Shibuya, T.
Osaka J. Math.
26 (1989), 483–490

THE ARF INVARIANT OF PROPER LINKS IN SOLID TORI

Dedicated to Professor Junzo Tao on his 60 th birthday

TETSUO SHIBUYA

(Received July 6, 1988)

Let $L=K_1\cup\cdots\cup K_n$ be a tame oriented link with n components in a 3-space R^3. L is said to be *proper* if the linking number of a knot K_i and $L-K_i$, denoted by $Link(K_i, L-K_i)\ (=\sum_{1\leq j\leq n, j\neq i} Link(K_i, K_j))$, is even for $i=1, \cdots, n$. The *total linking number* of L, denoted by $Link(L)$, means $\sum_{1\leq i<j\leq n} Link(K_i, K_j)$.

For two links L_1, L_2 in $R^3[a]$, $R^3[b]$ respectively for $a<b$, L_1 is said to be *related to* L_2 (or L_1 and L_2 are said to be *related*) if there is a locally flat proper surface F of genus zero in $R^3[a, b]$ with $F\cap R^3[a]=L_1$ and $F\cap R^3[b]=-L_2$, where $-L_2$ means the reflective inverse of L_2.

The *Arf invariant* of a proper link L, denoted by $\varphi(L)$, is defined to that of a knot related to L which is well-defined by Theorem 2 in [4].

Let V^*, V be solid tori with longitudes λ^*, λ respectively and μ a meridian of ∂V in R^3, where λ^* is a trivial knot, and f_m an orientation preserving onto homeomorphism of V^* onto V such that $f_m(\lambda^*)=\lambda+m\mu$ for an integer m. Especially f_0 is said to be *faithful*. For a link ℓ^* in V^*, $f_m(\ell^*)$ is called a link *T-congruent to* $\ell=f_0(\ell^*)$ (in V) and denoted by $\ell(m)$. The *winding number* of ℓ in V means the (algebraic) intersection number of ℓ and a meridian disk of V and is denoted by $w_V(\ell)$ or simply by $w(\ell)$.

Theorem 1. *Let $\ell, \ell(m)$ and $p=w(\ell)$ be those of the above. Suppose that p is odd or both p and m are even. Then ℓ is proper if and only if $\ell(m)$ is proper. Let ℓ be a proper link.*

(1) *Assume that p is odd. Then*

$$\begin{aligned}\varphi(\ell(m)) &= \varphi(\ell) && \text{if } m \text{ is even, or } m \text{ is odd and } p=8r\pm 1\\ &\equiv \varphi(\ell)+1 \pmod 2 && \text{if } m \text{ is odd and } p=8r\pm 3.\end{aligned}$$

(2) *Assume that p and m are even. Then*

$$\begin{aligned}\varphi(\ell(m)) &= \varphi(\ell) && \text{if } p=4r\\ &\equiv \varphi(\ell)+1 \pmod 2 && \text{if } p=4r+2,\end{aligned}$$

for an integer r.

If p is even and m is odd in Theorem 1, $\ell(m)$ is not always proper even though ℓ is proper.

Let $V_1^*, \cdots, V_n^*$ be mutually disjoint solid tori in R^3 with cores $c_1^*, \cdots, c_n^*$ respectively such that $\Gamma^*=c_1^* \cup \cdots \cup c_n^*$ is a trivial link. An orientation preserving homeomorphism f of $\mathcal{V}^*=V_1^* \cup \cdots \cup V_n^*$ onto $\mathcal{V}=V_1 \cup \cdots \cup V_n$ is said to be *faithful* if $f|_{V_i^*}: V_i^* \to V_i$ is faithful for $i=1, \cdots, n$. For a link $\ell^*=\ell_1^* \cup \cdots \cup \ell_n^*$ in $\mathcal{V}^*$, we write $f(\ell^*)$ (or $f(\ell_i^*)$) by ℓ(or ℓ_i), where ℓ_i^* is a link in V_i^*.

Theorem 2. *Let ℓ^*, $\ell=\ell_1 \cup \cdots \cup \ell_n$ and $\Gamma=f(\Gamma^*)$ be those of the above. Suppose that $w(\ell_i)\equiv w(\ell_j)$ $(=p)$ (mod 4) for $i, j=1, \cdots, n$ and $q=Link(\Gamma)$. If ℓ^* and Γ are proper, then ℓ is also proper and*

(1) $\varphi(\ell)\equiv\varphi(\ell^*)+\varphi(\Gamma)$ (*mod* 2) *if p is odd*

(2) $\varphi(\ell)=\varphi(\ell^*)$ *if p and q are even, or q is odd and $p=4m$*

$\equiv\varphi(\ell^*)+1$ (*mod* 2) *if q is odd and $p=4m+2$*

for some integer m.

Corollary 1. *Let ℓ^*, ℓ, Γ, p and q be those of Theorem 2. If q is even, then*

$$\begin{aligned}\varphi(\ell)&\equiv\varphi(\ell^*)+\varphi(\Gamma) \ (\textit{mod } 2) && \textit{if } p \textit{ is odd}\\ &=\varphi(\ell^*) && \textit{if } p \textit{ is even.}\end{aligned}$$

If $n=1$ in Theorem 2, namely Γ is a knot, we define that $Link(\Gamma)=0$. Hence we obtain the following.

Corollary 2. *If Γ is a knot,*

$$\begin{aligned}\varphi(\ell)&\equiv\varphi(\ell^*)+\varphi(\Gamma) \ (\textit{mod } 2) && \textit{if } p \textit{ is odd}\\ &=\varphi(\ell^*) && \textit{if } p \textit{ is even.}\end{aligned}$$

Theorem 3. *Let $\ell^*, \ell=\ell_1 \cup \cdots \cup \ell_n$ be those of the above. Suppose that $w(\ell_i)\equiv w(\ell_j)$ $(=p)$ (mod 2) and $Link(c_i, \Gamma-c_i)\equiv 0$ (mod 4) for $i=1, \cdots, n$. If ℓ^* is proper, then ℓ is proper and*

$$\begin{aligned}\varphi(\ell)&\equiv\varphi(\ell^*)+\varphi(\Gamma) \ (\textit{mod } 2) && \textit{if } p \textit{ is odd}\\ &=\varphi(\ell^*) && \textit{if } p \textit{ is even.}\end{aligned}$$

Theorem 4. *Let $\ell=\ell_1 \cup \cdots \cup \ell_n$ and Γ be those of the above. If Γ is a boundary link and ℓ_i is proper for $i=1, \cdots, n$, then $\varphi(\ell)\equiv \sum_{i=1}^{n} \varphi(\ell_i)$ (mod 2).*

The author thanks to Doctor H. Murakami for his helpful advice.

Proof of Theorems.

Lemma 1 is easily obtained by Theorem 2 in [4].

Lemma 1. *If two proper links L_1 and L_2 are related, then $\varphi(L_1)=\varphi(L_2)$.*

For a knot K, $\bar{K}$ means the knot orientation reversed to K. For a 2-component link $L_0=K_1\cup K_2$, let $L_0'=\bar{K}_1\cup K_2$ and $s=Link(K_1, K_2)$.

Lemma 2 ([2]). *$V_{L_0'}(t)=t^{-3s}\,V_{L_0}(t)$ for Jones polynomials of L_0, L_0'.*

For a link L, a relation between Jones polynomial and the Arf invariant of L is known by [3].

Lemma 3 ([3]). *For a n-component link L,*

$$V_L(\sqrt{-1})=\begin{cases}(\sqrt{2})^{n-1}\times(-1)^{\varphi(L)} & \text{if } L \text{ is proper}\\ 0 & \text{if } L \text{ is non-proper.}\end{cases}$$

By using the above Lemmas, we prove Lemma 4 which is effective to prove Theorems 1, 2 and 3.

Let $L=L_1\cup L_2$ be a link, where L_1, L_2 consist of m_1, m_2 knots $K_1, \cdots, K_{m_1}$, $K_{m_1+1}, \cdots, K_{m_1+m_2}$ respectively. The linking number of L_1 and L_2, denoted by $Link(L_1, L_2)$, means $\sum_{i=1}^{m_1}\sum_{j=m_1+1}^{m_1+m_2} Link(K_i, K_j)$. For a link $L_1=K_1\cup\cdots\cup K_{m_1}$, we denote that $\bar{L}_1=\bar{K}_1\cup\cdots\cup\bar{K}_{m_1}$.

Lemma 4. *Let $L=L_1\cup L_2$ be a proper link and $L'=\bar{L}_1\cup L_2$. Then L' is also proper and $Link(L_1, L_2)$ is even. Moreover*

$$\begin{aligned}\varphi(L')&=\varphi(L) && \text{if } Link(L_1, L_2)\equiv 0 \ (mod\ 4)\\ &\equiv\varphi(L)+1 \ (mod\ 2) && \text{if } Link(L_1, L_2)\equiv 2 \ (mod\ 4).\end{aligned}$$

Proof. Let $L_1=K_1\cup\cdots\cup K_{m_1}$ and $L_2=K_{m_1+1}\cup\cdots\cup K_{m_1+m_2}$. As L is proper, $Link(K_h, L-K_h)=2r_h$ for $K_h\subset L$ and some integer r_h. Then we see that $Link(\bar{K}_i, L'-\bar{K}_i)=2\,(r_i-Link(K_i, L_2))$ for $K_i\subset L_1$ and $Link(K_j, L'-K_j)=2\,(r_j-Link(K_j, L_1))$ for $K_j\subset L_2$ and that $Link(L_1, L_2)=2\,(r_1+\cdots+r_{m_1}-Link(L_1))$. Hence L' is also proper and $Link(L_1, L_2)$ is even.

Let $L_0=\kappa_1\cup\kappa_2$ be a 2-component link related to L such that κ_1, κ_2 are obtained by fusion (band sum) of L_1, L_2 respectively and let $L_0'=\bar{\kappa}_1\cup\kappa_2$ which is related to L'. As $Link(\kappa_1, \kappa_2)=Link(L_1, L_2)$ $(=s)$ is even, L_0 and L_0' are proper. So by Lemma 1, $\varphi(L_0)=\varphi(L)$ and $\varphi(L_0')=\varphi(L')$. As $Link(L_1, L_2)=Link(\kappa_1, \kappa_2)=s$, $V_{L_0'}(t)=t^{-3s}\,V_{L_0}(t)$ by Lemma 2 and hence $V_{L_0'}(\sqrt{-1})=(\sqrt{-1})^{-3s}\,V_{L_0}(\sqrt{-1})$. Therefore if $s\equiv 0\ (mod\ 4)$, then $\varphi(L')=\varphi(L_0')=\varphi(L_0)=\varphi(L)$ and if $s\equiv 2\ (mod\ 4)$, then $\varphi(L')=\varphi(L_0')\equiv\varphi(L_0)+1\equiv\varphi(L)+1\ (mod\ 2)$.

For a link L in a solid torus V, the minimum of intersection of L and a meridian disk in V is called the *order* of L (in V) and denoted by $o_V(L)$ or simply by $o(L)$.

To prove Theorem 1, we prepare Lemma 5.

Lemma 5. *Let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3, \mathcal{L}_4$ be torus links of type* $(8m\pm1, 8m\pm1)$, $(8m\pm3, 8m\pm3)$ *and* $(4m, 8m)$, $(4m+2, 8m+4)$ *for some integer m respectively.*

Then $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, $\mathcal{L}_4$ are proper. Furthermore if we orient $\mathcal{L}_i$ so that $o(\mathcal{L}_i)=w(\mathcal{L}_i)$ for each i, then $\varphi(\mathcal{L}_1)=\varphi(\mathcal{L}_3)=0$ and $\varphi(\mathcal{L}_2)=\varphi(\mathcal{L}_4)=1$.

Proof. It is easily seen that $\mathcal{L}_i$ is proper for $i=1, 2, 3, 4$.

Next suppose that $o(\mathcal{L}_i)=w(\mathcal{L}_i)$ for each i. $\mathcal{L}_1$ consists of $(8m\pm 1)$-component. Let $\mathcal{L}_{11}$, $\mathcal{L}_{12}$ be disjoint sublinks of $\mathcal{L}_1$ with $4m$, $(4m\pm 1)$-components respectively. Then $Link(\mathcal{L}_{11}, \mathcal{L}_{12})=4m\ (4m\pm 1)$. Hence $\varphi(\mathcal{L}_1)=\varphi(\bar{\mathcal{L}}_{11}\cup\mathcal{L}_{12})$ by Lemma 4. As $\bar{\mathcal{L}}_{11}\cup\mathcal{L}_{12}$ is related to a torus knot of type $(\pm 1, \pm 1)$, $\varphi(\mathcal{L}_1)=0$. By the same way as above, we see that $\varphi(\mathcal{L}_2)=1$, for the Arf invariant of torus link of type $(\pm 3, \pm 3)$ is 1.

$\mathcal{L}_3$ consists of $4m$-component and let $\mathcal{L}_{31}$, $\mathcal{L}_{32}$ be disjoint sublinks of $\mathcal{L}_3$ with $2m$, $2m$-components. Then $Link(\mathcal{L}_{31}, \mathcal{L}_{32})=8m^2$. Hence $\varphi(\mathcal{L}_3)=\varphi(\bar{\mathcal{L}}_{31}\cup\mathcal{L}_{32})$ by Lemma 4. As $\bar{\mathcal{L}}_{31}\cup\mathcal{L}_{32}$ is related to a trivial knot, $\varphi(\mathcal{L}_3)=0$. By the same way as above, we easily see that $\varphi(\mathcal{L}_4)=1$.

Proof of Theorem 1. We easily see that, when p is odd or both p and m are even, ℓ is proper if and only if $\ell(m)$ is proper.

Let n be $o(\ell)$. Then $\ell(m)$ is obtained by a fusion of ℓ and a torus link $\mathcal{L}_0$ of type (n, mn) split from ℓ in V and hence $\ell(m)$ is related to $\ell\circ\mathcal{L}_0$, where $\circ$ means that ℓ is split from $\mathcal{L}_0$. By the way, $\ell\circ\mathcal{L}_0$ is related to $\ell\circ\mathcal{L}$, where $\mathcal{L}$ is a torus link of type (p, mp) for $p=w(\mathcal{L})$. If ℓ and $\ell(m)$ are proper, $\mathcal{L}_0$, $\mathcal{L}$ are also proper and $\varphi(\ell(m))=\varphi(\ell\circ\mathcal{L}_0)=\varphi(\ell\circ\mathcal{L})$ by Lemma 1. Hence we obtain Theorem 1 by Lemma 5.

Let $\mathcal{V}=V_1\cup\cdots\cup V_n$ be the union of mutually disjoint solid tori in R^3 and Γ that of Theorem 2. For a core c_i, take a p_i-component link, denoted by $p_i\,c_i$, in V_i, each of which is parallel and homologous to c_i and non-twisted, namely $p_i\,c_i$ is contained on a non-twisted annulus A_i in V_i with $\partial A_i\supset c_i$, in V_i for $i=1, \cdots, n$. Especially if $p_i=p_j(=p)$, we denote $pc_1\cup\cdots\cup pc_n$ by $p\Gamma$.

In Lemma 6, we consider the case that $p=2$ which is used to prove Lemma 7.

Lemma 6. $\varphi(2\Gamma)=\begin{cases} 0 & \textit{if } q \textit{ is even} \\ 1 & \textit{if } q \textit{ is odd, where } q=Link(\Gamma). \end{cases}$

Proof. Let $2\Gamma=\Gamma\cup\Gamma'$. As $c_i\cup c_i'(\subset\Gamma\cup\Gamma')$ is non-twisted, $Link(\Gamma, \Gamma')=2q$. Hence if q is even, $\varphi(2\Gamma)=\varphi(\bar{\Gamma}\cup\Gamma')$ and if q is odd, $\varphi(2\Gamma)\equiv\varphi(\bar{\Gamma}\cup\Gamma')+1$ (*mod* 2) by Lemma 4. As $\bar{\Gamma}\cup\Gamma'$ is related to a trivial knot, we obtain Lemma 6 by Lemma 1.

Lemma 7. *If Γ is proper, $p\Gamma$ is also proper and*

(1) $\varphi(p\Gamma)=\varphi(\Gamma)$ *if p is odd*

(2) $\varphi(p\Gamma)=\begin{cases} 0 & \textit{if } p \textit{ and } q \textit{ are even, or } q \textit{ is odd and } p=4m \\ 1 & \textit{if } q \textit{ is odd and } p=4m+2 \end{cases}$

for some integer m and $q=Link(\Gamma)$. Hence if q is even,

$$\varphi(p\Gamma)=\begin{cases}\varphi(\Gamma) & \text{if } p \text{ is odd}\\ 0 & \text{if } p \text{ is even.}\end{cases}$$

Proof. As pc_i is non-twisted, we easily see that if Γ is proper, $p\Gamma$ is also proper.

Lemma 7 is clear if $p=0$. Hence we assume that $p>0$. Each pc_i consists of p components, say $c_{i1}, \cdots, c_{ip}$. Let $L_1=c_{11}\cup c_{21}\cup\cdots\cup c_{n1}$ and $L_2=p\Gamma-L_1$. Then we see that $Link(L_1, L_2)=2(p-1)q$.

If p is odd or q is even, $\varphi(p\Gamma)=\varphi(\bar{L}_1\cup L_2)$ by Lemma 4. As $\bar{L}_1\cup L_2$ is related to $(p-2)\Gamma$, $\varphi(p\Gamma)=\varphi((p-2)\Gamma)$ by Lemma 1. By doing this successively, if p is odd, $\varphi(p\Gamma)=\varphi(\Gamma)$ and if both p and q are even, $\varphi(p\Gamma)=\varphi(\mathcal{O})=0$ for a trivial knot $\mathcal{O}$.

Next we consider the case that q is odd and p is even. Then,

$$\varphi(p\Gamma)\equiv\varphi((p-2)\Gamma)+1\equiv\varphi((p-4)\Gamma) \quad (mod\ 2)$$

by Lemma 4. Hence if $p=4m$, $\varphi(p\Gamma)=\varphi(\mathcal{O})=0$ and if $p=4m+2$, $\varphi(p\Gamma)=\varphi(2\Gamma)=1$ by Lemma 6.

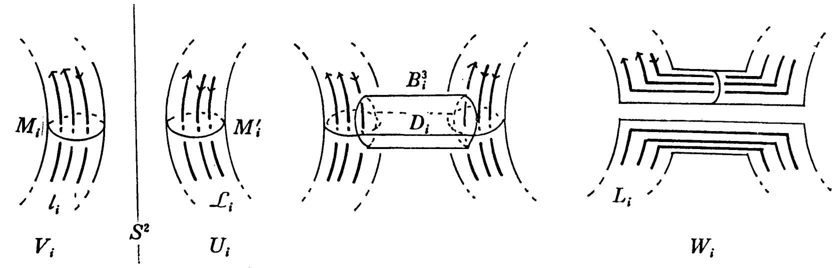
By the similar proof of Lemma 7, we obtain Lemma 8.

Lemma 8. *Let $p_i\equiv p_j$ (mod 4) and $p=Min\{p_1,\cdots,p_n\}$. Then $\varphi(p\Gamma)=\varphi(p_1c_1\cup\cdots\cup p_nc_n)$ for a proper link $\Gamma=c_1\cup\cdots\cup c_n$.*

Let $\mathcal{L}_i$ be a link in V_i with r_i components for some integer r_i such that $\mathcal{L}_i$ is non-twisted and parallel to c_i and $w(\mathcal{L}_i)=p_i(\leqq r_i)$ for $i=1,\cdots,n$. Then as $\mathcal{L}=\mathcal{L}_1\cup\cdots\cup\mathcal{L}_n$ is related to $p_1c_1\cup\cdots\cup p_nc_n$, we obtain Lemma 9.

Lemma 9. *If Γ is proper and $p_i\equiv p_j$ (mod 2), $\mathcal{L}$ is also proper and $\varphi(\mathcal{L})=\varphi(p_1c_1\cup\cdots\cup p_nc_n)$.*

Proof of Theorem 2. Let $\mathcal{U}=U_1\cup\cdots\cup U_n$ be the union of mutually disjoint solid tori in R^3 with core $-\Gamma=(-c_1)\cup\cdots\cup(-c_n)$, the reflective inverse of Γ, split from $\mathcal{V}$ by a 2-sphere S^2 and symmetric with respect to S^2. For $l=l_1\cup\cdots\cup l_n$ in $\mathcal{V}$, let $\tilde{\mathcal{L}}_i$ be a link with $r_i(=o(l_i))$ components in U_i such that $\tilde{\mathcal{L}}_i$ is non-twisted and parallel to $-c_i$ and $w(\tilde{\mathcal{L}}_i)=p_i(=w(l_i))$, $i=1,\cdots,n$. Attach a 3-ball B_i^3 to $V_i\cup U_i$ such that $V_i\cup U_i\cup B_i^3$ is symmetric with respect to S^2, Fig. 1(b) for each i. Let M_i, M_i' be meridian disks of V_i, U_i respectively such that $\#(l_i\cap M_i)=\#(\tilde{\mathcal{L}}_i\cap M_i')=p_i$ and $M_i\cap B_i=\partial M_i\cap\partial B_i(=\{\text{an arc } \alpha_i\})$, $M_i'\cap B_i=\partial M_i'\cap\partial B_i(=\{\text{an arc } \beta_i\})$, where $\#(X)$ means the number of points of X, see Fig. 1(a). Let D_i be a proper non-twisted disk in B_i with $\partial D_i\supset\alpha_i\cup\beta_i$ and $\Delta_i=M_i\cup M_i'\cup D_i$. For each i, perform the fusion of $l_i\circ\tilde{\mathcal{L}}_i$ along Δ_i and we obtain a link L_i which is contained in a solid torus $W_i=\overline{V_i\cup U_i\cup B_i-\Delta_i\times[-\varepsilon,\varepsilon]}$



(a) (b) (c)

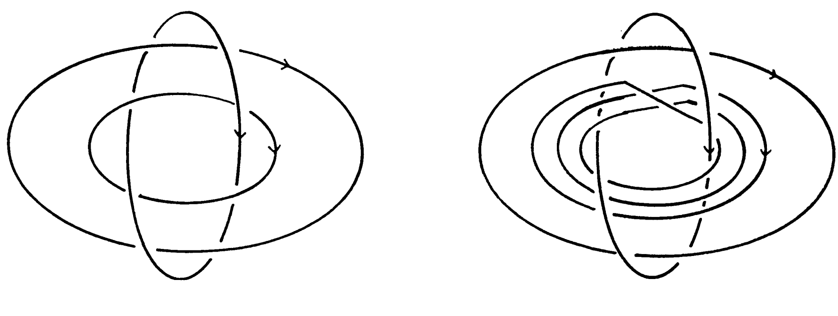
Fig. 1

for a small positive number ε, Fig. 1(c). Then $\mathcal{W}=W_1\cup\cdots\cup W_n$ is the union of disjoint solid tori which is symmetric with respect to S^2 by the construction. So the core of $\mathcal{W}$ is cobordant to zero by [1] and hence $L=L_1\cup\cdots\cup L_n$ is cobordant to $L^*=L_1^*\cup\cdots\cup L_n^*$ by [5], [6] for a faithful homeomorphism f_0 of $\mathcal{W}^*$ onto $\mathcal{W}$, where $L=f_0(L^*)$. As $\tilde{\mathcal{L}}_i$ is non-twisted, L^* is ambient isotopic to ℓ^*. As L is cobordant to ℓ^* and ℓ^* is proper, L is also proper. Moreover as Γ is proper, $\tilde{\mathcal{L}}=\tilde{\mathcal{L}}_1\cup\cdots\cup\tilde{\mathcal{L}}_n$ is proper. Hence we easily see that ℓ is also proper. As L and $\ell\circ\tilde{\mathcal{L}}$ are related,

$$\varphi(\ell)+\varphi(\tilde{\mathcal{L}})\equiv\varphi(L)=\varphi(L^*)=\varphi(\ell^*)\quad(mod\ 2)\,.$$

So we obtain Theorem 2 by Lemmas 7, 8 and 9.

REMARK 1. In Theorem 2, if we replace the condition "$p_i\equiv p_j$ (mod 4)" by "$p_i\equiv p_j$ $(mod\ 2)$", the conclusion is not true. For example, we consider the links Γ, ℓ illustrated in Fig. 2. Then $\varphi(\Gamma)=0$ and $\varphi(\ell)=1$, hence $\varphi(\ell)\not\equiv\varphi(\ell^*)+\varphi(\Gamma)$ $(mod\ 2)$.



Γ ℓ

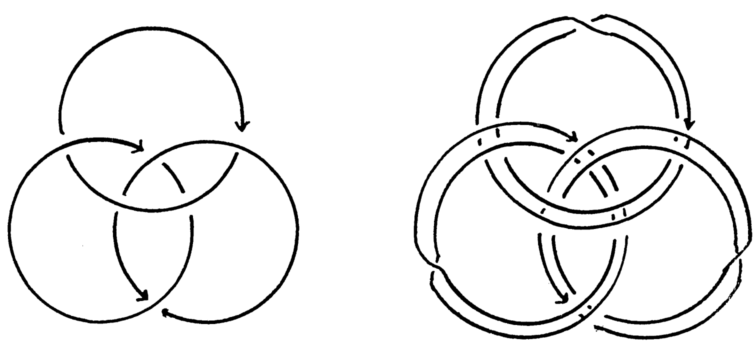
Fig. 2

Proof of Theorem 3. As $Link(c_i, \Gamma-c_i)\equiv 0$ $(mod\ 4)$, $Link(c_i, \Gamma-c_i)=4r_i$ for some integer r_i for each i. Then $2Link(\Gamma)=\sum_{i=1}^{n} Link(c_i, \Gamma-c_i)=4(r_1+\cdots+$

r_n). Hence $Link(\Gamma)$ is even. Therefore we obtain Theorem 3 by Lemma 7 and the proof of Theorem 2.

REMARK 2. The link in Fig. 2 is an example that the conclusion of Theorem 3 is not true if we replace that "$Link(c_i, \Gamma - c_i) \equiv 0$ (mod 4)" by "$Link(c_i, \Gamma - c_i) \equiv 0$ (mod 2)".

EXAMPLE 1. Let Γ, ℓ be links illustrated in Fig. 3. As $Link(\Gamma)=3$ and ℓ^* is a trivial link, $\varphi(\ell)=\varphi(\ell^*)+1=1$ by Theorem 2.



Γ ℓ

Fig. 3

EXAMPLE 2. Let ℓ be a link illustrated in Fig. 4. As Γ is the Whitehead link, $Link(\Gamma)=0$ and $\varphi(\Gamma)=1$ and ℓ^* is a trivial link. Hence $\varphi(\ell) \equiv \varphi(\ell^*)+\varphi(\Gamma) = 1$ by Theorem 3.

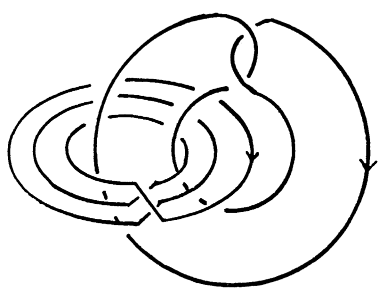


Fig. 4

Proof of Theorem 4. For a proper link ℓ_i in V_i, let k_i be a knot obtained by a fusion of ℓ_i in V_i for each i. As ℓ_i, ℓ are related to k_i, $\ell_0 = k_1 \cup \cdots \cup k_n$ respectively, $\varphi(\ell_i)=\varphi(k_i)$ and $\varphi(\ell)=\varphi(\ell_0)$ by Lemma 1. Furthermore as Γ is a boundary link, there are mutually disjoint surfaces $\mathcal{F}=F_1 \cup \cdots \cup F_n$ with $\partial\mathcal{F}=\ell_0$, $\partial F_i = k_i$. Then $\varphi(\ell_0)=\sum_{i=1}^{n}\varphi(k_i)$ (mod 2) by Theorem 3 in [4]. Hence we obtain that

$$\varphi(\ell) = \varphi(\ell_0) \equiv \sum_{i=1}^{n} \varphi(k_i) \equiv \sum_{i=1}^{n} \varphi(\ell_i) \quad (mod\ 2)\,.$$

References

[1] F. Hosokawa: *A concept of cobordism between links,* Ann. of Math. **86** (1967), 362–373.
[2] V.F.R. Jones: *A polynomial invariant for knots via von Neumann algebras,* Bull. Amer. Math. Soc. **12** (1985), 103–111.
[3] H. Murakami: *A recursive calculation of the Arf invariant of a link,* Math. Soc. Japan, **38** (1986), 335–338.
[4] R.A. Robertello: *An invariant of knot cobordism,* Comm. Pure Appl. Math., **18** (1965), 543–555.
[5] T. Shibuya: *On the cobordism of compound knots,* Math. Sem. Notes Kobe Univ., **8** (1980), 331–337.
[6] T. Shibuya: *A condition for links to be cobordant to zero,* Memoirs of Osaka Inst. Tech. **30** (1985), 11–16.

Department of Mathematics
Osaka Institute of Technology
Asahi, Osaka 535
Japan