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THE ARF INVARIANT OF PROPER LINKS
IN SOLID TORI

Dedicated to Professor Junzo Tao on his 60th birthday

TETSUO SHIBUYA

(Received July 6, 1988)

Let $L = K_1 \cup \cdots \cup K_n$ be a tame oriented link with $n$ components in a 3-space $\mathbb{R}^3$. $L$ is said to be proper if the linking number of a knot $K_i$ and $L - K_i$, denoted by $\text{Link}(K_i, L - K_i) = \sum_{1 \leq i < j \leq n} \text{Link}(K_i, K_j)$, is even for $i = 1, \ldots, n$. The total linking number of $L$, denoted by $\text{Link}(L)$, means $\sum_{1 \leq i < j \leq n} \text{Link}(K_i, K_j)$.

For two links $L_1, L_2$ in $\mathbb{R}^3[a], \mathbb{R}^3[b]$ respectively for $a < b$, $L_1$ is said to be related to $L_2$ (or $L_1$ and $L_2$ are said to be related) if there is a locally flat proper surface $F$ of genus zero in $\mathbb{R}^3[a, b]$ with $F \cap \mathbb{R}^3[a] = L_1$ and $F \cap \mathbb{R}^3[b] = -L_2$, where $-L_2$ means the reflective inverse of $L_2$.

The Arf invariant of a proper link $L$, denoted by $\varphi(L)$, is defined to that of a knot related to $L$ which is well-defined by Theorem 2 in [4].

Let $V^*, V$ be solid tori with longitudes $\lambda^*, \lambda$ respectively and $\mu$ a meridian of $\partial V$ in $\mathbb{R}^3$, where $\lambda^*$ is a trivial knot, and $f_m$ an orientation preserving onto homeomorphism of $V^*$ onto $V$ such that $f_m(\lambda^*) = \lambda + m\mu$ for an integer $m$. Especially $f_0$ is said to be faithful. For a link $\ell^*$ in $V^*$, $f_m(\ell^*)$ is called a link $T$-congruent to $\ell = f_0(\ell^*)$ (in $V$) and denoted by $\ell(m)$. The winding number of $\ell$ in $V$ means the (algebraic) intersection number of $\ell$ and a meridian disk of $V$ and is denoted by $w_v(\ell)$ or simply by $w(\ell)$.

**Theorem 1.** Let $\ell, \ell(m)$ and $p = w(\ell)$ be those of the above. Suppose that $p$ is odd or both $p$ and $m$ are even. Then $\ell$ is proper if and only if $\ell(m)$ is proper. Let $\ell$ be a proper link.

1. Assume that $p$ is odd. Then
   - $\varphi(\ell(m)) = \varphi(\ell)$ if $m$ is even, or $m$ is odd and $p = 8r \pm 1$
   - $\equiv \varphi(\ell) + 1 \pmod{2}$ if $m$ is odd and $p = 8r \pm 3$.

2. Assume that $p$ and $m$ are even. Then
   - $\varphi(\ell(m)) = \varphi(\ell)$ if $p = 4r$
   - $\equiv \varphi(\ell) + 1 \pmod{2}$ if $p = 4r + 2$,

for an integer $r$. 

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If \( p \) is even and \( m \) is odd in Theorem 1, \( \ell(m) \) is not always proper even though \( \ell \) is proper.

Let \( V_1^*, \ldots, V_n^* \) be mutually disjoint solid tori in \( \mathbb{R}^3 \) with cores \( c_1^*, \ldots, c_n^* \) respectively such that \( \Gamma^* = c_1^* \cup \cdots \cup c_n^* \) is a trivial link. An orientation preserving homeomorphism \( f \) of \( \mathcal{U}^* = V_1^* \cup \cdots \cup V_n^* \) onto \( \mathcal{V} = V_1 \cup \cdots \cup V_n \) is said to be faithful if \( f|_{V_i^*}: V_i^* \rightarrow V_i \) is faithful for \( i = 1, \ldots, n \). For a link \( l^* = l_1^* \cup \cdots \cup l_n^* \) in \( \mathcal{U}^* \), we write \( f(l^*) \) (or \( f(l_i^*) \)) by \( \ell(\text{or} \ l_i) \), where \( l_i^* \) is a link in \( V_i^* \).

**Theorem 2.** Let \( l^*, \ell = \ell_1 \cup \cdots \cup \ell_n \) and \( \Gamma = f(\Gamma^*) \) be those of the above. Suppose that \( \omega(l_i) \equiv \omega(l_j) \pmod{4} \) for \( i, j = 1, \ldots, n \) and \( q = \text{Link}(\Gamma) \). If \( l^* \) and \( \Gamma \) are proper, then \( \ell \) is also proper and

1. \( \varphi(\ell) \equiv \varphi(l^*) + \varphi(\Gamma) \pmod{2} \) if \( p \) is odd
2. \( \varphi(\ell) = \varphi(l^*) \equiv \varphi(l^*) + 1 \pmod{2} \) if \( q \) is odd and \( p = 4m + 2 \)

for some integer \( m \).

**Corollary 1.** Let \( l^*, \ell, \Gamma, p \) and \( q \) be those of Theorem 2. If \( q \) is even, then

\[
\varphi(\ell) \equiv \varphi(l^*) + \varphi(\Gamma) \pmod{2} \quad \text{if } p \text{ is odd}
\]

\[
\varphi(\ell) = \varphi(l^*) \quad \text{if } p \text{ is even.}
\]

If \( n = 1 \) in Theorem 2, namely \( \Gamma \) is a knot, we define that \( \text{Link}(\Gamma) = 0 \). Hence we obtain the following.

**Corollary 2.** If \( \Gamma \) is a knot,

\[
\varphi(\ell) \equiv \varphi(l^*) + \varphi(\Gamma) \pmod{2} \quad \text{if } p \text{ is odd}
\]

\[
\varphi(\ell) = \varphi(l^*) \quad \text{if } p \text{ is even.}
\]

**Theorem 3.** Let \( l^*, \ell = \ell_1 \cup \cdots \cup \ell_n \) be those of the above. Suppose that \( \omega(l_i) \equiv \omega(l_j) \pmod{4} \) and \( \text{Link}(c_i, \Gamma - c_i) \equiv 0 \pmod{4} \) for \( i = 1, \ldots, n \). If \( l^* \) is proper, then \( \ell \) is proper and

\[
\varphi(\ell) \equiv \varphi(l^*) + \varphi(\Gamma) \pmod{2} \quad \text{if } p \text{ is odd}
\]

\[
\varphi(\ell) = \varphi(l^*) \quad \text{if } p \text{ is even.}
\]

**Theorem 4.** Let \( \ell = \ell_1 \cup \cdots \cup \ell_n \) and \( \Gamma \) be those of the above. If \( \Gamma \) is a boundary link and \( \ell_i \) is proper for \( i = 1, \ldots, n \), then \( \varphi(\ell) \equiv \sum_{i=1}^{n} \varphi(\ell_i) \pmod{2} \).

The author thanks to Doctor H. Murakami for his helpful advice.

**Proof of Theorems.**

Lemma 1 is easily obtained by Theorem 2 in [4].

**Lemma 1.** If two proper links \( L_1 \) and \( L_2 \) are related, then \( \varphi(L_1) = \varphi(L_2) \).
For a knot $K$, $\overline{K}$ means the knot orientation reversed to $K$. For a 2-component link $L_0 = K_1 \cup K_2$, let $L'_0 = \overline{K}_1 \cup \overline{K}_2$ and $s = \text{Link}(K_1, K_2)$.

**Lemma 2** ([2]). \[ V_{L'_0}(t) = t^{-3s} V_{L_0}(t) \] for Jones polynomials of $L_0, L'_0$.

For a link $L$, a relation between Jones polynomial and the Arf invariant of $L$ is known by [3].

**Lemma 3** ([3]). For a n-component link $L$,
\[
V_L(\sqrt{-1}) = \begin{cases} 
(\sqrt{2})^{n-1} \times (-1)^{j(L)} & \text{if } L \text{ is proper} \\
0 & \text{if } L \text{ is non-proper.}
\end{cases}
\]

By using the above Lemmas, we prove Lemma 4 which is effective to prove Theorems 1, 2 and 3.

Let $L = L_1 \cup L_2$ be a link, where $L_1, L_2$ consist of $m_1, m_2$ knots $K_1, \ldots, K_{m_1}$, $K_{m_1+1}, \ldots, K_{m_1+m_2}$ respectively. The linking number of $L_1$ and $L_2$, denoted by $\text{Link}(L_1, L_2)$, means $\sum_{i=1}^{m_1} \sum_{j=m_1+1}^{m_1+m_2} \text{Link}(K_i, K_j)$. For a link $L = K_1 \cup \cdots \cup K_{m_1}$ we denote that $L_1 = K_1 \cup \cdots \cup K_{m_1}$.

**Lemma 4.** Let $L = L_1 \cup L_2$ be a proper link and $L' = L_1 \cup L_2$. Then $L'$ is also proper and $\text{Link}(L_1, L_2)$ is even. Moreover, if $\text{Link}(L_1, L_2) \equiv 0 \pmod{4}$, then $\varphi(L') = \varphi(L)$ and $\varphi(L'_0) = \varphi(L_0)$.

Proof. Let $L_1 = K_1 \cup \cdots \cup K_{m_1}$ and $L_2 = K_{m_1+1} \cup \cdots \cup K_{m_1+m_2}$. As $L$ is proper, $\text{Link}(K_1, L_2 - K_1) = 2r_i$ for $K_i \subset L$ and some integer $r_i$. Then we see that $\text{Link}(\overline{K}_1, L'_1 - \overline{K}_1) = 2(r_i - \text{Link}(K_i, L_1))$ for $K_i \subset L_1$ and $\text{Link}(K_1, L'_2 - K_1) = 2(r_i - \text{Link}(K_1, L_2))$ for $K_1 \subset L_2$ and that $\text{Link}(L_1, L_2) = 2(r_1 + \cdots + r_{m_1} - \text{Link}(L_1))$. Hence $L'$ is also proper and $\text{Link}(L_1, L_2)$ is even.

Let $L_0 = K_1 \cup \cdots \cup K_{m_1}$ be a 2-component link related to $L$ such that $\kappa_1, \kappa_2$ are obtained by fusion (band sum) of $L_1, L_2$ respectively and let $L_0' = \kappa_1 \cup \kappa_2$ which is related to $L'$. As $\text{Link}(\kappa_1, \kappa_2) = \text{Link}(L_1, L_2) = s$ is even, $L_0$ and $L_0'$ are proper. So by Lemma 1, $\varphi(L_0) = \varphi(L)$ and $\varphi(L_0') = \varphi(L')$. As $\text{Link}(L_1, L_2) = \text{Link}(\kappa_1, \kappa_2) = s$, $V_{L_0}(t) = t^{-3s} V_{L'_0}(t)$ by Lemma 2 and hence $V_{L_0}(\sqrt{-1}) = (\sqrt{-1})^{-3s} V_{L'_0}(\sqrt{-1})$. Therefore if $s \equiv 0 \pmod{4}$, then $\varphi(L') = \varphi(L'_0) = \varphi(L_0) = \varphi(L)$ and if $s \equiv 2 \pmod{4}$, then $\varphi(L') = \varphi(L'_0) = \varphi(L_0) + 1 = \varphi(L) + 1 \pmod{2}$.

For a link $L$ in a solid torus $V$, the minimum of intersection of $L$ and a meridian disk in $V$ is called the order of $L$ (in $V$) and denoted by $o(L)$ or simply by $o(L)$.

To prove Theorem 1, we prepare Lemma 5.

**Lemma 5.** Let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3, \mathcal{L}_4$ be torus links of type $(8m \pm 1, 8m \pm 1)$, $(8m \pm 3, 8m \pm 3)$ and $(4m, 8m)(4m+2, 8m+4)$ for some integer $m$ respectively.
Then $L_1, L_2$ and $L_3, L_4$ are proper. Furthermore if we orient $L_i$ so that $o(L_i)=w(L_i)$ for each $i$, then $\phi(L_1)=\phi(L_3)=0$ and $\phi(L_2)=\phi(L_4)=1$.

Proof. It is easily seen that $L_i$ is proper for $i=1, 2, 3, 4$.

Next suppose that $o(L_i)=w(L_i)$ for each $i$. $L_1$ consists of $(8m\pm 1)$-component. Let $L_{11}, L_{12}$ be disjoint sublinks of $L_1$ with $4m, (4m\pm 1)$-components respectively. Then $\text{Link}(L_{11}, L_{12})=4m (4m\pm 1)$. Hence $\phi(L_1)=\phi(L_{11} \cup L_{12})$ by Lemma 4. As $L_{11} \cup L_{12}$ is related to a torus knot of type $(\pm 1, \pm 1)$, $\phi(L_1)=0$. By the same way as above, we see that $\phi(L_2)=1$, for the Arf invariant of torus link of type $(\pm 3, \pm 3)$ is 1.

$L_3$ consists of $4m$-component and let $L_3, L_3'$ be disjoint sublinks of $L_3$ with $2m, 2m$-components. Then $\text{Link}(L_3, L_3')=8m^2$. Hence $\phi(L_3)=\phi(L_{31} \cup L_{32})$ by Lemma 4. As $L_{31} \cup L_{32}$ is related to a trivial knot, $\phi(L_3)=0$. By the same way as above, we easily see that $\phi(L_4)=1$.

Proof of Theorem 1. We easily see that, when $p$ is odd or both $p$ and $m$ are even, $l$ is proper if and only if $\ell(m)$ is proper.

Let $n$ be $o(l)$. Then $\ell(m)$ is obtained by a fusion of $l$ and a torus link $L_0$ of type $(n, mn)$ split from $l$ in $V$ and hence $\ell(m)$ is related to $l_0 L_0$, where $o$ means that $l$ is split from $L_0$. By the way, $l_0 L_0$ is related to $l_0 L$, where $L$ is a torus link of type $\langle p, mp \rangle$ for $p=w(L)$. If $l$ and $\ell(m)$ are proper, $L_0, L$ are also proper and $\phi(\ell(m))=\phi(l_0 L_0)=\phi(l_0 L)$ by Lemma 1. Hence we obtain Theorem 1 by Lemma 5.

Let $V=V_1 \cup \cdots \cup V_n$ be the union of mutually disjoint solid tori in $R^3$ and $\Gamma$ that of Theorem 2. For a core $c_i$, take a $p_i$-component link, denoted by $p_i, c_i$, in $V_i$, each of which is parallel and homologous to $c_i$ and non-twisted, namely $p_i, c_i$ is contained on a non-twisted annulus $A_i$ in $V_i$ with $\partial A_i \sim c_i$, in $V_i$ for $i=1, \cdots, n$. Especially if $p_i=p_j=(=p)$, we denote $p_{c_1} \cup \cdots \cup p_{c_n}$ by $p \Gamma$.

In Lemma 6, we consider the case that $p=2$ which is used to prove Lemma 7.

**Lemma 6.** $\phi(2\Gamma) = \begin{cases} 0 & \text{if } q \text{ is even} \\ 1 & \text{if } q \text{ is odd, where } q=\text{Link}(\Gamma) \end{cases}$

Proof. Let $2\Gamma=\Gamma \cup \Gamma'$. As $c_i \cup c_i' (\subset \Gamma \cup \Gamma')$ is non-twisted, $\text{Link}(\Gamma, \Gamma')=2q$. Hence if $q$ is even, $\phi(2\Gamma)=\phi(\Gamma \cup \Gamma')$ and if $q$ is odd, $\phi(2\Gamma) \equiv \phi(\Gamma \cup \Gamma')+1 \pmod{2}$ by Lemma 4. As $\Gamma \cup \Gamma'$ is related to a trivial knot, we obtain Lemma 6 by Lemma 1.

**Lemma 7.** If $\Gamma$ is proper, $p \Gamma$ is also proper and

1. $\phi(p \Gamma) = \phi(\Gamma)$ if $p$ is odd

2. $\phi(p \Gamma) = \begin{cases} 0 & \text{if } p \text{ and } q \text{ are even, or } q \text{ is odd and } p=4m \\ 1 & \text{if } q \text{ is odd and } p=4m+2 \end{cases}$
for some integer $m$ and $q = \text{Link}(\Gamma)$. Hence if $q$ is even,
\[
\varphi(p\Gamma) = \begin{cases} 
\varphi(\Gamma) & \text{if } p \text{ is odd} \\
0 & \text{if } p \text{ is even.}
\end{cases}
\]

Proof. As $pc_i$ is non-twisted, we easily see that if $\Gamma$ is proper, $p\Gamma$ is also proper.

Lemma 7 is clear if $p=0$. Hence we assume that $p>0$. Each $pc_i$ consists of $p$ components, say $c_{i_1}, \ldots, c_{i_p}$. Let $L_1 = c_{i_1} \cup c_{i_2} \cup \cdots \cup c_{i_m}$ and $L_2 = p\Gamma - L_1$. Then we see that $\text{Link}(L_1, L_2) = 2(\rho - 1)q$.

If $p$ is odd or $q$ is even, $\varphi(p\Gamma) = \varphi(L_1 \cup L_2)$ by Lemma 4. As $L_1 \cup L_2$ is related to $(p-2)\Gamma$, $\varphi(p\Gamma) = \varphi((p-2)\Gamma)$ by Lemma 1. By doing this successively, if $p$ is odd, $\varphi(p\Gamma) = \varphi(\Gamma)$ and if both $p$ and $q$ are even, $\varphi(p\Gamma) = \varphi(\emptyset) = 0$ for a trivial knot $\emptyset$.

Next we consider the case that $q$ is odd and $p$ is even. Then,
\[
\varphi(p\Gamma) \equiv \varphi((p-2)\Gamma) + 1 \equiv \varphi((p-4)\Gamma) \pmod{2}
\]
by Lemma 4. Hence if $p=4m$, $\varphi(p\Gamma) = \varphi(\emptyset) = 0$ and if $p=4m+2$, $\varphi(p\Gamma) = \varphi(2\Gamma) = 1$ by Lemma 6.

By the similar proof of Lemma 7, we obtain Lemma 8.

**Lemma 8.** Let $p_i \equiv p_j \pmod{4}$ and $p = \text{Min} \{p_1, \ldots, p_n\}$. Then $\varphi(p\Gamma) = \varphi(p_1 c_{i_1} \cup \cdots \cup p_n c_{i_n})$ for a proper link $\Gamma = c_{i_1} \cup \cdots \cup c_{i_n}$.

Let $L_i$ be a link in $V_i$ with $r_i$ components for some integer $r_i$ such that $L_i$ is non-twisted and parallel to $c_{i}$ and $w(L_i) = p_i (\leq r_i)$ for $i=1, \ldots, n$. Then as $L = L_1 \cup \cdots \cup L_n$ is related to $p_1 c_{i_1} \cup \cdots \cup p_n c_{i_n}$, we obtain Lemma 9.

**Lemma 9.** If $\Gamma$ is proper and $p_i \equiv p_j \pmod{2}$, $L$ is also proper and $\varphi(L) = \varphi(p_1 c_{i_1} \cup \cdots \cup p_n c_{i_n})$.

Proof of Theorem 2. Let $\mathcal{U} = U_1 \cup \cdots \cup U_n$ be the union of mutually disjoint solid tori in $\mathbb{R}^3$ with core $-\Gamma = (-c_1) \cup \cdots \cup (-c_n)$, the reflective inverse of $\Gamma$, split from $\mathcal{V}$ by a 2-sphere $S^2$ and symmetric with respect to $S^2$. For $i = 1, \ldots, n$ in $\mathcal{U}$, let $\tilde{L}_i$ be a link with $r_i (=o(L_i))$ components in $U_i$ such that $\tilde{L}_i$ is non-twisted and parallel to $c_i$ and $w(\tilde{L}_i) = p_i (\leq r_i)$, $i=1, \ldots, n$. Attach a 3-ball $B_i$ to $V_i \cup U_i$ such that $V_i \cup U_i \cup B_i$ is symmetric with respect to $S^2$, Fig. 1(b) for each $i$. Let $M_i, M_i'$ be meridian disks of $V_i, U_i$ respectively such that $\#(L_i \cap M_i) = \#(\tilde{L}_i \cap M_i') = p_i$ and $M_i \cap B_i = \partial M_i \cap \partial B_i (= \{\text{an arc } \alpha_i\}), M_i' \cap B_i = \partial M_i' \cap \partial B_i (= \{\text{an arc } \beta_i\})$, where $\#(X)$ means the number of points of $X$, see Fig. 1(a). Let $D_i$ be a proper non-twisted disk in $B_i$ with $\partial D_i \supset \alpha_i \cup \beta_i$ and $\Delta_i = M_i \cup M_i' \cup D_i$. For each $i$, perform the fusion of $L_i \circ \tilde{L}_i$ along $\Delta_i$ and we obtain a link $L_i$ which is contained in a solid torus $W_i = \tilde{V}_i \cup U_i \cup B_i - \Delta_i \times [-\varepsilon, \varepsilon]$.
for a small positive number $\varepsilon$, Fig. 1(c). Then $\mathcal{W}=W_1 \cup \cdots \cup W_n$ is the union of disjoint solid tori which is symmetric with respect to $S^2$ by the construction. So the core of $\mathcal{W}$ is cobordant to zero by [1] and hence $L=L_1 \cup \cdots \cup L_n$ is cobordant to $L^* = L_1^* \cup \cdots \cup L_n^*$ by [5], [6] for a faithful homeomorphism $f_0$ of $\mathcal{W}^*$ onto $\mathcal{W}$, where $L=f_0(L^*)$. As $\mathcal{L}_i$ is non-twisted, $L^*$ is ambient isotopic to $\mathcal{L}_i^*$. As $L$ is cobordant to $\mathcal{L}_i^*$ and $\mathcal{L}_i^*$ is proper, $L$ is also proper. Moreover as $\Gamma$ is proper, $\mathcal{L}=\mathcal{L}_1^* \cup \cdots \cup \mathcal{L}_n^*$ is proper. Hence we easily see that $\mathcal{L}$ is also proper. As $L$ and $\mathcal{L}$ are related, $\varphi(\mathcal{L}) + \varphi(\mathcal{L}^*) \equiv \varphi(L) = \varphi(L^*) = \varphi(\mathcal{L}_i^*) \pmod{2}$.

So we obtain Theorem 2 by Lemmas 7, 8 and 9.

**Remark 1.** In Theorem 2, if we replace the condition "$\rho_i \equiv \rho_j \pmod{4}$" by "$\rho_i \equiv \rho_j \pmod{2}$", the conclusion is not true. For example, we consider the links $\Gamma$, $\mathcal{L}$ illustrated in Fig. 2. Then $\varphi(\Gamma)=0$ and $\varphi(\mathcal{L})=1$, hence $\varphi(\mathcal{L}) \equiv \varphi(\mathcal{L}_i^*) + \varphi(\Gamma) \pmod{2}$.

**Proof of Theorem 3.** As $\text{Link}(c_i, \Gamma-c_i) \equiv 0 \pmod{4}$, $\text{Link}(c_i, \Gamma-c_i)=4r_i$ for some integer $r_i$ for each $i$. Then $2\text{Link}(\Gamma) = \sum_{i=1}^n \text{Link}(c_i, \Gamma-c_i) = 4(r_1+\cdots+}$
Hence Link(Γ) is even. Therefore we obtain Theorem 3 by Lemma 7 and the proof of Theorem 2.

REMARK 2. The link in Fig. 2 is an example that the conclusion of Theorem 3 is not true if we replace that \( \text{"Link}(c_i, \Gamma - c_i) \equiv 0 \pmod{4}\) by \( \text{"Link}(c_i, \Gamma - c_i) \equiv 0 \pmod{2}\).

EXAMPLE 1. Let Γ, ℓ be links illustrated in Fig. 3. As \( \text{Link}(Γ) = 3 \) and ℓ* is a trivial link, \( \varphi(ℓ) = \varphi(ℓ^*) + 1 = 1 \) by Theorem 2.

\[
\begin{align*}
\text{Fig. 3}
\end{align*}
\]

EXAMPLE 2. Let ℓ be a link illustrated in Fig. 4. As Γ is the Whitehead link, \( \text{Link}(Γ) = 0 \) and \( \varphi(Γ) = 1 \) and ℓ* is a trivial link. Hence \( \varphi(ℓ) = \varphi(ℓ^*) + \varphi(Γ) = 1 \) by Theorem 3.

\[
\begin{align*}
\text{Fig. 4}
\end{align*}
\]

Proof of Theorem 4. For a proper link \( ℓ_i \) in \( V_i \), let \( k_i \) be a knot obtained by a fusion of \( ℓ_i \) in \( V_i \) for each \( i \). As \( ℓ_i \), \( ℓ \) are related to \( k_i \), \( ℓ_0 = k_1 \cup \cdots \cup k_n \) respectively, \( \varphi(ℓ_i) = \varphi(k_i) \) and \( \varphi(ℓ) = \varphi(ℓ_0) \) by Lemma 1. Furthermore as \( Γ \) is a boundary link, there are mutually disjoint surfaces \( \mathcal{F} = F_1 \cup \cdots \cup F_n \) with \( \partial \mathcal{F} = ℓ_0 \), \( \partial F_i = k_i \). Then \( \varphi(ℓ_0) = \sum_{i=1}^{n} \varphi(k_i) \pmod{2} \) by Theorem 3 in [4]. Hence we obtain that
\[\varphi(\mathcal{L}) = \varphi(\mathcal{L}_0) \equiv \sum_{i=1}^{n} \varphi(k_i) \equiv \sum_{i=1}^{n} \varphi(l_i) \pmod{2}.\]

References


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