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THE ARF INVARIANT OF PROPER LINKS IN SOLID TORI

Dedicated to Professor Junzo Tao on his 60 th birthday

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Let $L=K_1\cup\cdots\cup K_n$ be a tame oriented link with n components in a 3-space R^3. L is said to be *proper* if the linking number of a knot K_i and $L-K_i$, denoted by $Link(K_i, L-K_i)\ (=\sum_{1\leq j\leq n, j\neq i} Link(K_i, K_j))$, is even for $i=1, \cdots, n$. The *total linking number* of L, denoted by $Link(L)$, means $\sum_{1\leq i<j\leq n} Link(K_i, K_j)$.

For two links L_1, L_2 in $R^3[a]$, $R^3[b]$ respectively for $a<b$, L_1 is said to be *related to* L_2 (or L_1 and L_2 are said to be *related*) if there is a locally flat proper surface F of genus zero in $R^3[a, b]$ with $F\cap R^3[a]=L_1$ and $F\cap R^3[b]=-L_2$, where $-L_2$ means the reflective inverse of L_2.

The *Arf invariant* of a proper link L, denoted by $\varphi(L)$, is defined to that of a knot related to L which is well-defined by Theorem 2 in [4].

Let V^*, V be solid tori with longitudes λ^*, λ respectively and μ a meridian of ∂V in R^3, where λ^* is a trivial knot, and f_m an orientation preserving onto homeomorphism of V^* onto V such that $f_m(\lambda^*)=\lambda+m\mu$ for an integer m. Especially f_0 is said to be *faithful*. For a link l^* in V^*, $f_m(l^*)$ is called a link *T-congruent to* $l=f_0(l^*)$ (in V) and denoted by $l(m)$. The *winding number* of l in V means the (algebraic) intersection number of l and a meridian disk of V and is denoted by $w_V(l)$ or simply by $w(l)$.

Theorem 1. *Let $l, l(m)$ and $p=w(l)$ be those of the above. Suppose that p is odd or both p and m are even. Then l is proper if and only if $l(m)$ is proper. Let l be a proper link.*

(1) *Assume that p is odd. Then*

$$\begin{aligned}\varphi(l(m)) &= \varphi(l) && \text{if } m \text{ is even, or } m \text{ is odd and } p=8r\pm1\\ &\equiv \varphi(l)+1 \pmod 2 && \text{if } m \text{ is odd and } p=8r\pm3.\end{aligned}$$

(2) *Assume that p and m are even. Then*

$$\begin{aligned}\varphi(l(m)) &= \varphi(l) && \text{if } p=4r\\ &\equiv \varphi(l)+1 \pmod 2 && \text{if } p=4r+2,\end{aligned}$$

for an integer r.

If p is even and m is odd in Theorem 1, $\ell(m)$ is not always proper even though ℓ is proper.

Let $V_1^*, \cdots, V_n^*$ be mutually disjoint solid tori in R^3 with cores $c_1^*, \cdots, c_n^*$ respectively such that $\Gamma^*=c_1^* \cup \cdots \cup c_n^*$ is a trivial link. An orientation preserving homeomorphism f of $\mathcal{V}^*=V_1^* \cup \cdots \cup V_n^*$ onto $\mathcal{V}=V_1 \cup \cdots \cup V_n$ is said to be *faithful* if $f|_{V_i^*}: V_i^* \to V_i$ is faithful for $i=1, \cdots, n$. For a link $\ell^*=\ell_1^* \cup \cdots \cup \ell_n^*$ in $\mathcal{V}^*$, we write $f(\ell^*)$ (or $f(\ell_i^*)$) by ℓ(or ℓ_i), where ℓ_i^* is a link in V_i^*.

Theorem 2. *Let ℓ^*, $\ell=\ell_1 \cup \cdots \cup \ell_n$ and $\Gamma=f(\Gamma^*)$ be those of the above. Suppose that $w(\ell_i)\equiv w(\ell_j)$ $(=p)$ (mod 4) for $i,j=1, \cdots, n$ and $q=Link(\Gamma)$. If ℓ^* and Γ are proper, then ℓ is also proper and*

(1) $\varphi(\ell)\equiv\varphi(\ell^*)+\varphi(\Gamma)$ (mod 2) *if p is odd*

(2) $\varphi(\ell)=\varphi(\ell^*)$ *if p and q are even, or q is odd and $p=4m$*

$\equiv\varphi(\ell^*)+1$ (mod 2) *if q is odd and $p=4m+2$*

for some integer m.

Corollary 1. *Let ℓ^*, ℓ, Γ, p and q be those of Theorem 2. If q is even, then*

$$\begin{aligned}\varphi(\ell)&\equiv\varphi(\ell^*)+\varphi(\Gamma) \pmod 2 && \text{if } p \text{ is odd}\\ &=\varphi(\ell^*) && \text{if } p \text{ is even.}\end{aligned}$$

If $n=1$ in Theorem 2, namely Γ is a knot, we define that $Link(\Gamma)=0$. Hence we obtain the following.

Corollary 2. *If Γ is a knot,*

$$\begin{aligned}\varphi(\ell)&\equiv\varphi(\ell^*)+\varphi(\Gamma) \pmod 2 && \text{if } p \text{ is odd}\\ &=\varphi(\ell^*) && \text{if } p \text{ is even.}\end{aligned}$$

Theorem 3. *Let $\ell^*, \ell=\ell_1 \cup \cdots \cup \ell_n$ be those of the above. Suppose that $w(\ell_i)\equiv w(\ell_j)$ $(=p)$ (mod 2) and $Link(c_i, \Gamma-c_i)\equiv 0$ (mod 4) for $i=1, \cdots, n$. If ℓ^* is proper, then ℓ is proper and*

$$\begin{aligned}\varphi(\ell)&\equiv\varphi(\ell^*)+\varphi(\Gamma) \pmod 2 && \text{if } p \text{ is odd}\\ &=\varphi(\ell^*) && \text{if } p \text{ is even.}\end{aligned}$$

Theorem 4. *Let $\ell=\ell_1 \cup \cdots \cup \ell_n$ and Γ be those of the above. If Γ is a boundary link and ℓ_i is proper for $i=1, \cdots, n$, then $\varphi(\ell)\equiv \sum_{i=1}^{n} \varphi(\ell_i)$ (mod 2).*

The author thanks to Doctor H. Murakami for his helpful advice.

Proof of Theorems.

Lemma 1 is easily obtained by Theorem 2 in [4].

Lemma 1. *If two proper links L_1 and L_2 are related, then $\varphi(L_1)=\varphi(L_2)$.*

For a knot K, $\bar{K}$ means the knot orientation reversed to K. For a 2-component link $L_0=K_1\cup K_2$, let $L_0'=\bar{K}_1\cup K_2$ and $s=Link(K_1, K_2)$.

Lemma 2 ([2]). *$V_{L_0'}(t)=t^{-3s}V_{L_0}(t)$ for Jones polynomials of L_0, L_0'.*

For a link L, a relation between Jones polynomial and the Arf invariant of L is known by [3].

Lemma 3 ([3]). *For a n-component link L,*

$$V_L(\sqrt{-1})=\begin{cases}(\sqrt{2})^{n-1}\times(-1)^{\varphi(L)} & \text{if } L \text{ is proper}\\ 0 & \text{if } L \text{ is non-proper.}\end{cases}$$

By using the above Lemmas, we prove Lemma 4 which is effective to prove Theorems 1, 2 and 3.

Let $L=L_1\cup L_2$ be a link, where L_1, L_2 consist of m_1, m_2 knots $K_1, \cdots, K_{m_1}$, $K_{m_1+1}, \cdots, K_{m_1+m_2}$ respectively. The linking number of L_1 and L_2, denoted by $Link(L_1, L_2)$, means $\sum_{i=1}^{m_1}\sum_{j=m_1+1}^{m_1+m_2}Link(K_i, K_j)$. For a link $L_1=K_1\cup\cdots\cup K_{m_1}$, we denote that $\bar{L}_1=\bar{K}_1\cup\cdots\cup\bar{K}_{m_1}$.

Lemma 4. *Let $L=L_1\cup L_2$ be a proper link and $L'=\bar{L}_1\cup L_2$. Then L' is also proper and $Link(L_1, L_2)$ is even. Moreover*

$$\begin{aligned}\varphi(L')&=\varphi(L) && \text{if } Link(L_1, L_2)\equiv 0 \pmod 4\\ &\equiv\varphi(L)+1 \pmod 2 && \text{if } Link(L_1, L_2)\equiv 2 \pmod 4.\end{aligned}$$

Proof. Let $L_1=K_1\cup\cdots\cup K_{m_1}$ and $L_2=K_{m_1+1}\cup\cdots\cup K_{m_1+m_2}$. As L is proper, $Link(K_h, L-K_h)=2r_h$ for $K_h\subset L$ and some integer r_h. Then we see that $Link(\bar{K}_i, L'-\bar{K}_i)=2(r_i-Link(K_i, L_2))$ for $K_i\subset L_1$ and $Link(K_j, L'-K_j)=2(r_j-Link(K_j, L_1))$ for $K_j\subset L_2$ and that $Link(L_1, L_2)=2(r_1+\cdots+r_{m_1}-Link(L_1))$. Hence L' is also proper and $Link(L_1, L_2)$ is even.

Let $L_0=\kappa_1\cup\kappa_2$ be a 2-component link related to L such that κ_1, κ_2 are obtained by fusion (band sum) of L_1, L_2 respectively and let $L_0'=\bar{\kappa}_1\cup\kappa_2$ which is related to L'. As $Link(\kappa_1, \kappa_2)=Link(L_1, L_2)$ $(=s)$ is even, L_0 and L_0' are proper. So by Lemma 1, $\varphi(L_0)=\varphi(L)$ and $\varphi(L_0')=\varphi(L')$. As $Link(L_1, L_2)=Link(\kappa_1, \kappa_2)$ $=s$, $V_{L_0'}(t)=t^{-3s}V_{L_0}(t)$ by Lemma 2 and hence $V_{L_0'}(\sqrt{-1})=(\sqrt{-1})^{-3s}V_{L_0}(\sqrt{-1})$. Therefore if $s\equiv 0 \pmod 4$, then $\varphi(L')=\varphi(L_0')=\varphi(L_0)=\varphi(L)$ and if $s\equiv 2 \pmod 4$, then $\varphi(L')=\varphi(L_0')\equiv\varphi(L_0)+1\equiv\varphi(L)+1 \pmod 2$.

For a link L in a solid torus V, the minimum of intersection of L and a meridian disk in V is called the *order* of L (in V) and denoted by $o_V(L)$ or simply by $o(L)$.

To prove Theorem 1, we prepare Lemma 5.

Lemma 5. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, $\mathcal{L}_4$ be torus links of type* $(8m\pm1, 8m\pm1)$, $(8m\pm3, 8m\pm3)$ *and* $(4m, 8m)$, $(4m+2, 8m+4)$ *for some integer m respectively.*

Then $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, $\mathcal{L}_4$ are proper. Furthermore if we orient $\mathcal{L}_i$ so that $o(\mathcal{L}_i)=w(\mathcal{L}_i)$ for each i, then $\varphi(\mathcal{L}_1)=\varphi(\mathcal{L}_3)=0$ and $\varphi(\mathcal{L}_2)=\varphi(\mathcal{L}_4)=1$.

Proof. It is easily seen that $\mathcal{L}_i$ is proper for $i=1, 2, 3, 4$.

Next suppose that $o(\mathcal{L}_i)=w(\mathcal{L}_i)$ for each i. $\mathcal{L}_1$ consists of $(8m\pm1)$-component. Let $\mathcal{L}_{11}$, $\mathcal{L}_{12}$ be disjoint sublinks of $\mathcal{L}_1$ with $4m$, $(4m\pm1)$-components respectively. Then $Link(\mathcal{L}_{11}, \mathcal{L}_{12})=4m\ (4m\pm1)$. Hence $\varphi(\mathcal{L}_1)=\varphi(\bar{\mathcal{L}}_{11}\cup\mathcal{L}_{12})$ by Lemma 4. As $\bar{\mathcal{L}}_{11}\cup\mathcal{L}_{12}$ is related to a torus knot of type $(\pm1, \pm1)$, $\varphi(\mathcal{L}_1)=0$. By the same way as above, we see that $\varphi(\mathcal{L}_2)=1$, for the Arf invariant of torus link of type $(\pm3, \pm3)$ is 1.

$\mathcal{L}_3$ consists of $4m$-component and let $\mathcal{L}_{31}$, $\mathcal{L}_{32}$ be disjoint sublinks of $\mathcal{L}_3$ with $2m$, $2m$-components. Then $Link(\mathcal{L}_{31}, \mathcal{L}_{32})=8m^2$. Hence $\varphi(\mathcal{L}_3)=\varphi(\bar{\mathcal{L}}_{31}\cup\mathcal{L}_{32})$ by Lemma 4. As $\bar{\mathcal{L}}_{31}\cup\mathcal{L}_{32}$ is related to a trivial knot, $\varphi(\mathcal{L}_3)=0$. By the same way as above, we easily see that $\varphi(\mathcal{L}_4)=1$.

Proof of Theorem 1. We easily see that, when p is odd or both p and m are even, ℓ is proper if and only if $\ell(m)$ is proper.

Let n be $o(\ell)$. Then $\ell(m)$ is obtained by a fusion of ℓ and a torus link $\mathcal{L}_0$ of type (n, mn) split from ℓ in V and hence $\ell(m)$ is related to $\ell\circ\mathcal{L}_0$, where $\circ$ means that ℓ is split from $\mathcal{L}_0$. By the way, $\ell\circ\mathcal{L}_0$ is related to $\ell\circ\mathcal{L}$, where $\mathcal{L}$ is a torus link of type (p, mp) for $p=w(\mathcal{L})$. If ℓ and $\ell(m)$ are proper, $\mathcal{L}_0$, $\mathcal{L}$ are also proper and $\varphi(\ell(m))=\varphi(\ell\circ\mathcal{L}_0)=\varphi(\ell\circ\mathcal{L})$ by Lemma 1. Hence we obtain Theorem 1 by Lemma 5.

Let $\mathcal{V}=V_1\cup\cdots\cup V_n$ be the union of mutually disjoint solid tori in R^3 and Γ that of Theorem 2. For a core c_i, take a p_i-component link, denoted by $p_i\,c_i$, in V_i, each of which is parallel and homologous to c_i and non-twisted, namely $p_i\,c_i$ is contained on a non-twisted annulus A_i in V_i with $\partial A_i\supset c_i$, in V_i for $i=1, \cdots, n$. Especially if $p_i=p_j(=p)$, we denote $pc_1\cup\cdots\cup pc_n$ by $p\Gamma$.

In Lemma 6, we consider the case that $p=2$ which is used to prove Lemma 7.

Lemma 6. $\varphi(2\Gamma)=\begin{cases}0 & \textit{if } q \textit{ is even}\\ 1 & \textit{if } q \textit{ is odd, where } q=Link(\Gamma).\end{cases}$

Proof. Let $2\Gamma=\Gamma\cup\Gamma'$. As $c_i\cup c_i'(\subset\Gamma\cup\Gamma')$ is non-twisted, $Link(\Gamma, \Gamma')=2q$. Hence if q is even, $\varphi(2\Gamma)=\varphi(\bar{\Gamma}\cup\Gamma')$ and if q is odd, $\varphi(2\Gamma)\equiv\varphi(\bar{\Gamma}\cup\Gamma')+1$ (*mod* 2) by Lemma 4. As $\bar{\Gamma}\cup\Gamma'$ is related to a trivial knot, we obtain Lemma 6 by Lemma 1.

Lemma 7. *If Γ is proper, $p\Gamma$ is also proper and*

(1) $\varphi(p\Gamma)=\varphi(\Gamma)$ *if p is odd*

(2) $\varphi(p\Gamma)=\begin{cases}0 & \textit{if } p \textit{ and } q \textit{ are even, or } q \textit{ is odd and } p=4m\\ 1 & \textit{if } q \textit{ is odd and } p=4m+2\end{cases}$

for some integer m and $q=Link(\Gamma)$. Hence if q is even,

$$\varphi(p\Gamma)=\begin{cases}\varphi(\Gamma) & \text{if } p \text{ is odd}\\ 0 & \text{if } p \text{ is even.}\end{cases}$$

Proof. As pc_i is non-twisted, we easily see that if Γ is proper, $p\Gamma$ is also proper.

Lemma 7 is clear if $p=0$. Hence we assume that $p>0$. Each pc_i consists of p components, say $c_{i1}, \cdots, c_{ip}$. Let $L_1=c_{11}\cup c_{21}\cup\cdots\cup c_{n1}$ and $L_2=p\Gamma-L_1$. Then we see that $Link(L_1, L_2)=2(p-1)q$.

If p is odd or q is even, $\varphi(p\Gamma)=\varphi(\bar{L}_1\cup L_2)$ by Lemma 4. As $\bar{L}_1\cup L_2$ is related to $(p-2)\Gamma$, $\varphi(p\Gamma)=\varphi((p-2)\Gamma)$ by Lemma 1. By doing this successively, if p is odd, $\varphi(p\Gamma)=\varphi(\Gamma)$ and if both p and q are even, $\varphi(p\Gamma)=\varphi(\mathcal{O})=0$ for a trivial knot $\mathcal{O}$.

Next we consider the case that q is odd and p is even. Then,

$$\varphi(p\Gamma)\equiv\varphi((p-2)\Gamma)+1\equiv\varphi((p-4)\Gamma) \quad (mod\ 2)$$

by Lemma 4. Hence if $p=4m$, $\varphi(p\Gamma)=\varphi(\mathcal{O})=0$ and if $p=4m+2$, $\varphi(p\Gamma)=\varphi(2\Gamma)=1$ by Lemma 6.

By the similar proof of Lemma 7, we obtain Lemma 8.

Lemma 8. *Let $p_i\equiv p_j$ (mod 4) and $p=Min\{p_1,\cdots,p_n\}$. Then $\varphi(p\Gamma)=\varphi(p_1c_1\cup\cdots\cup p_nc_n)$ for a proper link $\Gamma=c_1\cup\cdots\cup c_n$.*

Let $\mathcal{L}_i$ be a link in V_i with r_i components for some integer r_i such that $\mathcal{L}_i$ is non-twisted and parallel to c_i and $w(\mathcal{L}_i)=p_i(\leqq r_i)$ for $i=1,\cdots,n$. Then as $\mathcal{L}=\mathcal{L}_1\cup\cdots\cup\mathcal{L}_n$ is related to $p_1c_1\cup\cdots\cup p_nc_n$, we obtain Lemma 9.

Lemma 9. *If Γ is proper and $p_i\equiv p_j$ (mod 2), $\mathcal{L}$ is also proper and $\varphi(\mathcal{L})=\varphi(p_1c_1\cup\cdots\cup p_nc_n)$.*

Proof of Theorem 2. Let $\mathcal{U}=U_1\cup\cdots\cup U_n$ be the union of mutually disjoint solid tori in R^3 with core $-\Gamma=(-c_1)\cup\cdots\cup(-c_n)$, the reflective inverse of Γ, split from $\mathcal{V}$ by a 2-sphere S^2 and symmetric with respect to S^2. For $l=l_1\cup\cdots\cup l_n$ in $\mathcal{V}$, let $\tilde{\mathcal{L}}_i$ be a link with $r_i(=o(l_i))$ components in U_i such that $\tilde{\mathcal{L}}_i$ is non-twisted and parallel to $-c_i$ and $w(\tilde{\mathcal{L}}_i)=p_i(=w(l_i))$, $i=1,\cdots,n$. Attach a 3-ball B_i^3 to $V_i\cup U_i$ such that $V_i\cup U_i\cup B_i^3$ is symmetric with respect to S^2, Fig. 1(b) for each i. Let M_i, M_i' be meridian disks of V_i, U_i respectively such that $\#(l_i\cap M_i)=\#(\tilde{\mathcal{L}}_i\cap M_i')=p_i$ and $M_i\cap B_i=\partial M_i\cap\partial B_i(=\{\text{an arc } \alpha_i\})$, $M_i'\cap B_i=\partial M_i'\cap\partial B_i(=\{\text{an arc } \beta_i\})$, where $\#(X)$ means the number of points of X, see Fig. 1(a). Let D_i be a proper non-twisted disk in B_i with $\partial D_i\supset\alpha_i\cup\beta_i$ and $\Delta_i=M_i\cup M_i'\cup D_i$. For each i, perform the fusion of $l_i\circ\tilde{\mathcal{L}}_i$ along Δ_i and we obtain a link L_i which is contained in a solid torus $W_i=\overline{V_i\cup U_i\cup B_i-\Delta_i\times[-\varepsilon,\varepsilon]}$

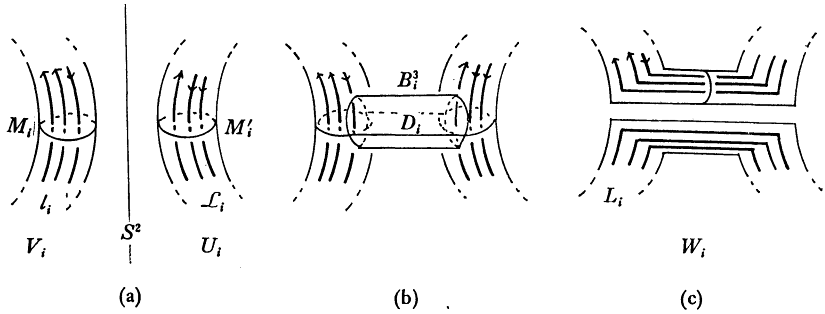


Fig. 1

for a small positive number ε, Fig. 1(c). Then $\mathcal{W}=W_1\cup\cdots\cup W_n$ is the union of disjoint solid tori which is symmetric with respect to S^2 by the construction. So the core of $\mathcal{W}$ is cobordant to zero by [1] and hence $L=L_1\cup\cdots\cup L_n$ is cobordant to $L^*=L_1^*\cup\cdots\cup L_n^*$ by [5], [6] for a faithful homeomorphism f_0 of $\mathcal{W}^*$ onto $\mathcal{W}$, where $L=f_0(L^*)$. As $\tilde{\mathcal{L}}_i$ is non-twisted, L^* is ambient isotopic to ℓ^*. As L is cobordant to ℓ^* and ℓ^* is proper, L is also proper. Moreover as Γ is proper, $\tilde{\mathcal{L}}=\tilde{\mathcal{L}}_1\cup\cdots\cup\tilde{\mathcal{L}}_n$ is proper. Hence we easily see that ℓ is also proper. As L and $\ell\circ\tilde{\mathcal{L}}$ are related,

$$\varphi(\ell)+\varphi(\tilde{\mathcal{L}})\equiv\varphi(L)=\varphi(L^*)=\varphi(\ell^*)\quad(mod\ 2)\,.$$

So we obtain Theorem 2 by Lemmas 7, 8 and 9.

REMARK 1. In Theorem 2, if we replace the condition "$p_i\equiv p_j$ (mod 4)" by "$p_i\equiv p_j$ (*mod* 2)", the conclusion is not true. For example, we consider the links Γ, ℓ illustrated in Fig. 2. Then $\varphi(\Gamma)=0$ and $\varphi(\ell)=1$, hence $\varphi(\ell)\not\equiv\varphi(\ell^*)+\varphi(\Gamma)$ (*mod* 2).

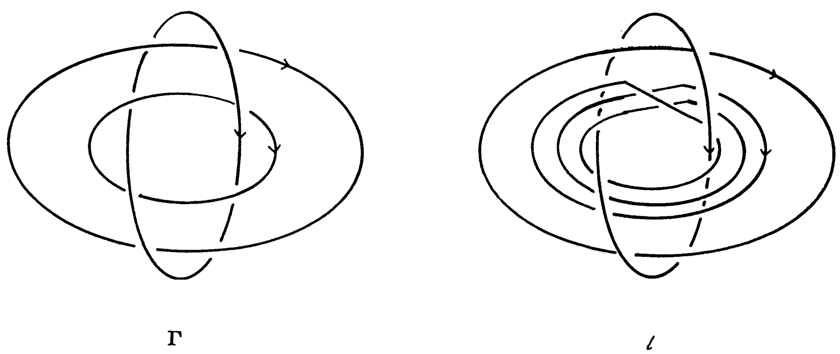


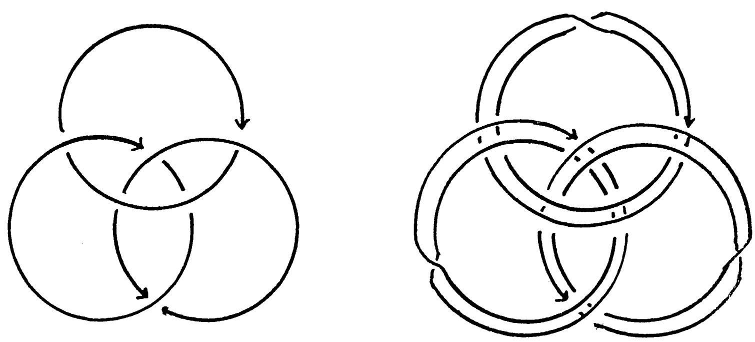
Fig. 2

Proof of Theorem 3. As $Link(c_i, \Gamma-c_i)\equiv 0$ (*mod* 4), $Link(c_i, \Gamma-c_i)=4r_i$ for some integer r_i for each i. Then $2Link(\Gamma)=\sum_{i=1}^{n} Link(c_i, \Gamma-c_i)=4\,(r_1+\cdots+$

r_n). Hence $Link(\Gamma)$ is even. Therefore we obtain Theorem 3 by Lemma 7 and the proof of Theorem 2.

REMARK 2. The link in Fig. 2 is an example that the conclusion of Theorem 3 is not true if we replace that "$Link(c_i, \Gamma-c_i)\equiv 0$ (*mod* 4)" by "$Link(c_i, \Gamma-c_i)\equiv 0$ (*mod* 2)".

EXAMPLE 1. Let Γ, ℓ be links illustrated in Fig. 3. As $Link(\Gamma)=3$ and ℓ^* is a trivial link, $\varphi(\ell)=\varphi(\ell^*)+1=1$ by Theorem 2.



Γ ℓ

Fig. 3

EXAMPLE 2. Let ℓ be a link illustrated in Fig. 4. As Γ is the Whitehead link, $Link(\Gamma)=0$ and $\varphi(\Gamma)=1$ and ℓ^* is a trivial link. Hence $\varphi(\ell)\equiv\varphi(\ell^*)+\varphi(\Gamma)=1$ by Theorem 3.

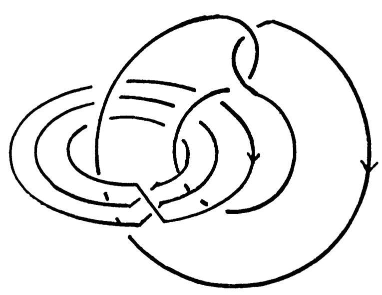


Fig. 4

Proof of Theorem 4. For a proper link ℓ_i in V_i, let k_i be a knot obtained by a fusion of ℓ_i in V_i for each i. As ℓ_i, ℓ are related to k_i, $\ell_0=k_1\cup\cdots\cup k_n$ respectively, $\varphi(\ell_i)=\varphi(k_i)$ and $\varphi(\ell)=\varphi(\ell_0)$ by Lemma 1. Furthermore as Γ is a boundary link, there are mutually disjoint surfaces $\mathcal{F}=F_1\cup\cdots\cup F_n$ with $\partial\mathcal{F}=\ell_0$, $\partial F_i=k_i$. Then $\varphi(\ell_0)=\sum_{i=1}^{n}\varphi(k_i)$ (*mod* 2) by Theorem 3 in [4]. Hence we obtain that

$$\varphi(\ell) = \varphi(\ell_0) \equiv \sum_{i=1}^{n} \varphi(k_i) \equiv \sum_{i=1}^{n} \varphi(\ell_i) \quad (mod\ 2)\,.$$

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