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THE FIXED POINT SET OF C ACTIONS ON A COMPACT COMPLEX SPACE

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Introduction

Let X be a connected complete algebraic variety defined over the complex number field C. Suppose that a connected unipotent linear algebraic group G acts regulary on X. Let X^G be the set of fixed points of this action. Then in [3] Horrocks has shown that X^G is connected and that the inclusion $X^G \hookrightarrow X$ induces an isomorphism of the algebraic fundamental groups, i.e., the profinite completions of the fundamental groups, of both spaces. In the complex analytic category, by Carrell and Sommese this result was partially generalized in [1], where they have shown that if X is a general connected complex manifold and G is a unipotent algebraic group as above which acts biholomorphically and meromorphically on X (cf. 1.2 for the definition), then the set of fixed points X^G is connected.

In this note we push this analogy a little further to one on the compartison of the fundamental groups. Namely, we show that when X is a compact complex space, the inclusion $X^G \hookrightarrow X$ induces and isomorphism of the (topological) fundamental groups of these spaces (cf. Theorem 3.1).

1. Preliminaries

In this section we summarize some of the known results on the fundamental groups and the meromorphic actions of algebraic groups on a complex space.

1.1. First we consider the fundamental groups. Let X be a topological space. We denote by $\pi_0(X)$ (resp. $\pi_1(X)$) the set of connected components (resp. the fundamental group with respect to some reference point) of X. (We shall be sloppy about the base points of fundamental groups throughout the paper.) We shall use the following terminology:

DEFINITION 1.1. Let $f: X \to Y$ be a continuous map of X into another topological space Y. Then we say that

1) f is *0-connected* if the induced map $f_{0*}: \pi_0(X) \to \pi_0(Y)$ is bijective, and

2) f is $\frac{1}{2}$-*conneceted* (resp. 1-*connected*) if it is 0-connected and the induced homomorphism of the fundamental groups of the corresponding connected components is surjective (resp. isomorphic).

In general we denote by $\mathscr{UC}(X)$ the category of all the unramified coverings of X and continuous maps over X. Then a continuous map f as above defines a contravariant functor f^*: $\mathscr{UC}(Y) \to \mathscr{UC}(X)$ by taking the fibered products. Let $\tilde{Y} \to Y$ be the universal covering of Y and $\tilde{X} := f^*(\tilde{Y}) = X \times_Y \tilde{Y}$ its pull-back to X. The follwing is then standard.

Lemma 1.1. 1) *The following conditions are equivalent*:

a) f *is* $\frac{1}{2}$-*connected*,

b) *the natural map* $\tilde{X} \to \tilde{Y}$ *is* 0-*connected, and*

c) *the functor* $f^*: \mathscr{UC}(Y) \to \mathscr{UC}(X)$ *is fully faithful.*

2) *The following two conditions are also equivalent*:

a) f *is* 1-*connected, and*

b) *the functor* $f^*: \mathscr{UC}(Y) \to \mathscr{UC}(X)$ *is an equivalence of categories.* □

Using this criterion, as in [3, Prop. 1.2] we can easily prove the following:

Lemma 1.2. *Let* $f: X \to Y$ *be as above. Suppose that* X *and* Y *are unions of two subspaces* X_1 *and* X_2 *and* Y_1 *and* Y_2 *respectively such that* $f(X_i) \subseteq Y_i, i=1,2$. *Suppose further that the induced maps* $f_i: X_i \to Y_i$ *are* 1-*connected and the induced map* $f_{12}: X_1 \cap X_2 \to Y_1 \cap Y_2$ *is* $\frac{1}{2}$-*connected, then* f *is* 1-*conneced.* □

Now we pass to the category of complex spaces. (In this note all the complex spaces are assumed to have countable topology.) The following is more or less well-known and can be proved by a dirct topological method. Here, we give an analytic proof using Lemma 1.1 above.

Lemma 1.3. *Let* X *and* Y *be complex spaces and* $f: X \to Y$ *a proper morphism of complex spaces.*

1) *Suppose that each fiber of* f *is connected. Then* f *is* $\frac{1}{2}$-*connected.*

2) *If, further, each fiber of* f *is simply connected, then* f *is* 1-*connected.*

Proof. 1) We may assume that Y is connected. Let $r_Y: \tilde{Y} \to Y$ be the universal covering of Y. Then $\tilde{X} = f^*\tilde{Y}$ is a fiber space over $\tilde{Y}$ with connected fibers. Hence, $\tilde{X}$ also is connected. The assertion then follows from Lemma 1.1.

2) Let $r_X: \tilde{X} \to X$ be any unramified covering of X and $\tilde{f} = f r_X$ the induced map. Again by Lemma 1.1 it suffices to show that r_X descends to an unramified covering $\tilde{Y} \to Y$ so that $f^*(\tilde{Y}) = \tilde{X}$. For this purpose we put an equivalence relation $\sim$ on the space $\tilde{X}$ as follows; for any two points x and x' of $\tilde{X}$, $x \sim x'$ if

and only if they both belong to one and the same connected component of some fiber of $\tilde{f}$. Let $\tilde{Y}$ be the quotient space of $\tilde{X}$ (as a topological space) with respect to this equivalence relation. We then claim that $\tilde{Y}$ gives an unramified covering of Y to which $\tilde{X}$ descends. Indeed, by construction we have a natural continuous map $r_Y: \tilde{Y} \to Y$. Since the map f is proper and the fibers are simply connected, for any point y of Y there exists a neighborhood V of y such that $\tilde{X}$ is trivial over $f^{-1}(V)$, i.e., isomorphic to the product $f^{-1}(V) \times \pi_1(X)$ over $f^{-1}(V)$. Hence, over each V, r_Y is homemorphic to the projection $V \times \pi_1(X) \to V$; this implies that r_Y is a desired unramified covering. □

1.2. We next recall some definitions and results on the actions of algebraic groups on complex spaces. Let G be a (complex) linear algebraic group acting biholoomorphically on a complex space X. Let G^* be any algebraic compactification of G. We say that the action of G on X is *meromorphic* if the morphism of complex spaces $\sigma: G \times X \to X$ defining the action extends to a meromorphic map $\sigma^*: G^* \times X \to X$. (The condition is independent of the compactification G^*.) We denote by $\boldsymbol{C}$ the additive group of complex numbers considered as a 1-dimensional complex algebraic group. In this case we can take $G^* = \boldsymbol{P}$, the complex projective line.

We shall use the following lemma which is essentially proved in [2, §4] (cf. also [1]).

Lemma 1.4. *Let G be a connected algebraic group acting biholomorphically and meromorphically on a compact irreducible complex space X. Then there exists a diagram of morphisms of irreducible compact complex spaces*

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ f \downarrow & & \\ T & & \end{array}$$

with the following properties:

1) *G acts on each space biholomrphiically and meromorphically, making the morphisms G-equivariant, where the action on X is the given one and that on T is the trivial one.*

2) *There exists a Zariski open subset V of T such that for any $v \in V$ the fiber $Z_v := f^{-1}(v)$ of f is mapped isomorphically by π onto a closed subspace of X which is the closure of a G-orbit in X.*

3) *π is bimeromorphic.*

Proof. The proof is essentially the same as in [2, Th. 4.1]. So we shall recall only briefly the construction of the diagram. Let $\sigma^*: G^* \times X \to X$ define the meromorphic action of G on X with respect to the natural open embedding $G \hookrightarrow G^*$. Take the graph $\Gamma \subseteq G^* \times X \times X$ of σ^*, and Γ' its projection in $X \times X$. Regard $\Gamma' \subseteq X \times X$ as defining a family of complex subspaces of X parametrized by X with respect to the first projection. Then we get the universal meromorphic map $\tau : X \to D_X$ of X into the Douady space D_X of compact subspaces of X. Then T is by definition the minimal analytic subspace of D_X containing the (meromorphic) image of τ, and the diagram is obtained by restricting the universal family over D_X to T.

Roughly speaking, for a general point x of X the corresponding subspace Γ'_x of X is the closure $\overline{Gx}$ of the orbit Gx of x, and τ is defined at such a point x with the point $\tau(x) \in D_X$ corresponding to the subspace $\overline{Gx}$ itself. It follows that with respect to the natural action of G on D_X, $\tau(x)$ belongs to the fixed point set D_X^G. Therefore, we conclude that T also is contained in D_X^G. Finally, since π is bijective on the union of $Gx \subseteq Z_{\tau(x)}$ for all the general points x as above, we can deduce that π is bimeromorphic. □

2. Simple connectivity of fibers

2.1. First we give a general lemma relating the fundamental group of a special fiber to that of a general fiber in a certain fiber space.

Let B be a polycylinder of sufficiently small radius in $C^n = C^n(z_1, \cdots, z_n)$ for some $n > 0$. Define a hypersurface A in B by the equation $z_1 \cdots z_l = 0$ for some $1 \le l \le n$. We set $U = B - A$. Let Z be an irreducible complex space and $f : Z \to B$ a proper morphism of complex spaces with connected fibers. We assume that f is smooth over U. For any point $b \in B$ we write $Z_b = f^{-1}(b)$. We write $Y = Z_{0,red}$, where 0 is the origin of B, and $_{red}$ means taking the underlying reduced subspace. Fixing any point $b_o \in U$ we set $F = Z_{b_o}$.

In order to state the lemma exactly we introduce the notion of multiplicity of the map f along each irreducible component $A_i := \{z_i = 0\}$, $1 \le i \le l$, of A. Take any resolution $r : \hat{Z} \to Z$ of the singularities of Z according to Hironaka so that the exceptional set of r is a divisor with only normal crossings in $\hat{Z}$. We take a point $a^i = (a_1^i, \cdots, a_n^i)$ of $A_i - \cup_{j \ne i} A_j$ and consider the restriction $f_i : \hat{Z}_i \to D_i$ of fr over the 1-dimensional disc $D_i = \{z_j = a_j^i ; j \ne i\}$. If we take a^i sufficiently general, then $\hat{Z}_i$ is nonsingular and the fiber $Y_i := f_i^{-1}(a^i)$ over a^i of f_i is a divisor with only normal crossings in $\hat{Z}_i$. Let $Y_{i\mu}$, $1 \le \mu \le d_i$, be the irreducible components of Y_i and $m_{i\mu}$ the multiplicity of $Y_{i\mu}$ in $f_i^{-1}(a^i)$. Then we call the greatest common divisor m_i of $m_{i\mu}$ the *multiplicity* of f along A_i. (It is standard to check that m_i is independent of the choice of a resolution r and a point a^i as above (and even of the coordinates z_i).) Then our lemma is stated as follows.

Lemma 2.1. *Let the notations and assumptions be as above. Suppose that Z is locally irreducible. Then, possibly after restricting B around the origin, there* *exist a quotient group H of $\pi_1(F)$ and positive integers k_i which are divisors of m_i, $1 \leq i \leq l$, such that $\pi_1(Y)$ is isomorphic to a group which is an extension by H of the finite abelian group $\oplus_i \boldsymbol{Z}/k_i\boldsymbol{Z}$ so that we have an exact sequence (of non-commutative groups) of the form*

$$1 \to H \to \pi_1(Y) \xrightarrow{q} \oplus_i \boldsymbol{Z}/k_i\boldsymbol{Z} \to 1. \tag{1}$$

Corollary 2.2. *If $\pi_1(F)$ is finite, then so is $\pi_1(Y)$. Moreover, if F is simply connected, $\pi_1(Y)$ is a finite abelian group with at most l generators.*

REMARK 2.1. 1) If Z is nonsingular, we can actually take $k_i = m_i$. In general, k_i can be smaller than m_i.

2) The abelian unramified covering corresponding to the quotient map q is described as follows. Define a covering map $h: B' \to B$ by $h(z_1, \cdots, z_n) = (z_1^{k_1}, \cdots, z_l^{k_l}, z_{l+1}, \cdots, z_n)$ for an appropriate polycylinder B'. Then, if $f': Z' \to B$ is the normalized pullback of $Z \to B$ by h, the induced morphism $Z' \to Z$ turns out to be unramified and the induced unramified covering $Y' \to Y$ is the desired abelian covering, where $Y' = Z'_{0,red}$; $\pi_1(Y')$ is thus isomorphic to the group H of the above lemma.

Proof of Lemma 2.1. Restricting B if necessary, we may assume that the natural map $\pi_1(Y) \to \pi_1(Z)$ induced by the inclusion is isomorphic. Let $V = f^{-1}(U)$. Then the natural map $t: \pi_1(V) \to \pi_1(Z)$ is surjective. Indeed, let $w: \tilde{Z} \to Z$ be the universal covering map. Then, since Z is locally irreducible, $\tilde{Z}$ is irreducible and hence $w^{-1}(V)$ is connected. Then by Lemma 1.1, 1) applied to the inclusion $V \hookrightarrow Z$ it follows that t is surjective. On the other hand, since the restriction $f_U: V \to U$ of f to V is topologically a fiber bundle, we get an exact sequence of the form

$$1 \to L \to \pi_1(V) \xrightarrow{u} \pi_1(U) \to 1, \tag{2}$$

where L is the natural image of $\pi_1(F)$ in $\pi_1(V)$. We claim that the image by u of the kernel K of t in $\pi_1(U)$ ($\cong \oplus_i \boldsymbol{Z}$) contains the subgroup $\oplus_i m_i \boldsymbol{Z}$. (Note that the class γ_i defined by a closed path which turns once around the a^i counterclockwise in the punctured disc $D_i - a^i$ with D_i and a^i as above give canonical generators of $\pi_1(U)$.) Using the notations introduced before the lemma, take a 1-dimensional disc $D_{i\mu}$ in $\hat{Z}_i$ which intersects transversally with $Y_{i\mu}$ at a general point $y_{i\mu}$ and then consider the class in $\pi_1(V)$ defined by a closed path in $D_{i\mu}$ turning once around $y_{i\mu}$. Since the induced map $D_{i\mu} \to D_i$ is an $m_{i\mu}$-ple covering ramified at $y_{i\mu}$, the image by u of this class in $\pi_1(U)$ generate the subgroup $m_{i\mu}\boldsymbol{Z}$ of the i-th component of $\pi_1(U)$. This consideration for all i and μ gives the desired claim. Then by taking the quotient of the sequence (2) by K we get an exact sequence of teh form (1) with $H=L/L\cap K$.

When the general fiber F is a nonsingular rational curve, we can obtain a little more precise result.

Corollary 2.3. *In* Lemma 2.1 *suppose that F is a nonsingular rational curve. Then Y is simply connected.*

Proof. It is well-known that Y is a union of irreducible rational curves, say $Y_1,\cdots,Y_m$. Suppose that Y is not simply connected. Then either the dual graph associated to Y contains a cycle, or some irreducible component Y_i has a node as its singularity. In any case, the fundamental group $\pi_1(Y)$ of Y is necessarily infinite. This contradicts Corollary 2.2. (Note that we may also show directly that $m_i=1$ for all i.) □

2.2. We now apply the above corollary to a quotient map of a C-action. Namely, combining Corollary 2.3 and Lemma 1.4 we shall prove the following:

Lemma 2.4. *Let X be a compact irreducible complex space on which a 1-dimensional connected algebraic group G acts biholomorphically and meromorphically. Then there exists a diagram of morphisms of irreducible compact complex spaces*

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ {\scriptstyle f}\downarrow & & \\ T & & \end{array}$$

with Z normal and with the following properties:

1) *G acts on each space biholomorphically and meromorphically making the morphisms G-equivariant, where the action on X is the given one and that on T is the trivial one,*

2) *a general fiber of f is a nonsingular rational curve, and every fiber of f is a connected and simply connected 1-dimensional subspace of Z, and*

3) *π is bimeromorphic.*

Proof. Consider the diagram of complex spaces

$$\begin{array}{ccc} & Z_0 & \xrightarrow{\pi_0} X \\ f_0 & \downarrow & \\ & T_0 & \end{array}$$

associated to the given action as in Lemma 1.4. Let $n: Z_1 \to Z_0$ be the normalization of Z_0 and $f_1: Z_1 \to T_0$ the induced morphism. By the property 2) in Lemma 1.4 of the above diagram we see that the general fibers of f_1 are (connected) nonsingular rational curves. Let N be the maximal Zariski open subset of T such that f_1 is smooth over N. Let $S = T_0 - N$. Take by Hironaka a proper bimeromorphic morphism $g: T \to T_0$ such that T is nonsingular and $\hat{S} := g^{-1}(S)_{red}$ is a divisor with only normal crossings in T. Let Z be the normalization of the pull-back $Z_1 \times_{T_0} T$, and $f: Z \to T$ and $\pi: Z \to X$ the induced morphisms. We have the naturally induced G-action on Z which makes the morphisms f and π G-equivariant, where the action on T is the trivial one. Clearly every fiber of f is connected and 1-dimensional and π is bimeromorphic. So what remains to be shown is that every fiber of f is simply connected. The problem is local with respect to T. So take any point t of T. Then, if we restrict the morphism f over a polycylindrical neighborhood of t, the resulting morphism satisfies all the conditions of Corollary 2.3. Thus the desired simple connecitvity follows from that corollary. □

For the total space Z of the fiber space $f: Z \to T$ as in the above lemma we can now prove the 1-connectivity of the inclusion $Z^G \hookrightarrow Z$, to which the proof in the general case will be reduced by using the above lemma.

Lemma 2.5. *Let $f: Z \to T$ be a morphism of compact irreducible complex spaces such that its general fibers are nonsingular and every fiber is connected, simply connected and is of dimension one. Suppose that there exist biholomorphic and meromorphic actions of the additive group $G = C$ on Z and T making f G-eququvariant, where the action on T is the trivial one. Then the natural inclusion $\iota: Z^G \quad Z$ is 1-connected.*

Proof. We may assume that the action is not trivial. First by Lemma 1.3 and by our assumption we get that $f: Z \to T$ is 1-connected. So it suffices to show that the induced morphism $f_G: Z^G \to T$ also is 1-connected. Let U be a Zariski open subset of T over which f is smooth. Then $f_G^{-1}(U)$ is mapped isomorphiically onto U by f_G. Hence, there exists a unique irreducible component Z_0^G of Z^G which is mapped surjectively onto T. On the other hand, for any point $t \in T$ each fiber $(Z^G)_t = (Z_t)^G$ of f_G is a connected subspace of Z_t, Z_t being of dimension one. It follows that Z^G itself is connected. Moreover, as a connected subspace of the 1-dimensional simply connected space Z_t, $(Z^G)_t$ also is simply connected. Thus, again by Lemma

1.3 f_G is 1-connected, which completes the proof of the lemma. □

3. Theorem

We consider a connected linear algebraic group G which admits an increasing sequence of connected normal algebraic subgroups

$$G_0=\{e\} \subseteq \cdots \subseteq G_i \subseteq \cdots \subseteq G_m=G \tag{3}$$

such that all the quotient groups G_i/G_{i+1} are isomorphic to the additive group $\boldsymbol{C}$. Such a group is necessarily unipotent and vice versa. So we refer to such a group simply as a connected unipotent algebraic group in what follows. On the fixed point set of the action of such a group we prove the following result which is a generalization of a result of Horrocks in [3] (cf. Remark 3.1 below).

Theorem 3.1. *Let G be a connected unipotent linear algebraic group and X a compact complex space. Suppose that G acts biholomorphiically and meromorphically on X. Let X^G be the set of fixed points of this action. Then the natural inclusion $\iota: X^G \to X$ is 1-connected* (cf. Def. 1.1).

REMARK 3.1. When X is an analytic space underlying some complete algebraic variety (defined over $\boldsymbol{C}$), Horrocks [3] has proved that ι is 0-connected and induces an isomorpphism of the algebraic fundamental groups (i.e., the profinite completions of π_1) of the corresponding connected components of X^G and X. As a partial generalization of this, Carrell and Sommese [1] have shown that ι is 0-connected even if X is a general complex space which is not necessarily compact.

Proof of Theorem 3.1. Using the sequence (3) and the fact that $X^{G_i}=(X^{G_{i-1}})^{G_i/G_{i-1}}$ the proof is reduced easily to the case where G is the additive group $\boldsymbol{C}$ (cf. [3, proof of Theorem 6.2] and [1]). Then, we proceed by induction on $n:=\dim X$. First of all, the assertion is trivial when $n=0$. So suppose that $n>0$ and that the theorem is true for any meromorphic G-action on a complex space of smaller dimension.

The fact that ι is $\frac{1}{2}$-connected follows just as in the proof of Theorem 5.1 of [3]: Let $r:\tilde{X}\to X$ be the universal covering of X. Then the action of G on X lifts to a meromorphic action on $\tilde{X}$, G being simply connected. Since r is a discrete map and G is connected, we have $(\tilde{X})^G=r^{-1}(X^G)$. Then by the above mentioned result of [1], we get that $r^{-1}(X^G)$ is connected. Thus, ι is $\frac{1}{2}$-connected by Lemma 1.1.

On the other hand, suppose that X is a union of two G-invariant closed analytic subsets X_1 and X_2 such that $\dim X_1\cap X_2<n$. Then by induction hypothesis the inclusion $(X_1\cap X_2)^G=X_1^G\cap X_2^G \hookrightarrow X_1\cap X_2$ is 1-connected. Thus, if the theorem is true for the induced G-actions on X_i, by Lemma 1.2 the same is also true for our G-action on X. Using this inductively we may easily reduce the proof to the

case where X is irreducible. So suppose that X is irreducible. We then consider a diagram of G-equivariant morphisms of irreducible compact complex spaces

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ f\downarrow & & \\ T & & \end{array}$$

obtained in Lemma 2.4 associated to the given G-action on X. In view of the properties 1) and 2) in that lemma, we conclude from Lemma 2.5 that the natural inclusion $Z^G \hookrightarrow Z$ is 1-connected.

We shall now derive the 1-connectedness of the inclusion $\iota : X^G \hookrightarrow X$ from this. Let $r_G : \tilde{X}^G \to X^G$ be any unramified covering of X^G. Since we know already that ι is $\frac{1}{2}$-connected, we have only to show that there exists an unramfied covering $r : \tilde{X} \to X$ which induces r_G over X^G.

First, consider the maximal Zariski open subset U of X such that $\pi : \pi^{-1}(U) \to U$ is isomorphic. Set $S = X - U$ and $M = \pi^{-1}(S)$. Since π is G-equivariant, S and M are G-invariant. Moreover, since the dimensions of S and M are less than n, by the induction hypothesis the inclusions $S^G \hookrightarrow S$ and $M^G \hookrightarrow M$ are both 1-connected. Next, let $S' = S \cup X^G$ and $M' = M \cup Z^G$ so that we have $M' = \pi^{-1}(S')$. Then the inclusions $X^G = S'^G \hookrightarrow S'$ and $Z^G = M'^G \hookrightarrow M'$ are also 1-connected by Lemma 1.2.

Take now a small neighborhood W of S' in X (resp. Q of M' in Z) such that $\pi(Q) \subseteq W$ and that the inclusions $S' \hookrightarrow W$ and $M' \hookrightarrow Q$ are 1-connected. By what we have shown, it then follows that r_G extends to a (unique) unramified covering $r_W : \tilde{W} \to W$ of W and that the pull-back $r_Q : \tilde{Q} \to Q$ of r_W to Q by π extends to a unique unramified covering $r_Z : \tilde{Z} \to Z$ of Z. Write $U' = X - S'$ and $V' = Z - M'$. Since $V' \cong U'$, the restriction of r_Z to V' descends to an unramified covering $r_{U'} : \tilde{U} \to U'$, which is isomorphic over $\pi(Q) - S' (\subseteq W - S')$ to r_W by the definition of r_Z. Hence, we can patch together $r_{U'}$ and $r_W | \pi(Q)$ over $\pi(Q) - S'$ by some isomorphism and get a desired unramified covering which extends r_G to X. The proof of the theorem is now complete. $\square$

Suppose that X is a manifold and $G = C$. Then a shorter proof using only Lemma 2.1 and without using induction is possible. In fact, in this case we can show the next lemma, which is enough for the proof of the theorem as one sees easily from the above proof in the general case.

Lemma 3.2. *Let Z be a normal compact complex space on which G acts biholomorphically and meromorphically. Let $\pi : Z \to X$ be a G-equivariant bimeromorphic morphism. Suppose that the natural inclusion $\iota_Z : Z^G \to Z$ is* 1*-connected. Then $\iota_X : X^G \to X$ also is* 1*-connected.*

Proof. We may assume that X is connected. First we show that the homomorphism $\pi_*:\pi_1(Z)\to\pi_1(X)$ is isomorphic. In fact, take any resolution $r:\tilde{Z}\to Z$ and let $\tilde{\pi}:\tilde{Z}\to X$ be the induced bimeromorphic morphism. Since both $\tilde{Z}$ and X are nonsingular, $\tilde{\pi}$ induces an isomorphism $\pi_1(\tilde{Z})\cong\pi_1(X)$ as is well-known. On the other hand, since Z is normal, the fibers of r is connected, and hence the induced homomorphism $\pi_1(\tilde{Z})\to\pi_1(Z)$ is surjective (cf. Lemma 1.3). This implies that π_* is isomorphic as claimed.

On the other hand, since $Z\to X$ is G-equivariant, the induced map $\pi_G: Z^G\to X^G$ is surjective with connected fibers. In fact, each fiber $\pi^{-1}(x)$ of a point $x\in X^G$ is connected and G-invariant, and hence by the $\frac{1}{2}$-connected theorem (which is proved independently of the induction) $Z^G\cap\pi^{-1}(x)$ is nonempty and connected. Hence, again by Lemma 1.3,1), we get that $\pi_{G_*}:\pi_1(Z^G)\to\pi_1(X^G)$ is surjective. The lemma then follows from the following commutative diagram of homomorphisms of fundamental groups

$$\begin{array}{ccc} \pi_1(Z^G) & \xrightarrow{\pi_{G*}} & \pi_1(X^G) \\ {\scriptstyle \iota_{Z_*}}\downarrow & & \downarrow{\scriptstyle \iota_{X_*}} \\ \pi_1(Z) & \xrightarrow[\pi_*]{} & \pi_1(X). \end{array}$$

□

Once Theorem 3.1 is obtained, by the same argument as in [3, §7], where the case of algebraic fundamental group was treated, we get the following:

Theorem 3.3. *Let P^n be the complex projective space of dimension n. For nonnegative integers d and q with $0\le q\le n$ denote by $C(n,d)$ the Chow variety of P^n of q-cycles of degree d. Then $C(n,d)$ is connected and simply connected.* □

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