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ON THE HOMOTOPY GROUP $\pi_{2n+9}(U(n))$ FOR $n \geq 6$

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The homotopy groups $\pi_{2n+i}(U(n))$ of the unitary group $U(n)$ for $0 \leq i \leq 8$, $i=10$ and 12 were determined by Borel and Hirzebruch [2], Bott [3], Kervaire [7], Toda [22, 23], Matsunaga [8-12], Mimura and Toda [13], Mosher [14, 15], and Imanishi [6]. For $n \geq 5$ and $i=9, 11$ or 13 the odd components were determined by [12] and [6], but the 2-component had not been completely determined. Indeed Mosher [15] has not determined some group extensions which appear in case of $i=9$ only if $n \equiv 2, 4$ or $6 \pmod{8}$ and $n \geq 6$. In this note we shall determine these group extensions for $i=9$. $\pi_{2n+9}(U(n))$ for $n \leq 5$ was determined by [6], [13], [15] and [23]. Therefore we shall complete the computation of $\pi_{2n+9}(U(n))$. While the group $\pi_{2n+9}(U(n))$ has been computed by Vastersavendts [24] for $n \equiv 0 \pmod{4}$, $6 \pmod{8}$ or $2 \pmod{16}$, her results contradict Mosher's [15] and ours for $n \equiv 0 \pmod{16}$ and $n \equiv 6 \pmod{8}$ respectively.

We shall prove

Theorem. *The 2-component of $\pi_{2n+9}(U(n))$ for $n \equiv 2, 4$ or $6 \pmod{8}$ and $n \geq 6$ is given by the following table:*

$n \pmod{(\)}$	$\pi_{2n+9}(U(n))$
2(16)	$Z_2 \oplus Z_4 \oplus Z_2$
10(32)	$Z_2 \oplus Z_4 \oplus Z_4$
26(64)	$Z_2 \oplus Z_4 \oplus Z_8$
58(64)	$Z_2 \oplus Z_4 \oplus Z_{16}$
4(8)	$Z_2 \oplus Z_2 \oplus Z_8$
6(8)	$Z_2 \oplus Z_4$

where $Z_m = Z/mZ$ is the cyclic group of order m .

We shall use the notations and terminologies defined in [20] or the book of Toda [23] without any reference.

1. Method of computation

By Theorem 4.3 of Toda [22] we know that $\pi_{2n+9}(U(n))$ is isomorphic to the stable homotopy group $\pi_{2n+9}^s(P_{n+6,6})$ of the stunted complex projective space $P_{n+6,6} = P_{n+6}/P_n$ if $n \geq 5$. We shall compute $\pi_{2n+9}^s(P_{n+6,6})$.

Consider the canonical cofibration

$$S^{2(n+k)-3} \xrightarrow{p_{n+k-1,k-1}} P_{n+k-1,k-1} \xrightarrow{i_1} P_{n+k,k} \xrightarrow{q_{k-1}} S^{2(n+k)-2}$$

and the associated exact sequence

$$(S)_k: \quad \cdots \rightarrow G_{i-2k+2} \xrightarrow{p_*} \pi_{2n-1+i}^s(P_{n+k-1,k-1}) \xrightarrow{i_{1*}} \pi_{2n-1+i}^s(P_{n+k,k}) \xrightarrow{q_*} G_{i-2k+1} \xrightarrow{p_*} \cdots.$$

We set the two steps of computation:

- (1) determine the G_* -module structure of $\pi_{2n-1+i}^s(P_{n+k-1,k-1})$,
- (2) describe $p_{n+k-1,k-1} \in \pi_{2(n+k)-3}^s(P_{n+k-1,k-1})$ explicitly.

If these two are possible, we know $\pi_{2n-1+i}^s(P_{n+k,k})$ up to group extension

$$0 \rightarrow \text{Cokernel of } p_* \rightarrow \pi_{2n-1+i}^s(P_{n+k,k}) \rightarrow \text{Kernel of } p_* \rightarrow 0.$$

To determine this group extension, we prepare a lemma.

Lemma 1 (cf. Theorem 2.1 of [13]). *Let $A \xrightarrow{f} X \xrightarrow{i} C(f)$ be a cofibration and*

$$\cdots \rightarrow \pi_n^s(X) \xrightarrow{i_*} \pi_n^s(C(f)) \xrightarrow{\Delta} \pi_{n-1}^s(A) \xrightarrow{f_*} \pi_{n-1}^s(X) \rightarrow \cdots$$

an associated stable exact sequence. Assume that $\alpha \in \pi_{n-1}^s(A)$ satisfies $f_(\alpha) = 0$, and the order of α is k . For an arbitrary element β of $\langle f, \alpha, k\alpha \rangle \subset \pi_n^s(X)$, there exists an element $[\alpha] \in \pi_n^s(C(f))$ such that*

$$\Delta([\alpha]) = \alpha \quad \text{and} \quad i_*(\beta) = -k[\alpha].$$

Proof. By definition of Toda bracket, there exists a commutative stable diagram with $\beta = a \circ b$:

$$\begin{array}{ccccccc} & & S^n & & & & \\ & & \downarrow b & & & & \\ S^{n-1} & \xrightarrow{\alpha} & A & \longrightarrow & C(\alpha) & \longrightarrow & S^n \xrightarrow{-\alpha} EA \\ = \downarrow & & f \downarrow & & a \downarrow & & = \downarrow \\ A & \xrightarrow{f} & X & \longrightarrow & C(f) & \longrightarrow & EA \end{array}$$

Then we may put $[\alpha] = -a'$.

For the above (2), we consider $(S)_k$ for $i=2k-2$:

$$\pi_{2(n+k)-2}^s(P_{n+k,k}) \xrightarrow{q_*} G_0 \xrightarrow{p_*} \pi_{2(n+k)-3}^s(P_{n+k-1,k-1}).$$

The exactness of this shows that

$$\#\#p_{n+k-1,k-1} = \#(\text{Cokernel of } q_*).$$

On the other hand by (4.5) of [20] we know that

$$\begin{aligned} \#(\text{Cokernel of } q_*) &= Q^s\{n+k, k\} \\ &= C\{jM_k(C) - n - k, k\} \quad \text{for large } j \end{aligned}$$

and this number was determined for $k \leq 8$ in (3.1) of [20]. We shall need the 2-component of this number for $k=5$ and 6. Let $\nu_2(m)$ be the exponent of 2 in the factorization of an integer m into the prime powers.

Lemma 2 ((3.1) of [20]). $\nu_2(\#\#p_{n+4,4})$ and $\nu_2(\#\#p_{n+5,5})$ are given by the following table:

$\nu_2(\#\#p_{n+4,4})$	$n \bmod (\)$	$\nu_2(\#\#p_{n+5,5})$	$n \bmod (\)$
4	4, 6(8)	4	4, 6(8)
3	0(8), 2(16)	3	0, 2(16)
2	10(16)	2	8(16), 10(32)
		1	26(64)
		0	58(64)

Considering the above (1) and (2), we shall compute inductively $\pi_{2n-1+i}^s(P_{n+k,k})$ for $k \leq 6$ and some $i \leq 10$. Since the suspension $EP_{n+k,k}$ is $2n$ -connected and the pair $(W_{n+k,k}, EP_{n+k,k})$ is $(4n+3)$ -connected, it follows that $\pi_{2n-1+i}^s(P_{n+k,k})$ is isomorphic to $\pi_{2n+i}(W_{n+k,k})$ for $i \leq 2n$, where $W_{n+k,k} = U(n+k)/U(n)$ is the complex Stiefel manifold. Nomura and Furukawa [16] have computed $\pi_{2n+i}(W_{n+k,k})$ for $k=2, 3$ and $i \leq 21, 19$ respectively. Therefore we already know $\pi_{2n-1+i}^s(P_{n+k,k})$ for $2 \leq k \leq 3$ and $i \leq 10$. But informations for (1) from [16] are not sufficient for our purpose. So we shall recompute some $\pi_{2n-1+i}^s(P_{n+k,k})$ for $k \leq 3$.

2. Computation

From now on, n means always an even integer ≥ 6 , $\pi_*^s(\)$ and G_* often denote only the 2-primary component of itself. We work in the stable category of pointed spaces and stable maps between them.

Since $p_{n+1,1}=n\eta=0$, it follows that $P_{n+2,2}=S^{2n}\vee S^{2n+2}$. Let $s: S^{2n+2}\rightarrow P_{n+2,2}$ be an inclusion map which is a right inverse of q_1 . Then

$$(2.1) \quad i_{1*}+s_*: G_{i-1}\oplus G_{i-3}\rightarrow \pi_{2n-1+i}^s(P_{n+2,2}) \quad \text{is an isomorphism.}$$

By the proof of (1.11), (i) of (1.13) and (1.14) of [20], we have

$$p_{n+2,2}=(n/2)i_{1*}(\nu+\alpha_1)+s_*\eta: S^{2n+3}\rightarrow P_{n+2,2}=S^{2n}\vee S^{2n+2}.$$

Put

$$e_n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{4} \\ 2 & \text{if } n \equiv 2 \pmod{4} \end{cases}.$$

Then by (2.1) and $(S)_3$ for $i=8$, we have a short exact sequence

$$0 \rightarrow Z_{16}\{i_{2*}\sigma\} \rightarrow \pi_{2n+7}^s(P_{n+3,3}) \rightarrow Z_{8/e_n}\{e_n\nu\} \rightarrow 0.$$

We have

$$\begin{aligned} \langle p_{n+2,2}, e_n\nu, (8/e_n)\iota \rangle &= \langle (n/2)i_{1*}\nu, e_n\nu, (8/e_n)\iota \rangle + \langle s_*\eta, e_n\nu, (8/e_n)\iota \rangle \\ &\supseteq i_{1*}\langle (n/2)\nu, e_n\nu, (8/e_n)\iota \rangle + s_*\langle \eta, e_n\nu, (8/e_n)\iota \rangle \\ &\supseteq i_{1*}\{(ne_n/4)\langle (2/e_n)\nu, e_n\nu, (8/e_n)\iota \rangle\} \\ &\supseteq 0 \end{aligned}$$

since $\langle \eta, e_n\nu, (8/e_n)\iota \rangle \subset G_5=0$ and $\langle (2/e_n)\nu, e_n\nu, (8/e_n)\iota \rangle \supseteq 0$ (see e.g. [16]). Therefore by Lemma 1 the above short exact sequence splits, that is, there exists $[e_n\nu] \in \pi_{2n+7}^s(P_{n+3,3})$ with $q_{2*}[e_n\nu]=e_n\nu$ and

$$(2.2) \quad \pi_{2n+7}^s(P_{n+3,3}) = Z_{16}\{i_{2*}\sigma\} \oplus Z_{8/e_n}\{[e_n\nu]\}.$$

It follows from $(S)_3$ for $i=9$ that $i_{1*}: \pi_{2n+8}^s(P_{n+2,2}) \rightarrow \pi_{2n+8}^s(P_{n+3,3})$ is an isomorphism. Hence by (2.1) we have

$$(2.3) \quad \pi_{2n+8}^s(P_{n+3,3}) = Z_2\{i_{2*}\varepsilon\} \oplus Z_2\{i_{2*}\bar{\nu}\} \oplus Z_2\{i_{1*}s_*\nu^2\}.$$

From (2.1) and $(S)_3$ for $i=10$ it follows that

$$(2.4) \quad \pi_{2n+9}^s(P_{n+3,3}) = Z_{16}\{i_{1*}s_*\sigma\} \oplus Z_2\{i_{2*}\mu\} \oplus Z_2\{i_{2*}\eta\varepsilon\} \oplus Z_{2/e_n}\{i_{2*}\nu^3\}.$$

Analysing $p_{n+k,k}$ for $k=3, 4$ and 5 , we consider the followings. Put

$$L_{m,k} = \begin{cases} 1 & \text{if } m+k \equiv 1 \pmod{2} \\ 2 & \text{if } m+k \equiv 0 \pmod{2} \end{cases}.$$

Then, since $L_{m,k}(m+k-1) \equiv 0 \pmod{2}$, $q_{l-1*}(L_{m,k}p_{m+k,l})=L_{m,k}(m+k-1)\eta=0$ and hence $i_{1*}^{-1}(L_{m,k}p_{m+k,l})$ is not empty for $1 < l < m+k$, and

$$(T)_k \quad i_1^{-1}(L_{m,k}p_{m+k,k}) = i_1^{-1}(L_{m,k}q_{m-1,k}p_{m+k}) \supset q_{m-1,k}i_1^{-1}(L_{m,k}p_{m+k})$$

and by (1.15) of [20]

$$\begin{aligned} (T)'_k & q_{k-2} * q_{m-1,k} i_1^{-1}(L_{m,k}p_{m+k}) \\ &= q_{m+k-3} * i_1^{-1}(L_{m,k}p_{m+k}) \\ &= \begin{cases} (m+k-2)(\nu+\alpha_1) & \text{if } m+k \equiv 0 \pmod{2} \\ \{(1/2)(m+k+1)(\nu+\alpha_1), (1/2)(m+k+1)(\nu+\alpha_1)+4\nu\} & \text{if } m+k \equiv 1 \pmod{2} \end{cases} \end{aligned}$$

Now $q_{1*} = s_*^{-1}: \pi_{2n+5}^s(P_{n+2,2}) \xrightarrow{\cong} \pi_{2n+5}^s(S^{2n+2}) = G_3$ by (2.1), since $q_1 \circ s = 1$.

Then by $(T)'_3$

$$q_{n-1,k} i_1^{-1}(p_{n+3}) \supset ((n+4)/2)s_*(\nu+\alpha_1)$$

and by $(T)_3$

$$p_{n+3,3} = ((n+4)/2)i_1 s_*(\nu+\alpha_1)$$

so that $p_{n+3,3} \circ \eta = 0$ and

$$\langle p_{n+3,3}, \eta, 2\iota \rangle \supset i_1 s_* \langle ((n+4)/2)\nu, \eta, 2\iota \rangle = 0$$

and by Lemma 1 there exists $[\eta] \in \pi_{2n+7}^s(P_{n+4,4})$ with $q_{3*}[\eta] = \eta$ and

$$(2.5) \quad \pi_{2n+7}^s(P_{n+4,4}) = Z_{16}\{i_{3*}\sigma\} \oplus Z_{8/e_n}\{i_{1*}[e_n\nu]\} \oplus Z_2\{[\eta]\}.$$

We have also the following from (2.3) and $(S)_4$ for $i=9$

$$(2.6) \quad \pi_{2n+8}^s(P_{n+4,4}) = Z_2\{i_{3*}\varepsilon\} \oplus Z_2\{i_{3*}\bar{\nu}\} \oplus Z_{2/e_n}\{i_{2*}s_*\nu^2\} \oplus Z_2\{[\eta]\eta\}.$$

By the same argument as the proof of (2.2) we know that there exists $[[e_n\nu]] \in \pi_{2n+9}^s(P_{n+4,4})$ with $q_{3*}[[e_n\nu]] = e_n\nu$ and

$$\begin{aligned} (2.7) \quad \pi_{2n+9}^s(P_{n+4,4}) &= Z_{16}\{i_{2*}s_*\sigma\} \oplus Z_2\{i_{3*}\mu\} \oplus Z_2\{i_{3*}\eta\varepsilon\} \\ &\quad \oplus Z_{2/e_n}\{i_{3*}\nu^3\} \oplus Z_{8/e_n}\{[[e_n\nu]]\}. \end{aligned}$$

To compute $\pi_{2n+9}^s(P_{n+5,5})$ we shall prepare four lemmas.

Remember that in [20] we used the notations: $HP_{m+k} = HP_{m+k}/HP_m$, the stunted quaternionic projective space; $\pi: P_{2m+2k, 2k} \rightarrow HP_{m+k}$, the canonical quotient map;

$$(2.8) \quad S^{2n+4k-1} \xrightarrow{\tilde{P}_{(n/2)+k,k}^H} HP_{(n/2)+k,k} \xrightarrow{i_1^H} HP_{(n/2)+k+1,k+1}$$

the canonical cofibration.

Lemma 3. *We have*

- (i) $\pi_{2n+7}^s(HP_{(n/2)+2,2}) = Z_{16}\{i_{1*}^H\sigma\} \oplus Z_{8/e_n}\{\pi_*i_{1*}[e_n\nu]\},$
- (ii) $\pi_{2n+8}^s(HP_{(n/2)+2,2}) = Z_2\{i_{1*}^H\varepsilon\} \oplus Z_2\{i_{1*}^H\bar{\nu}\},$
- (iii) $\pi_{2n+9}^s(HP_{(n/2)+2,2}) = Z_2\{i_{1*}^H\mu\} \oplus Z_2\{i_{1*}^H\eta\varepsilon\} \oplus Z_{2/e_n}\{i_{1*}^H\nu^3\},$
- (iv) $p_{(n/2)+2,2}^H \circ \eta = \begin{cases} i_{1*}^H\varepsilon & \text{if } n \equiv 2 \pmod{8} \\ i_{1*}^H\bar{\nu} & \text{if } n \equiv 4 \pmod{8} \\ i_{1*}^H(\varepsilon + \bar{\nu}) & \text{if } n \equiv 6 \pmod{8} \\ 0 & \text{if } n \equiv 0 \pmod{8}. \end{cases}$

Proof. Considering the stable homotopy exact sequence associated with (2.8) for $k=1$, we obtain (ii) and (iii) immediately since $G_4=G_5=0$ and $p_{(n/2)+1,1}^H=(n/2)(\nu+\alpha_1)$ and by Lemma 1 we have a split exact sequence:

$$0 \rightarrow Z_{16}\{i_{1*}^H\sigma\} \rightarrow \pi_{2n+7}^s(HP_{(n/2)+2,2}) \rightarrow Z_{8/e_n}\{e_n\nu\} \rightarrow 0.$$

Then the following commutative diagram induces (i):

$$\begin{array}{ccccccc}
 & & S^{2n+7} & \xrightarrow{e_n\nu} & S^{2n+4} & & \\
 & & \downarrow [e_n\nu] & & \downarrow q_2 & & \\
 S^{2n} & \xrightarrow{i_1} & P_{n+2,2} & \xrightarrow{i_1} & P_{n+3,3} & \xrightarrow{i_1} & S^{2n+4} \\
 \downarrow = & & \downarrow \pi & & \downarrow \pi & & \downarrow = \\
 S^{2n} & = & HP_{(n/2)+1,1} & \xrightarrow{i_1^H} & HP_{(n/2)+2,2} & \xrightarrow{q_1^H} & S^{2n+4}
 \end{array}$$

Since $q_1^H \circ p_{(n/2)+2,2}^H \circ \eta = ((n/2)+1)(\nu+\alpha_1)\eta = 0$, there exists a map $f: S^{2n+8} \rightarrow S^{2n}$ with $i_1^H \circ f = p_{(n/2)+2,2}^H \circ \eta$. It is easily seen that $i_1^{H*}: \{HP_{(n/2)+2,2}, S^{2n-1}\} \rightarrow \{S^{2n}, S^{2n-1}\} = Z_2\{\eta\}$ is an isomorphism. Let $h \in \{HP_{(n/2)+2,2}, S^{2n-1}\}$ be the element with $h \circ i_1^H = \eta$. It follows from (2.7) of [21] that

$$h \circ p_{(n/2)+2,2}^H = \begin{cases} \varepsilon & \text{if } n \equiv 2 \pmod{8} \\ \bar{\nu} & \text{if } n \equiv 4 \pmod{8} \\ \varepsilon + \bar{\nu} & \text{if } n \equiv 6 \pmod{8} \\ 0 & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

Since $\eta \circ f = h \circ p_{(n/2)+2,2}^H \circ \eta$ and $\eta: G_8 \rightarrow G_9$ is a monomorphism, we obtain (iv). This completes the proof of Lemma 3.

Lemma 4. *For suitably chosen $[[e_n\nu]]$ it holds that $[\eta]\eta^2 = (4/e_n)[[e_n\nu]]$.*

Proof. By Proposition 1.4 of Toda [23]

$$(2.9) \quad [\eta]\eta^2 = [\eta] \circ \langle 2\iota, \eta, 2\iota \rangle = \langle [\eta], 2\iota, \eta \rangle \circ 2\iota.$$

Let $\text{Indet } \langle \alpha, \beta, \gamma \rangle$ be an indeterminacy of a Toda bracket $\langle \alpha, \beta, \gamma \rangle$. Then

$$\begin{aligned}\pi_{2n+9}^s(P_{n+4,4}) \supset \text{Indet } \langle [\eta], 2\iota, \eta \rangle &= [\eta] \circ \{S^{2n+9}, S^{2n+7}\} + \pi_{2n+8}^s(P_{n+4,4}) \circ \eta \\ &= Z_2\{[\eta]\eta^2\} + Z_2\{i_{3*}\eta\varepsilon\} + Z_{2/e_n}\{i_{3*}\nu^3\}\end{aligned}$$

and

$$q_{3*} \text{Indet } \langle [\eta], 2\iota, \eta \rangle = Z_2\{\eta^3\} = Z_2\{4\nu\} = \text{Indet } \langle \eta, 2\iota, \eta \rangle$$

and, since $q_{3*}\langle [\eta], 2\iota, \eta \rangle \subset \langle q_{3*}[\eta], 2\iota, \eta \rangle = \langle \eta, 2\iota, \eta \rangle$, we have

$$q_{3*}\langle [\eta], 2\iota, \eta \rangle = \langle \eta, 2\iota, \eta \rangle = \{2\nu, 6\nu\}.$$

Hence there exists an element in $\langle [\eta], 2\iota, \eta \rangle$ which is mapped to 2ν by q_{3*} . By (2.7) this element has a form as $(2/e_n)[[e_n\nu]] + i_{1*}x$ for some $x \in \pi_{2n+9}^s(P_{n+3,3})$, and from (2.9) it follows that $4i_{1*}x = 0$. Then by (2.7) $2i_{1*}x$ is divisible by 8, that is, $2i_{1*}x = 8i_{1*}y$ for some $y \in \pi_{2n+9}^s(P_{n+3,3})$. Then

$$\begin{aligned}[\eta]\eta^2 &= 2\{(2/e_n)[[e_n\nu]] + i_{1*}x\} \\ &= (4/e_n)([[e_n\nu]] + 2e_n i_{1*}y).\end{aligned}$$

Since $q_{3*}([[e_n\nu]]) + 2e_n i_{1*}y = e_n\nu$ and the order of $[[e_n\nu]] + 2e_n i_{1*}y$ is $8/e_n$, we may change $[[e_n\nu]]$ for $[[e_n\nu]] + 2e_n i_{1*}y$. So the conclusion follows.

Appointment: From now on we assume that $[[e_n\nu]]$ satisfies $[\eta]\eta^2 = (4/e_n)[[e_n\nu]]$.

Since $q_{3*}p_{n+4,4} = (n+3)\eta = \eta$, by (2.5) we can put

$$p_{n+4,4} = a_n i_{3*}\sigma + b_n i_{1*}[e_n\nu] + [\eta] + \text{odd torsion}$$

for some integers a_n and b_n . By Lemma 2 and (2.5) we have

$$(2.10) \quad a_n \equiv \begin{cases} 1 \pmod{2} & \text{if } n \equiv 4 \text{ or } 6 \pmod{8} \\ 0 \pmod{2} & \text{if } n \equiv 0 \text{ or } 2 \pmod{8}. \end{cases}$$

By $(T)_4$, and $(T)'_4$, for any $p' \in q_{n-1*}i_{1*}^{-1}(2p_{n+4}) \subset \pi_{2n+7}^s(P_{n+3,3})$ we have

$$i_1 \circ p' = 2p_{n+4,4} \quad \text{and} \quad q_2 \circ p' = (n+2)(\nu + \alpha_1).$$

Then $p' = 2a_n i_{2*}\sigma + 2b_n [e_n\nu] + \text{odd torsion}$. Applying q_{2*} to this equation we know that $2b_n e_n \equiv n+2 \pmod{8}$, and

$$(2.11) \quad b_n \equiv \begin{cases} 1 \pmod{2} & \text{if } n \equiv 0 \pmod{4} \text{ or } 2 \pmod{8} \\ 0 \pmod{2} & \text{if } n \equiv 6 \pmod{8}. \end{cases}$$

Lemma 5. *We have*

$$[2\nu]\eta = \begin{cases} i_{2*}\varepsilon & \text{if } n \equiv 2 \pmod{8} \\ i_{2*}\varepsilon \text{ or } i_{2*}\bar{\nu} & \text{if } n \equiv 6 \pmod{8} \end{cases}, \text{ and}$$

$$[\nu]\eta = (n/4)i_{2*}\varepsilon + i_{1*}s_*\nu^2 \text{ if } n \equiv 0 \pmod{4}.$$

Proof. By Lemma 1 we can easily construct a commutative diagram:

$$\begin{array}{ccccccc}
 & & S^{2n+8} & & & & \\
 & & \downarrow b & \searrow \eta & & & \\
 S^{2n+6} \xrightarrow{e_n\nu} & S^{2n+3} \longrightarrow & C(e_n\nu) \longrightarrow & S^{2n+7} \xrightarrow{-e_n\nu} & S^{2n+4} & & \\
 \downarrow = & & \downarrow a & & \downarrow -[e_n\nu] & & \downarrow = \\
 S^{2n+3} \longrightarrow & P_{n+2,2} \xrightarrow{i_1} & P_{n+3,3} \longrightarrow & S^{2n+4} & & & \\
 \downarrow p_{n+2,2} & & & & & &
 \end{array}$$

Then $a \circ b \in \langle p_{n+2,2}, e_n\nu, \eta \rangle$ and this Toda bracket is a coset of

$$\begin{aligned}
 & \pi_{2n+8}^*(P_{n+2,2})/\pi_{2n+7}^*(P_{n+2,2}) \circ \eta \\
 & = [Z_2\{i_{1*}\varepsilon\} \oplus Z_2\{i_{1*}\bar{\nu}\} / \{0, i_{1*}(\varepsilon + \bar{\nu})\}] \oplus Z_2\{s_*\nu^2\}.
 \end{aligned}$$

We have

$$\begin{aligned}
 \langle p_{n+2,2}, e_n\nu, \eta \rangle & = \langle (n/2)i_{1*}\nu, e_n\nu, \eta \rangle + \langle s_*\eta, e_n\nu, \eta \rangle \\
 & \supseteq i_{1*}\{(ne_n/4)\langle (2/e_n)\nu, e_n\nu, \eta \rangle\} + s_*\langle \eta, e_n\nu, \eta \rangle \\
 & \ni (ne_n/4)i_{1*}\varepsilon + e_n s_*\nu^2
 \end{aligned}$$

since $\langle (2/e_n)\nu, e_n\nu, \eta \rangle = \varepsilon + G_7 \circ \eta$ and $\langle \eta, e_n\nu, \eta \rangle = e_n\nu^2$ by Toda [23]. Hence

$$\begin{aligned}
 (2.12) \quad [e_n\nu]\eta & = i_{1*}(a \circ b) \\
 & = (ne_n/4)i_{2*}\varepsilon + e_n i_{1*}s_*\nu^2 \text{ or } ((ne_n/4) + 1)i_{2*}\varepsilon + i_{2*}\bar{\nu} + e_n i_{1*}s_*\nu^2.
 \end{aligned}$$

Thus Lemma 5 follows if $n \equiv 6 \pmod{8}$. By Lemma 4

$$(2.13) \quad p_{n+4,4} \circ \eta^2 = a_n i_{3*}(\eta\varepsilon + \nu^3) + b_n i_{1*}[e_n\nu]\eta^2 + (4/e_n)[[e_n\nu]]$$

and by (iii) of Lemma 3, the fact $4/e_n \equiv 0 \pmod{2}$ and the commutativity of the diagram in the proof of Lemma 3 it follows that

$$\begin{aligned}
 p_{(n/2)+2,2}^H \circ \eta^2 & = \pi \circ p_{n+4,4} \circ \eta^2 \\
 & = a_n i_{1*}^H(\eta\varepsilon + \nu^3) + b_n \pi_* i_{1*}[e_n\nu]\eta^2.
 \end{aligned}$$

Then the conclusions for $n \not\equiv 6 \pmod{8}$ follow from (iii) and (iv) of Lemma 3, (2.10), (2.11) and (2.12). This completes the proof of Lemma 5.

Lemma 6. *We have*

$$p_{n+4,4} \circ \eta^2 = \begin{cases} i_{3*} \eta \varepsilon + 2[[2\nu]] & \text{if } n \equiv 2 \pmod{4} \\ (n/4) i_{3*} \nu^3 + 4[[\nu]] & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Proof. The conclusion follows from (2.7), (2.10), (2.11), Lemma 5 and (2.13).

Now we compute $\pi_{2n+9}^s(P_{n+5,5})$. Since $p_{n+4,4} \cdot \eta = [\eta]\eta + (\text{other term})$ is non-zero, it follows from (2.7), Lemma 6 and (S)₅ for $i=10$ that

$$(2.14) \quad \pi_{2n+9}^s(P_{n+5,5}) = Z_{16}\{i_3 * s * \sigma\} \oplus Z_2\{i_4 * \mu\} \oplus H,$$

where

By $(T)_5'$

$$q_3 * q_{n-1} * i_1^{-1}(p_{n+5}) = \{((n+6)/2)(\nu + \alpha_1), ((n+6)/2)(\nu + \alpha_1) + 4\nu\}$$

and hence we can choose a map $\tilde{p} \in q_{n-1*} i_{1*}^{-1}(p_{n+5}) \subset \pi_{2n+9}^s(P_{n+4,4})$ with

$$q_3 \circ \tilde{p} = \begin{cases} ((n+6)/2)(\nu + \alpha_1) + 4\nu & \text{if } n \equiv 2 \pmod{16} \\ ((n+6)/2)(\nu + \alpha_1) & \text{otherwise} \end{cases}$$

and then by (T)₅

$$i_1 \circ \tilde{p} = p_{n+5,5}.$$

By (2.7) we can put

$$(2.15) \quad \tilde{p} = a'_n i_{2*} s_* \sigma + b'_n i_{3*} \mu + c_n i_{3*} \eta \varepsilon + d'_n i_{3*} \nu^3 + d_n [[e_n \nu]] + \text{odd torsion}$$

for some integers a'_n, b'_n, c_n, d'_n and d_n . Remark that $i_{3*}v^3=0$ if $n \equiv 2 \pmod{4}$. We have

$$d_n e_n v + \text{odd torsion} = q_3 \circ \tilde{p} \\ = \begin{cases} (((n+6)/2)+4)v + \text{odd torsion} & \text{if } n \equiv 2 \pmod{16} \\ ((n+6)/2)v + \text{odd torsion} & \text{otherwise} \end{cases}$$

and

$$(2.16) \quad d_n \equiv \begin{cases} 1 \pmod{2} & \text{if } n \equiv 0 \pmod{4} \text{ or } 6 \pmod{8} \\ 0 \pmod{4} & \text{if } n \equiv 2 \pmod{8}. \end{cases}$$

Put $p_{n+5,5} = a'_n i_{3*} s_* \sigma + b'_n i_{4*} \mu + \hat{p}$. Then the 2-primary part of \hat{p} is contained in H_n . Hence by Lemma 2 and (2.14) we have

$$a'_n \equiv 1 \pmod{2} \quad \text{if } n \equiv 4 \text{ or } 6 \pmod{8}.$$

Then by (2.14), (2.16) and $(S)_6$ for $i=10$ we have

$$(2.17) \quad \pi_{2n+9}^s(P_{n+6,6}) = Z_{8/e_n}\{i_{4*} s_* \sigma\} \oplus Z_2\{i_{5*} \mu\} \oplus Z_{2/e_n} \quad \text{if } n \equiv 4 \text{ or } 6 \pmod{8}$$

where if $n \equiv 4 \pmod{8}$, $Z_{2/e_n} = Z_2$ is generated by $i_{5*} \eta \varepsilon$.

Next suppose that $n \equiv 2 \pmod{8}$. Let l be the odd component of the order of \tilde{p} . Of course l is an odd integer. Put $\hat{p} = l a'_n s_* \sigma + b'_n i_{1*} \mu + c_n i_{1*} \eta \varepsilon$. Then by (2.15) and (2.16)

$$l \tilde{p} = i_{2*} \hat{p}$$

and we have a commutative diagram in which the each horizontal sequences are cofibrations and l denotes a multiplication by l :

$$\begin{array}{ccccc}
 S^{2n+9} & \xrightarrow{\tilde{p}_{n+5,5}} & P_{n+5,5} & \longrightarrow & P_{n+6,6} \\
 \uparrow = & & \uparrow i_1 & & \uparrow i_1 \\
 S^{2n+9} & \xrightarrow{\tilde{p}} & P_{n+4,4} & \longrightarrow & C(\tilde{p}) \\
 \downarrow = & & \downarrow l & & \downarrow \bar{l} \\
 S^{2n+9} & \xrightarrow{l \tilde{p}} & P_{n+4,4} & \longrightarrow & C(l \tilde{p}) \\
 \uparrow = & & \uparrow i_2 & & \uparrow i_2 \\
 S^{2n+9} & \xrightarrow{\hat{p}} & P_{n+2,2} & \longrightarrow & C(\hat{p}) \\
 \downarrow = & & \downarrow \pi & & \downarrow \bar{\pi} \\
 S^{2n+9} & \xrightarrow{\pi \circ \hat{p}} & HP_{(n/2)+1,1} & \xrightarrow{j} & C(\pi \circ \hat{p})
 \end{array}$$

We calculate the Adams' e_C and e_R invariants of $\pi \circ \hat{p} \in G_9$.

Lemma 7. *We have*

$$(i) \quad e_C(\pi \circ \hat{p}) = 0 \text{ and } b'_n \equiv 0 \pmod{2},$$

$$(ii) \quad e_R(\pi \circ \hat{p}) = \begin{cases} 1 & \text{if } n \equiv 2 \pmod{16} \\ 0 & \text{if } n \equiv 10 \pmod{16} \end{cases} \text{ and}$$

$$c_n = \begin{cases} 1 \pmod{2} & \text{if } n \equiv 2 \pmod{16} \\ 0 \pmod{2} & \text{if } n \equiv 10 \pmod{16} \end{cases}.$$

Proof. Applying \tilde{K} to the above diagram, we can show the first part of (i) by the similar method as the proof of (1.12) of [20]. Then the second part of (i) follows, since $\pi \circ s = \eta^2$ or 0, $e_C(\eta^2 \sigma) = e_C(\eta \varepsilon) = 0$ and $e_C(\mu) \neq 0$ by [1].

Put $n=8m+2$. Applying \widetilde{KO}^{-4} to the above diagram, we have the following commutative diagram in which the horizontal sequences are exact:

diagram (2.18)

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+7,5}) & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+8,6}) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0 \\
 & & \downarrow i_1^* & & \downarrow i_1^* & & \downarrow = \\
 0 & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+6,4}) & \longleftarrow & \widetilde{KO}^{-4}(C(\tilde{p})) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0 \\
 & & \uparrow l & & \uparrow l^* & & \uparrow = \\
 0 & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+6,4}) & \longleftarrow & \widetilde{KO}^{-4}(C(l\tilde{p})) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0 \\
 & & \downarrow i_2^* & & \downarrow i_2^* & & \downarrow = \\
 0 & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+4,2}) & \longleftarrow & \widetilde{KO}^{-4}(C(\tilde{p})) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0 \\
 & & \uparrow \pi^* & & \uparrow \bar{\pi}^* & & \uparrow = \\
 0 & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+4}) & \longleftarrow & \widetilde{KO}^{-4}(C(\pi \circ \tilde{p})) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0
 \end{array}$$

By Theorem 2 of Fujii [4] it is easily seen that

$$\begin{aligned}
 \widetilde{KO}^{-4}(P_{8m+8,6}) &= Z\{z_2z_0^{4m}, z_2z_0^{4m+1}, z_2z_0^{4m+2}\} \oplus Z_2\{z_2z_0^{4m+3}\} \\
 \widetilde{KO}^{-4}(P_{8m+7,5}) &= Z\{z_2z_0^{4m}, z_2z_0^{4m+1}, z_2z_0^{4m+2}\} \\
 \widetilde{KO}^{-4}(P_{8m+6,4}) &= Z\{z_2z_0^{4m}, z_2z_0^{4m+1}\} \\
 \widetilde{KO}^{-4}(P_{8m+4,2}) &= Z\{z_2z_0^{4m}\} \oplus Z_2\{z_2z_0^{4m+1}\}.
 \end{aligned}$$

Also note that a generator d of $\widetilde{KO}^{-4}(S^{16m+4})=Z$ satisfies

$$\pi^*d = z_2z_0^{4m} + xz_2z_0^{4m+1}$$

for some integer x . We shall not need the explicit value of x . Here we regard $\widetilde{KO}^{-4}(X/A)$ as a subgroup of $\widetilde{KO}^{-4}(X)$ if the quotient map $X \rightarrow X/A$ induces a monomorphism. Similar remarks shall hold in the forthcoming proof of (A). By chasing diagram, we know that there exist elements $[z_2z_0^{4m}]$ and $[z_2z_0^{4m+1}]$ in $\widetilde{KO}^{-4}(C(l\tilde{p}))$ such that

$$\bar{l}^*[z_2z_0^{4m}] = l_1^*z_2z_0^{4m} \quad \text{and} \quad \bar{l}^*[z_2z_0^{4m+1}] = l_1^*z_2z_0^{4m+1}.$$

Put $a' = [z_2z_0^{4m}] + x[z_2z_0^{4m+1}]$. Then there exists an element $a \in \widetilde{KO}^{-4}(C(\pi \circ \tilde{p}))$ such that

$$\bar{\pi}^*a = i_2^*a' \quad \text{and} \quad j^*a = d.$$

Let $b \in \widetilde{KO}^{-4}(C(\pi \circ \tilde{p}))$ and $b' \in \widetilde{KO}^{-4}(C(l\tilde{p}))$ be the images of the generator of $\widetilde{KO}^{-4}(S^{16m+14})=Z_2$.

Now we assume the followings which shall be proved later:

(A) $i_1^* z_2 z_0^{4m+2} = e_{2m} \bar{l}^* b'$,
 (B) the order of $i_2^* [z_2 z_0^{4m+1}]$ is 2.

Remark that $e_{2m} = 1$ if $m \equiv 0 \pmod{2}$, or 2 if $m \equiv 1 \pmod{2}$, and $\bar{l}^* b'$ is the generator of the 2-torsion of $\widetilde{KO}^{-4}(C(\tilde{p}))$. We have

$$\psi^3 a = 3^{8m+4} a + \lambda b$$

for some $\lambda \in \mathbb{Z}_2$, and

$$e_R(\pi \circ \hat{p}) = \lambda$$

and

$$\pi^* \psi^3 a = \pi^* (3^{8m+4} a + \lambda b) = i_2^* (3^{8m+4} a' + \lambda b') .$$

On the other hand

$$\pi^* \psi^3 a = \psi^3 \pi^* a = \psi^3 i_2^* a' = i_2^* \psi^3 a'$$

and

$$(2.19) \quad i_2^* (3^{8m+4} a' + \lambda b') = i_2^* \psi^3 a' .$$

Since the order of $i_1^* z_2 z_0^{4m+3}$ is 2 and $i_1^* z_2 z_0^{4m+2} = e_{2m} \bar{l}^* b'$ by (A), we have

$$\begin{aligned} \bar{l}^* \psi^3 a' &= \psi^3 \bar{l}^* a' \\ &= \psi^3 \{ \bar{l}^* (z_2 z_0^{4m} + x z_2 z_0^{4m+1}) \} \\ &= \bar{l}^* \psi^3 (z_2 z_0^{4m} + x z_2 z_0^{4m+1}) \\ &= \bar{l}^* \{ 3^{8m+4} z_2 z_0^{4m} + ((8m+2) 3^{8m+3} + x 3^{8m+6}) z_2 z_0^{4m+1} \\ &\quad + ((4m+1)(8m+1) 3^{8m+2} + x(8m+4) 3^{8m+5}) z_2 z_0^{4m+2} \\ &\quad + ((4m+1)(8m+1) 8m 3^{8m} + x(4m+2)(8m+3) 3^{8m+4}) z_2 z_0^{4m+3} \} \\ &= \bar{l}^* \{ 3^{8m+4} [z_2 z_0^{4m}] + ((8m+2) 3^{8m+3} + x 3^{8m+6}) [z_2 z_0^{4m+1}] + e_{2m} b' \} . \end{aligned}$$

Then, since \bar{l}^* is a monomorphism,

$$\psi^3 a' = 3^{8m+4} [z_2 z_0^{4m}] + ((8m+2) 3^{8m+3} + x 3^{8m+6}) [z_2 z_0^{4m+1}] + e_{2m} b'$$

and by (B)

$$i_2^* \psi^3 a' = 3^{8m+4} i_2^* [z_2 z_0^{4m}] + x i_2^* [z_2 z_0^{4m+1}] + e_{2m} i_2^* b'$$

also by (2.19) this equals to

$$\begin{aligned} i_2^* (3^{8m+4} a' + \lambda b') &= i_2^* \{ 3^{8m+4} ([z_2 z_0^{4m}] + x [z_2 z_0^{4m+1}]) + \lambda b' \} \\ &= 3^{8m+4} i_2^* [z_2 z_0^{4m}] + x i_2^* [z_2 z_0^{4m+1}] + \lambda i_2^* b' \end{aligned}$$

and, since $i_2^* b'$ is non-zero,

$$\lambda = e_{2m} \quad \text{in } \mathbb{Z}_2$$

and this implies the first part of (ii). Since $\pi \circ s = 0$ or η^2 , and $a'_n \equiv 0 \pmod{2}$ by Lemma 2, it follows that by the second part of (i) we have

$$\pi \circ \hat{p} = c_n \eta \varepsilon$$

and the above proof of the first part of (ii) shows that

$$c_n \equiv \begin{cases} 1 \pmod{2} & \text{if } n \equiv 2 \pmod{16} \\ 0 \pmod{2} & \text{if } n \equiv 10 \pmod{16} \end{cases}.$$

This implies the second part of (ii).

We shall give the proofs of (A) and (B).

The proof of (A): We have the following commutative diagram:

$$\begin{array}{ccccccc}
 S^{16m+13} & \longrightarrow & P_{8m+7,5} & \longrightarrow & P_{8m+8,6} & \longrightarrow & S^{16m+14} \\
 \downarrow \begin{matrix} \nearrow \\ \tilde{p} \end{matrix} & & \downarrow & & \downarrow i_1 & & \downarrow \begin{matrix} \nearrow \\ q_3 \end{matrix} \\
 S^{16m+13} & \xrightarrow{\tilde{p}} & P_{8m+6,4} & \longrightarrow & C(\tilde{p}) & \longrightarrow & S^{16m+14} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 S^{16m+13} & \longrightarrow & P_{8m+7,2} & \longrightarrow & P_{8m+8,3} & \longrightarrow & S^{16m+14} \\
 \downarrow q_3 & & \downarrow \bar{q}_3 & & \downarrow i_1 & & \downarrow q_2 \\
 S^{16m+13} & \longrightarrow & P_{8m+6,1} & \longrightarrow & C(q_3 \circ \tilde{p}) & \longrightarrow & S^{16m+14}
 \end{array}$$

We have

$$\begin{aligned}
 \widetilde{KO}^{-4}(P_{8m+8,3}) &= Z\{z_2 z_0^{4m+2}\} \oplus Z_2\{z_2 z_0^{4m+3}\}, \\
 q_3^* z_2 z_0^{4m+2} &= z_2 z_0^{4m+2}.
 \end{aligned}$$

It suffices for our purpose to compute $i_1^* z_2 z_0^{4m+2}$, since $i_1^* z_2 z_0^{4m+2}$ is contained in the image of $\widetilde{KO}^{-4}(S^{16m+14})$, and \bar{q}_3^* induces an isomorphism between the images of $\widetilde{KO}^{-4}(S^{16m+14})$. We have chosen \tilde{p} such that $q_3 \circ \tilde{p} = (m+1)\alpha_1$. Let u_m be the order of $q_3 \circ \tilde{p}$. Then $u_m = 1$ or 3 . Applying $\pi_{16m+14}^*(\)$ to the above diagram, we know easily that there exists uniquely an element $u \in \pi_{16m+14}^*(P_{8m+8,3})$ such that $q_2 \circ u = u_m \iota$, moreover there exists $\hat{u} \in \pi_{16m+14}^*(C(q_3 \circ \tilde{p}))$ such that $\hat{q}_2^* \hat{u} = u_m \iota$ and $u = i_1^* \hat{u}$, where ι is the identity map of S^{16m+14} . Since $\hat{q}_2^* : \widetilde{KO}^{-4}(S^{16m+14}) \rightarrow \widetilde{KO}^{-4}(C(q_3 \circ \tilde{p}))$ is an isomorphism, and $\hat{u}^* \hat{q}_2^*$ is the multiplication by u_m which is the identity homomorphism of $\widetilde{KO}^{-4}(S^{16m+14}) = Z_2$, it follows that $\hat{u}^* : \widetilde{KO}^{-4}(C(q_3 \circ \tilde{p})) \rightarrow \widetilde{KO}^{-4}(S^{16m+14})$ is the inverse of \hat{q}_2^* . Thus

$$(2.20) \quad i_1^* z_2 z_0^{4m+2} = \hat{q}_2^* \hat{u}^* i_1^* z_2 z_0^{4m+2} = \hat{q}_2^* u^* z_2 z_0^{4m+2}.$$

Next we determine $u^* z_2 z_0^{4m+2}$. Consider the commutative diagram:

$$\begin{array}{ccc}
\tilde{K}(P_{8m+8,3}) & \xrightarrow{u^*} & \tilde{K}(S^{16m+14}) \\
\downarrow \cong & & \downarrow \cong \\
\tilde{K}^{-4}(P_{8m+8,3}) & \xrightarrow{u^*} & \tilde{K}^{-4}(S^{16m+14}) \\
\downarrow r & & \downarrow r \\
\widetilde{KO}^{-4}(P_{8m+8,3}) & \xrightarrow{u^*} & \widetilde{KO}^{-4}(S^{16m+14})
\end{array}$$

Recall that $\tilde{K}(P_{8m+8,3})=Z\{z^{8m+5}, z^{8m+6}, z^{8m+7}\}$ and the real restriction homomorphism r in the right hand side is an epimorphism. We can prove the followings:

$$(2.21) \quad \begin{cases} r(g_C^2 z^{8m+5}) = z_2 z^{4m+3} + (8m+5)z_2 z_0^{4m+2} \\ r(g_C^2 z^{8m+6}) = z_2 z_0^{4m+3} + 2z_2 z_0^{4m+2} \\ r(g_C^2 z^{8m+7}) = z_2 z_0^{4m+3}, \end{cases}$$

$$(2.22) \quad r(g_C^2(z^{8m+5} - (4m+2)z^{8m+6} + z^{8m+7})) = z_2 z_0^{4m+2},$$

$$(2.23) \quad \begin{cases} u^* z^{8m+5} = (1/3)(8m+5)(3m+2)u_m \beta \\ u^* z^{8m+6} = (4m+3)u_m \beta \\ u^* z^{8m+7} = u_m \beta \end{cases}$$

where $\beta \in \tilde{K}(S^{16m+14})=Z$ is the generator such that $q_2^* \beta = z^{8m+7}$. (2.23) follows from the relation $\psi^2 u^* = u^* \psi^2$. For (2.21) we consider the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{K}^{-4}(P_{8m+8,3}) & \xhookrightarrow{q^*} & \tilde{K}^{-4}(P_{8m+8}) & \xleftarrow{i_1^*} & \tilde{K}^{-4}(P_{8m+9}) \\
\downarrow r & & \downarrow r & & \downarrow r \\
\widetilde{KO}^{-4}(P_{8m+8,3}) & \xhookrightarrow{q^*} & \widetilde{KO}^{-4}(P_{8m+8}) & \xleftarrow{i_1^*} & \widetilde{KO}^{-4}(P_{8m+9})
\end{array}$$

Since $\widetilde{KO}^{-4}(P_{8m+9})=Z\{z_2, z_2 z_0, \dots, z_2 z_0^{4m+3}\}$ is torsion free (see [4]), by the aid of the complexification homomorphism we can describe r in the right hand side explicitly. In particular we have

$$\begin{aligned}
r(g_C^2 z^{8m+5}) &= (1/3)(8m+5)(8m^2+10m+3)z_2 z_0^{4m+3} + (8m+5)z_2 z_0^{4m+2}, \\
r(g_C^2 z^{8m+6}) &= (4m+3)^2 z_2 z_0^{4m+3} + 2z_2 z_0^{4m+2}, \\
r(g_C^2 z^{8m+7}) &= (8m+7)z_2 z_0^{4m+3}.
\end{aligned}$$

Hence r in the left hand side satisfies (2.21). Then (2.22) follows from (2.21). By (2.22) and (2.23)

$$\begin{aligned}
u^* z_2 z_0^{4m+2} &= r(g_C^2 u^*(z^{8m+5} - (4m+2)z^{8m+6} + z^{8m+7})) \\
&= v_m r(g_C^2 \beta)
\end{aligned}$$

where $v_m = ((1/3)(8m+5)(3m+2) - (4m+2)(4m+3) + 1)u_m$.

Now

$$\begin{aligned}
 i_1^* z_2 z_0^{4m+2} &= i_1^* q_3^* z_2 z_0^{4m+2} \\
 &= \bar{q}_3^* i_1^* z_2 z_0^{4m+2} \\
 &= \bar{q}_3^* \hat{q}_2^* u^* z_2 z_0^{4m+2} \\
 &= v_m \bar{q}_3^* \hat{q}_2^* r(g_C^2 \beta) \\
 &= v_m \bar{l}^* b'
 \end{aligned}$$

where the third equality follows from (2.20). Therefore (A) follows since $v_m \equiv e_{2m} \pmod{2}$.

The proof of (B): It suffices to show that the second short exact sequence from the bottom on the diagram (2.18) splits. Naturally we have a commutative diagram in which the horizontal sequences are exact:

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+4,2}) & \longleftarrow & \widetilde{KO}^{-4}(C(\hat{p})) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0 \\
 & & \uparrow q_1^* & & \uparrow & & \uparrow = \\
 0 & \longleftarrow & \widetilde{KO}^{-4}(P_{8m+4,1}) & \longleftarrow & \widetilde{KO}^{-4}(C(q_1 \circ \hat{p})) & \longleftarrow & \widetilde{KO}^{-4}(S^{16m+14}) \longleftarrow 0
 \end{array}$$

It is easily seen that q_1^* is a monomorphism. By Propositions 3.3 and 7.1 of Adams [1] we have a homomorphism

$$e: G_7 = \pi_{16m+13}(S^{16m+6}) \rightarrow \text{Ext}^1(\widetilde{KO}^{-4}(S^{16m+6}), \widetilde{KO}^{-4}(S^{16m+14})) = Z_2.$$

Since $q_1 \circ \hat{p} = la'_n \sigma$, and $a'_n \equiv 0 \pmod{2}$ by Lemma 2, it follows that $q_1 \circ \hat{p}$ is divisible by 2, and $e(q_1 \circ \hat{p}) = 0$. This implies that the above lower sequence splits (see [1]), and also the upper one does. Then (B) follows and the proof of Lemma 7 is completed.

Now we proceed the computation of $\pi_{2n+9}^s(P_{n+6,6})$ for $n \equiv 2 \pmod{8}$. By (2.15), (2.16) and Lemma 7

$$p_{n+5,5} = i_1 \circ \tilde{p} = a'_n i_3 s_* \sigma + c_n i_4 s_* \eta \varepsilon + \text{odd torsion}.$$

Then we obtain the following table by Lemma 2

$n \pmod{8}$	$\nu_2(\# p_{n+5,5})$	a'_n
2(16)	3	2(4)
10(32)	2	4(8)
26(64)	1	8(16)
58(64)	0	0(16)

Put

$$e'_n = \begin{cases} 2 & \text{if } n \equiv 2 \pmod{16} \\ 2^2 & \text{if } n \equiv 10 \pmod{32} \\ 2^3 & \text{if } n \equiv 26 \pmod{64} \\ 2^4 & \text{if } n \equiv 58 \pmod{64} \end{cases}.$$

Then from (2.14) and $(S)_6$ for $i=10$ it follows

$$(2.24) \quad \pi_{2n+9}^s(P_{n+6,6}) = Z_{e'_n} \oplus Z_2 \{i_{5*}\mu\} \oplus Z_4 \{i_{2*}[[2\nu]]\} \quad \text{if } n \equiv 2 \pmod{8}$$

where $Z_{e'_n}$ is generated by $i_{4*}s_*\sigma$ if $n \equiv 10 \pmod{16}$, or $i_{4*}s_*\sigma + i_{2*}[[2\nu]]$ if $n \equiv 2 \pmod{16}$.

(2.17) and (2.24) give the proof of Theorem.

Added in proof. Professor Y. Furukawa has pointed out to the author that in [5], [17], [18] and [19] the stable homotopy groups $\pi_{2n+i}(W_{n+k,k})$ have been calculated for $k \leq 4$ and $i \leq 36$, and K. Oguchi [19] partly treated them for $k=5$.

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