\nu, \eta \rangle = \varepsilon + G_7 \circ \eta$ and $\langle \eta, e_n\nu, \eta \rangle = e_n\nu^2$ by Toda [23]. Hence

(2.12)
$$[e_n\nu]\eta = i_{1*}(a \circ b)$$

= $(ne_n/4)i_{2*}\varepsilon + e_ni_{1*}s_*\nu^2 \text{ or } ((ne_n/4)+1)i_{2*}\varepsilon + i_{2*}\bar{\nu} + e_ni_{1*}s_*\nu^2.$

Thus Lemma 5 follows if $n \equiv 6 \mod(8)$. By Lemma 4

(2.13)
$$p_{n+4,4}\circ\eta^2 = a_n i_{3*}(\eta \varepsilon + \nu^3) + b_n i_{1*}[e_n\nu]\eta^2 + (4/e_n)[[e_n\nu]]$$

and by (iii) of Lemma 3, the fact $4/e_n \equiv 0 \mod (2)$ and the commutativity of the diagram in the proof of Lemma 3 it follows that

$$p^{H}_{(n/2)+2,2} \circ \eta^{2} = \pi \circ p_{n+4,4} \circ \eta^{2}$$

= $a_{n} i^{H}_{1*} (\eta \mathcal{E} + \nu^{3}) + b_{n} \pi_{*} i_{1*} [e_{n} \nu] \eta^{2}.$

Then the conclusions for $n \equiv 6 \mod (8)$ follow from (iii) and (iv) of Lemma 3, (2.10), (2.11) and (2.12). This completes the proof of Lemma 5.

Lemma 6. We have

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$$p_{n+4,4} \circ \eta^2 = \begin{cases} i_{3*} \eta \varepsilon + 2\lfloor [2\nu] \rfloor & \text{if } n \equiv 2 \mod (4) \\ (n/4) i_{3*} \nu^3 + 4 \llbracket [\nu] \rfloor & \text{if } n \equiv 0 \mod (4) \end{cases}.$$

Proof. The conclusion follows from (2.7), (2.10), (2.11), Lemma 5 and (2.13).

Now we compute $\pi_{2n+9}^{s}(P_{n+5,5})$. Since $p_{n+4,4}\circ\eta = [\eta]\eta + (other term)$ is non-zero, it follows from (2.7), Lemma 6 and (S)₅ for i=10 that

(2.14)
$$\pi_{2n+9}^{s}(P_{n+5,5}) = Z_{16}\{i_{3*}s_{*}\sigma\} \oplus Z_{2}\{i_{4*}\mu\} \oplus H_{n}$$

where

$$H_{n} = \begin{cases} Z_{4}\{i_{1*}[[2\nu]]\} \text{ with the relations } i_{4*}\eta\varepsilon = 2i_{1*}[[2\nu]] \text{ and } i_{4*}\nu^{3} = 0 \\ & \text{if } n \equiv 2 \mod (4) \\ Z_{2}\{i_{4*}\eta\varepsilon\} \oplus Z_{2}\{i_{4*}\nu^{3}\} \oplus Z_{4}\{i_{1*}[[\nu]]\} \text{ if } n \equiv 0 \mod (8) \\ Z_{2}\{i_{4*}\eta\varepsilon\} \oplus Z_{8}\{i_{1*}[[\nu]]\} \text{ with the relation } i_{4*}\nu^{3} = 4i_{1*}[[\nu]] \\ & \text{if } n \equiv 4 \mod (8) . \end{cases}$$

By $(T)_5'$

$$q_{3*}q_{n-1*}i_{1*}^{-1}(p_{n+5}) = \{((n+6)/2)(\nu+\alpha_1), ((n+6)/2)(\nu+\alpha_1)+4\nu\}$$

and hence we can choose a map $\tilde{p} \in q_{n-1*} i_{1*}^{-1}(p_{n+5}) \subset \pi_{2n+9}^s(P_{n+4,4})$ with

$$q_3 \circ \tilde{p} = \begin{cases} ((n+6)/2)(\nu+\alpha_1) + 4\nu & \text{if } n \equiv 2 \mod (16) \\ ((n+6)/2)(\nu+\alpha_1) & \text{otherwise} \end{cases}$$

and then by $(T)_5$

$$i_1 \circ \tilde{p} = p_{n+5,5}$$
.

By (2.7) we can put

(2.15)
$$\tilde{p} = a'_{n}i_{2*}s_{*}\sigma + b'_{n}i_{3*}\mu + c_{n}i_{3*}\eta\varepsilon + d'_{n}i_{3*}\nu^{3} + d_{n}[[e_{n}\nu]] + odd \ torsion$$

for some integers a'_n , b'_n , c_n , d'_n and d_n . Remark that $i_{3*}\nu^3 = 0$ if $n \equiv 2 \mod (4)$. We have

$$d_{n}e_{n}\nu + odd \ torsion = q_{3} \circ \tilde{p}$$

$$= \begin{cases} (((n+6)/2) + 4)\nu + odd \ torsion & if \ n \equiv 2 \ mod \ (16) \\ ((n+6)/2)\nu + odd \ torsion & otherwise \end{cases}$$

and

(2.16)
$$d_n \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 0 \mod (4) \text{ or } 6 \mod (8) \\ 0 \mod (4) & \text{if } n \equiv 2 \mod (8) . \end{cases}$$

Put $p_{n+5,5} = a'_n i_{3*} s_* \sigma + b'_n i_{4*} \mu + \check{p}$. Then the 2-primary part of \check{p} is contained in H_n . Hence by Lemma 2 and (2.14) we have

 $a'_n \equiv 1 \mod (2)$ if $n \equiv 4 \text{ or } 6 \mod (8)$.

Then by (2.14), (2.16) and $(S)_6$ for i=10 we have

$$(2.17) \quad \pi_{2n+9}^{s}(P_{n+6,6}) = Z_{8/e_{n}}\{i_{4*}s_{*}\sigma\} \oplus Z_{2}\{i_{5*}\mu\} \oplus Z_{2/e_{n}} \qquad \text{if } n \equiv 4 \text{ or } 6 \mod (8)$$

where if $n \equiv 4 \mod (8)$, $Z_{2/e_n} = Z_2$ is generated by $i_{5*}\eta \mathcal{E}$.

Next suppose that $n \equiv 2 \mod (8)$. Let *l* be the odd component of the order of \tilde{p} . Of course *l* is an odd integer. Put $\hat{p} = la'_n s_* \sigma + b'_n i_{1*} \mu + c_n i_{1*} \eta \varepsilon$. Then by (2.15) and (2.16)

$$l\widetilde{p} = i_{2*}\hat{p}$$

and we have a commutative diagram in which the each horizontal sequences are cofibrations and l denotes a multiplication by l:

$$S^{2n+9} \xrightarrow{p_{n+5,5}} P_{n+5,5} \longrightarrow P_{n+6,6}$$

$$\uparrow = \qquad \uparrow i_1 \qquad \uparrow i_1$$

$$S^{2n+9} \xrightarrow{\tilde{p}} P_{n+4,4} \longrightarrow C(\tilde{p})$$

$$\downarrow = \qquad \downarrow l \qquad \downarrow l \qquad \downarrow \bar{l}$$

$$S^{2n+9} \xrightarrow{\tilde{p}} P_{n+4,4} \longrightarrow C(l\tilde{p})$$

$$\uparrow = \qquad \uparrow i_2 \qquad \uparrow i_2$$

$$S^{2n+9} \xrightarrow{\hat{p}} P_{n+2,2} \longrightarrow C(\hat{p})$$

$$\downarrow = \qquad \downarrow \pi \qquad \downarrow \pi \qquad \downarrow \pi$$

$$S^{2n+9} \xrightarrow{\pi \circ \hat{p}} HP_{(n/2)+1,1} \xrightarrow{j} C(\pi \circ \hat{p})$$

We calculate the Adams' e_c and e_R invariants of $\pi \circ \hat{p} \in G_9$.

Lemma 7. We have

(i)
$$e_{c}(\pi \circ \hat{p}) = 0$$
 and $b'_{n} \equiv 0 \mod (2)$,
(ii) $e_{R}(\pi \circ \hat{p}) = \begin{cases} 1 & \text{if } n \equiv 2 \mod (16) \\ 0 & \text{if } n \equiv 10 \mod (16) \end{cases}$ and
 $c_{n} \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 2 \mod (16) \\ 0 \mod (2) & \text{if } n \equiv 10 \mod (16) \end{cases}$

Proof. Applying \tilde{K} to the above diagram, we can show the first part of (i) by the similar method as the proof of (1.12) of [20]. Then the second part of (i) follows, since $\pi \circ s = \eta^2$ or 0, $e_c(\eta^2 \sigma) = e_c(\eta \mathcal{E}) = 0$ and $e_c(\mu) \neq 0$ by [1].

Put n=8m+2. Applying \widetilde{KO}^{-4} to the above diagram, we have the following commutative diagram in which the horizontal sequences are exact:

diagram (2.18)

$$0 \leftarrow \widetilde{KO}^{-4}(P_{8m+7,5}) \leftarrow \widetilde{KO}^{-4}(P_{8m+8,6}) \leftarrow \widetilde{KO}^{-4}(S^{16m+14}) \leftarrow 0$$

$$\downarrow i_1^* \qquad \downarrow i_1^* \qquad \downarrow =$$

$$0 \leftarrow \widetilde{KO}^{-4}(P_{8m+6,4}) \leftarrow \widetilde{KO}^{-4}(C(\tilde{p})) \leftarrow \widetilde{KO}^{-4}(S^{16m+14}) \leftarrow 0$$

$$\downarrow i_2^* \qquad \downarrow i_2^* \qquad \downarrow =$$

$$0 \leftarrow \widetilde{KO}^{-4}(P_{8m+4,2}) \leftarrow \widetilde{KO}^{-4}(C(\tilde{p})) \leftarrow \widetilde{KO}^{-4}(S^{16m+14}) \leftarrow 0$$

$$\downarrow i_2^* \qquad \downarrow i_2^* \qquad \downarrow =$$

$$0 \leftarrow \widetilde{KO}^{-4}(P_{8m+4,2}) \leftarrow \widetilde{KO}^{-4}(C(\tilde{p})) \leftarrow \widetilde{KO}^{-4}(S^{16m+14}) \leftarrow 0$$

$$\downarrow i_2^* \qquad \downarrow i_2^* \qquad \downarrow =$$

$$0 \leftarrow \widetilde{KO}^{-4}(S^{16m+4}) \leftarrow \widetilde{KO}^{-4}(C(\pi \circ \hat{p})) \leftarrow \widetilde{KO}^{-4}(S^{16m+14}) \leftarrow 0$$

By Theorem 2 of Fujii [4] it is easily seen that

$$\begin{split} \widetilde{KO}^{-4}(P_{8m+8,6}) &= Z \left\{ z_2 z_0^{4m}, \, z_2 z_0^{4m+1}, \, z_2 z_0^{4m+2} \right\} \oplus Z_2 \left\{ z_2 z_0^{4m+3} \right\} \\ \widetilde{KO}^{-4}(P_{8m+7,5}) &= Z \left\{ z_2 z_0^{4m}, \, z_2 z_0^{4m+1}, \, z_2 z_0^{4m+2} \right\} \\ \widetilde{KO}^{-4}(P_{8m+6,4}) &= Z \left\{ z_2 z_0^{4m}, \, z_2 z_0^{4m+1} \right\} \\ \widetilde{KO}^{-4}(P_{8m+4,2}) &= Z \left\{ z_2 z_0^{4m} \right\} \oplus Z_2 \left\{ z_2 z_0^{4m+1} \right\} \,. \end{split}$$

Also note that a generator d of $\widetilde{KO}^{-4}(S^{16m+4}) = Z$ satisfies

$$\pi^*d = z_2 z_0^{4m} + x z_2 z_0^{4m+1}$$

for some integer x. We shall not need the explicit value of x. Here we regard $\widetilde{KO}^{-4}(X|A)$ as a subgroup of $\widetilde{KO}^{-4}(X)$ if the quotient map $X \to X|A$ induces a monomorphism. Similar remarks shall hold in the forthcoming proof of (A). By chasing diagram, we know that there exist elements $[x_2 z_0^{4m}]$ and $[x_2 z_0^{4m+1}]$ in $\widetilde{KO}^{-4}(C(l\tilde{p}))$ such that

$$\overline{l}^*[z_2 z_0^{4m}] = l \overline{l}_1^* z_2 z_0^{4m}$$
 and $\overline{l}^*[z_2 z_0^{4m+1}] = l \overline{l}_1^* z_2 z_0^{4m+1}$.

Put $a' = [z_2 z_0^{4m}] + x[z_2 z_0^{4m+1}]$. Then there exists an element $a \in \widetilde{KO}^{-4}(C(\pi \circ \hat{p}))$ such that

 $\bar{\pi}^* a = \bar{i}_2^* a' \quad and \quad j^* a = d.$

Let $b \in \widetilde{KO}^{-4}(C(\pi \circ \hat{p}))$ and $b' \in \widetilde{KO}^{-4}(C(l\tilde{p}))$ be the images of the generator of $\widetilde{KO}^{-4}(S^{16m+14}) = \mathbb{Z}_2$.

Now we assume the followings which shall be proved later:

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(A) $i_1^* z_2 z_0^{4m+2} = e_{2m} \bar{l}^* b'$, (B) the order of $i_2^* [z_2 z_0^{4m+1}]$ is 2. Remark that $e_{2m} = 1$ if $m \equiv 0 \mod (2)$, or 2 if $m \equiv 1 \mod (2)$, and $\bar{l}^* b'$ is the generator of the 2-torsion of $\widetilde{KO}^{-4}(C(\tilde{p}))$. We have

$$\psi^3 a = 3^{8m+4}a + \lambda b$$

for some $\lambda \in \mathbb{Z}_2$, and

$$e_{R}(\pi\circ\hat{p})=\lambda$$

and

$$ar{\pi}^*\psi^3 a = ar{\pi}^*(3^{8m+4}a\!+\!\lambda b) = ar{i}_2^*(3^{8m+4}a'\!+\!\lambda b')$$
 .

On the other hand

$$ar{\pi}^*\psi^3a=\psi^3ar{\pi}^*a=\psi^3ar{i}_2^*a'=ar{i}_2^*\psi^3a'$$

and

(2.19)
$$i_2^*(3^{8m+4}a'+\lambda b') = i_2^*\psi^3a'.$$

Since the order of $\bar{i}_1^* z_2 z_0^{4m+3}$ is 2 and $\bar{i}_1^* z_2 z_0^{4m+2} = e_{2m} \bar{l}^* b'$ by (A), we have

$$\begin{split} \bar{l}^{*}\psi^{3}a' &= \psi^{3}\bar{l}^{*}a' \\ &= \psi^{3}\{l\bar{i}_{1}^{*}(z_{2}z_{0}^{4m} + xz_{2}z_{0}^{4m+1})\} \\ &= l\bar{i}_{1}^{*}\psi^{3}(z_{2}z_{0}^{4m} + xz_{2}z_{0}^{4m+1}) \\ &= l\bar{i}_{1}^{*}\{3^{8m+4}z_{2}z_{0}^{4m} + ((8m+2)3^{8m+3} + x3^{8m+6})z_{2}z_{0}^{4m+1} \\ &+ ((4m+1)(8m+1)3^{8m+2} + x(8m+4)3^{8m+5})z_{2}z_{0}^{4m+2} \\ &+ ((4m+1)(8m+1)8m3^{8m} + x(4m+2)(8m+3)3^{8m+4})z_{2}z_{0}^{4m+3}\} \\ &= \bar{l}^{*}\{3^{8m+4}[z_{2}z_{0}^{4m}] + ((8m+2)3^{8m+3} + x3^{8m+6})[z_{2}z_{0}^{4m+1}] + e_{2m}b'\} \;. \end{split}$$

Then, since \bar{l}^* is a monomorphism,

$$\psi^{3}a' = 3^{8m+4}[z_{2}z_{0}^{4w}] + ((8m+2)3^{8m+3} + x3^{8m+6})[z_{2}z_{0}^{4m+1}] + e_{2m}b'$$

and by (B)

$$i_2^*\psi^3 a' = 3^{8m+4}i_2^*[z_2 z_0^{4m}] + xi_2^*[z_2 z^{4m+1}] + e_{2m}i_2^*b'$$

also by (2.19) this equals to

$$\ddot{i}_{2}^{*}(3^{8m+4}a'+\lambda b') = \ddot{i}_{2}^{*}\{3^{8m+4}([z_{2}z_{0}^{4m}]+x[z_{2}z_{0}^{4m+1}])+\lambda b'\} \\ = 3^{8m+4}\dot{i}_{2}^{*}[z_{2}z_{0}^{4m}]+x\dot{i}_{2}^{*}[z_{2}z_{0}^{4m+1}]+\lambda \dot{i}_{2}^{*}b'$$

and, since i_2^*b' is non-zero,

$$\lambda = e_{2m} \quad in \ Z_2$$

and this implies the first part of (ii). Since $\pi \circ s = 0$ or η^2 , and $a'_n \equiv 0 \mod (2)$ by Lemma 2, it follows that by the second part of (i) we have

$$\pi \circ \hat{p} = c_n \eta \delta$$

and the above proof of the first part of (ii) shows that

$$c_{n} \equiv \begin{cases} 1 \mod (2) & \text{if } n \equiv 2 \mod (16) \\ 0 \mod (2) & \text{if } n \equiv 10 \mod (16) \end{cases}$$

This implies the second part of (ii).

We shall give the proofs of (A) and (B).

The proof of (A): We have the following commutative diagram:



We have

$$\widetilde{KO}^{-4}(P_{8m+8,3}) = Z\{z_2 z_0^{4m+2}\} \oplus Z_2\{z_2 z_0^{4m+3}\},\ q_3^* z_2 z_0^{4m+2} = z_2 z_0^{4m+2}.$$

It suffices for our purpose to compute $\hat{i}_1^* z_2 z_0^{4m+2}$, since $\tilde{i}_1^* z_2 z_0^{4m+2}$ is contained in the image of $\widetilde{KO}^{-4}(S^{16m+14})$, and \overline{q}_3^* induces an isomorphism between the images of $\widetilde{KO}^{-4}(S^{16m+14})$. We have chosen \widetilde{p} such that $q_3 \circ \widetilde{p} = (m+1)\alpha_1$. Let u_m be the order of $q_3 \circ \widetilde{p}$. Then $u_m = 1$ or 3. Applying $\pi_{16m+14}^*()$ to the above diagram, we know easily that there exists uniquely an element $u \in \pi_{16m+14}^*(P_{8m+8,3})$ such that $q_2 \circ u = u_m \iota$, moreover there exists $\hat{u} \in \pi_{16m+14}^*(C(q_3 \circ \widetilde{p}))$ such that $\hat{q}_2 \circ \hat{u} = u_m \iota$ and $u = \hat{i}_1 \circ \hat{u}$, where ι is the identity map of S^{16m+14} . Since \hat{q}_2^* : $\widetilde{KO}^{-4}(S^{16m+14}) \to \widetilde{KO}^{-4}(C(q_3 \circ \widetilde{p}))$ is an isomorphism, and $\hat{u}^* \hat{q}_2^*$ is the multiplication by u_m which is the identity homomorphism of $\widetilde{KO}^{-4}(S^{16m+14}) = Z_2$, it follows that $\hat{u}^* : \widetilde{KO}^{-4}(C(q_3 \circ \widetilde{p})) \to \widetilde{KO}(S^{16m+14})$ is the inverse of \hat{q}_2^* . Thus

(2.20)
$$\hat{i}_1^* z_2 z_0^{4m+2} = \hat{q}_2^* \hat{u}^* \hat{i}_1^* z_2 z_0^{4m+2} = \hat{q}_2^* u^* z_2 z_0^{4m+2}.$$

Next we determine $u^*z_2z_0^{4m+2}$. Consider the commutative diagram:

$$\begin{array}{cccc}
\widetilde{K}(P_{8m+8,3}) & \stackrel{u^*}{\longrightarrow} \widetilde{K}(S^{16m+14}) \\
\downarrow \cong & \downarrow \cong \\
\widetilde{K}^{-4}(P_{8m+8,3}) & \stackrel{u^*}{\longrightarrow} \widetilde{K}^{-4}(S^{16m+14}) \\
\downarrow r & \downarrow r \\
\widetilde{KO}^{-4}(P_{8m+8,3}) & \stackrel{u^*}{\longrightarrow} \widetilde{KO}^{-4}(S^{16m+14})
\end{array}$$

Recall that $\tilde{K}(P_{8m+8,3}) = Z\{z^{8m+5}, z^{8m+6}, z^{8m+7}\}$ and the real restriction homomorphism r in the right hand side is an epimorphism. We can prove the followings:

(2.21)
$$\begin{cases} r(g_c^2 z^{8m+5}) = z_2 z^{4m+3} + (8m+5) z_2 z_0^{4m+2} \\ r(g_c^2 z^{8m+6}) = z_2 z_0^{4m+3} + 2 z_2 z_0^{4m+2} \\ r(g_c^2 z^{8m+7}) = z_2 z_0^{4m+3}, \end{cases}$$

(2.22)
$$r(g_c^2(z^{8m+5}-(4m+2)z^{8m+6}+z^{8m+7})) = z_2 z_0^{4m+2},$$
$$(u^* z^{8m+5} = (1/3)(8m+5)(3m+2)u_m \beta$$

(2.23)
$$\begin{cases} u \ z \ = (1/3)(3m+$$

where $\beta \in \tilde{K}(S^{16m+14}) = Z$ is the generator such that $q_2^*\beta = z^{8m+7}$. (2.23) follows from the relation $\psi^2 u^* = u^* \psi^2$. For (2.21) we consider the following commutative diagram:

$$\begin{array}{c}
\widetilde{K}^{-4}(P_{8m+8,3}) & \stackrel{q^{*}}{\subset} \widetilde{K}^{-4}(P_{8m+8}) & \stackrel{i^{*}_{1}}{\longleftarrow} \widetilde{K}^{-4}(P_{8m+9}) \\
\downarrow^{r} & \downarrow^{r} & \downarrow^{r} \\
\widetilde{KO}^{-4}(P_{8m+8,3}) & \stackrel{q^{*}}{\subset} \widetilde{KO}^{-4}(P_{8m+8}) & \stackrel{i^{*}_{1}}{\longleftarrow} \widetilde{KO}^{-4}(P_{8m+9})
\end{array}$$

Since $\widetilde{KO}^{-4}(P_{8m+9}) = Z\{z_2, z_2z_0, \dots, z_2z_0^{4m+3}\}$ is torsion free (see [4]), by the aid of the complexification homomorphism we can describe r in the right hand side explicitly. In particular we have

$$\begin{aligned} r(g_c^2 z^{8m+5}) &= (1/3)(8m+5)(8m^2+10m+3)z_2 z_0^{4m+3}+(8m+5)z_2 z_0^{4m+2}, \\ r(g_c^2 z^{8m+6}) &= (4m+3)^2 z_2 z_0^{4m+3}+2 z_2 z_0^{4m+2}, \\ r(g_c^2 z^{8m+7}) &= (8m+7) z_2 z_0^{4m+3}. \end{aligned}$$

Hence r in the left hand side satisfies (2.21). Then (2.22) follows from (2.21). By (2.22) and (2.23)

$$u^*z_2z_0^{4m+2} = r(g_c^2u^*(z^{8m+5}-(4m+2)z^{8m+6}+z^{8m+7}))$$

= $v_mr(g_c^2\beta)$

where $v_m = ((1/3)(8m+5)(3m+2)-(4m+2)(4m+3)+1)u_m$.

Now

$$egin{aligned} &oldsymbol{i}_1^{oldsymbol{i}_1^*} x_2 x_0^{oldsymbol{a}_{0}m+2} &=oldsymbol{i}_1^* x_2 x_0^{oldsymbol{a}_{0}m+2} \ &=oldsymbol{a}_1^* oldsymbol{i}_1^* x_2 x_0^{oldsymbol{a}_{0}m+2} \ &=oldsymbol{a}_1^* oldsymbol{a}_1^* oldsymbol{a}_2 x_0^{oldsymbol{a}_{0}m+2} \ &=oldsymbol{a}_1^* oldsymbol{a}_1^* oldsymbol{a}_2 x_0^{oldsymbol{a}_{0}m+2} \ &=oldsymbol{a}_1^* oldsymbol{a}_1^* oldsymbol{a}_2 x_0^{oldsymbol{a}_{0}m+2} \ &=oldsymbol{a}_1^* oldsymbol{a}_2^* oldsymbol{a}_1^* oldsymbol{a}_2 x_0^{oldsymbol{a}_{0}m+2} \ &=oldsymbol{a}_1^* oldsymbol{a}_2^* oldsymbol{a}_1^* oldsymbol{a}_2^* oldsymbol{a}_1^* oldsymbol{a}_2^* oldsymbol{a}_1^* oldsymbol{a}_2^* oldsymbol{a}_1^* oldsymbol{a}_2^* oldsymbol{a}_1^* oldsymbo$$

where the third equality follows from (2.20). Therefore (A) follows since $v_m \equiv e_{2m} \mod (2)$.

The proof of (B): It suffices to show that the second short exact sequence from the bottom on the diagram (2.18) splits. Naturally we have a commutative diagram in which the horizontal sequences are exact:

$$0 \longleftarrow \widetilde{KO}^{-4}(P_{8m+4,2}) \longleftarrow \widetilde{KO}^{-4}(C(\hat{p})) \longleftarrow \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0$$

$$0 \longleftarrow \widetilde{KO}^{-4}(P_{8m+4,1}) \longleftarrow \widetilde{KO}^{-4}(C(q_1 \circ \hat{p})) \longleftarrow \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0$$

It is easily seen that q_1^* is a monomorphism. By Propositions 3.3 and 7.1 of Adams [1] we have a homomorphism

$$e: G_7 = \pi_{16m+13}(S^{16m+6}) \to \operatorname{Ext}^1(\widetilde{KO}^{-4}(S^{16m+6}), \widetilde{KO}^{-4}(S^{16m+14})) = Z_2$$

Since $q_1 \circ \hat{p} = la'_n \sigma$, and $a'_n \equiv 0 \mod (2)$ by Lemma 2, it follows that $q_1 \circ \hat{p}$ is divisible by 2, and $e(q_1 \circ \hat{p}) = 0$. This implies that the above lower sequence splits (see [1]), and also the upper one does. Then (B) follows and the proof of Lemma 7 is completed.

Now we proceed the computation of $\pi_{2n+9}^{s}(P_{n+6,6})$ for $n \equiv 2 \mod (8)$. By (2.15), (2.16) and Lemma 7

$$p_{n+5,5} = i_1 \circ \tilde{p} = a'_n i_{3*} s_* \sigma + c_n i_{4*} \eta \varepsilon + odd \ torsion.$$

Then we obtain the following table by Lemma 2

<i>n</i> mod ()	$\nu_2(\#p_{n+5,5})$	a'_n
2(16)	3	2(4)
10(32)	2	4(8)
26(64)	1	8(16)
58(64)	0	0(16)

Put

$$e'_{n} = \begin{cases} 2 & \text{if } n \equiv 2 \mod (16) \\ 2^{2} & \text{if } n \equiv 10 \mod (32) \\ 2^{3} & \text{if } n \equiv 26 \mod (64) \\ 2^{4} & \text{if } n \equiv 58 \mod (64) . \end{cases}$$

Then from (2.14) and (S)₆ for i=10 it follows

 $(2.24) \quad \pi_{2n+9}^{s}(P_{n+6,6}) = Z_{e'_{n}} \oplus Z_{2}\{i_{5*}\mu\} \oplus Z_{4}\{i_{2*}[[2\nu]]\} \qquad \text{if } n \equiv 2 \mod (8)$

where $Z_{e'_n}$ is generated by $i_{4*}s_*\sigma$ if $n \equiv 10 \mod (16)$, or $i_{4*}s_*\sigma + i_{2*}[[2\nu]]$ if $n \equiv 2 \mod (16)$.

(2.17) and (2.24) give the proof of Theorem.

Added in proof. Professor Y. Furukawa has pointed out to the author that in [5], [17], [18] and [19] the stable homotopy groups $\pi_{2n+i}(W_{n+k,k})$ have been calculated for $k \leq 4$ and $i \leq 36$, and K. Oguchi [19] partly treated them for k=5.

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