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PROJECTIVE DIMENSION OF COMPLEX BORDISM MODULES OF CW-SPECTRA, II

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In the previous paper [I] with the same title we tried to extend some results of [6, 8 and 9] to connective CW-spectra X. And we gave necessary and sufficient conditions that the Thom homomorphism

$$\mu=\mu\langle 0\rangle\colon MU_*(X)\to MU\langle 0\rangle_*(X)\simeq H_*(X)$$

is an epimorphism and that the homomorphism

$$\zeta=\mu_{Td}\langle 1\rangle\colon MU_*(X)\to MU_{Td}\langle 1\rangle_*(X)\simeq k_*(X)$$

(lifting the Thom homomorphism $\mu_C\colon MU_*(X)\to K_*(X)$) is an epimorphism.

In the present paper we study conditions that

$$\mu\langle n\rangle\colon MU_*(X)\to MU\langle n\rangle_*(X)$$

is an epimorphism for a general $n\geqq 0$.

As our main results we have

Theorem 1. *Let X be a connective CW-spectrum and $0\leqq n<\infty$. The following conditions are equivalent*:

I) $\mu\langle n\rangle\colon MU_*(X)\to MU\langle n\rangle_*(X)$ *is an epimorphism*;

II) $\mu\langle n\rangle$ *induces an isomorphism* $\tilde{\mu}\langle n\rangle\colon MU\langle n\rangle_*\otimes_{MU_*} MU_*(X)\to MU\langle n\rangle_*(X)$;

III) $\mathrm{Tor}^{MU_*}_{p,*}(MU\langle n\rangle_*, MU_*(X))=0$ *for all* $p\geqq 1$;

III)′ $\mathrm{Tor}^{MU_*}_{1,*}(MU\langle n\rangle_*, MU_*(X))=0$.

Theorem 2. *Let X be a connective CW-spectrum and $0\leqq n<\infty$. If one of the equivalent conditions stated in Theorem 1 is satisfied, then*

0) $\mathrm{hom\ dim}_{MU_*} MU_*(X)\leqq n+1$.

We use all notations and notions defined in [I] and quote the theorem of [I] in such a form as "Theorem I. 4".

1. Let X be a l-connected CW-spectrum and $\{X^p\}$ the skeleton filtration of X. For $p \geqq l+1$ and $n \geqq 0$ we consider the following commutative diagram

$$\begin{array}{ccccc} MU_{j+1}(X^p/X^{p-1}) & \to & MU_j(X^{p-1}) & \to & MU_j(X^p) \\ \downarrow & & \downarrow & & \downarrow \\ MU\langle n\rangle_{j+1}(X^p/X^{p-1}) & \to & MU\langle n\rangle_j(X^{p-1}) & \to & MU\langle n\rangle_j(X^p) \end{array}$$

$$\begin{array}{ccc} \to MU_j(X^p/X^{p-1}) & \to & MU_{j-1}(X^{p-1}) \\ \downarrow & & \downarrow \\ \to MU\langle n\rangle_j(X^p/X^{p-1}) & \to & MU\langle n\rangle_{j-1}(X^{p-1}) \end{array}$$

with exact rows. $\mu\langle n\rangle$: $MU_j(X^p/X^{p-1}) \to MU\langle n\rangle_j(X^p/X^{p-1})$ is an epimorphism for each j and particularly an isomorphism for each $j \leqq 2n+p+1$. By an induction on p we can show that

$$\mu\langle n\rangle\colon MU_j(X^p) \to MU\langle n\rangle_j(X^p)$$

is an isomorphism for each $j \leqq 2n+l+2$. Passing to the direct limit, we get

Lemma 1. *Let X be a l-connected CW-spectrum and $n \geqq 0$. Then $\mu\langle n\rangle$: $MU_j(X) \to MU\langle n\rangle_j(X)$ is an isomorphism for each $j \leqq 2n+l+2$.*

Lemma 2. *Let X be a connective CW-spectrum and $0 \leqq n < \infty$. If $MU\langle n\rangle_j(X)$ is (torsion) free as a Z-module for each $j \leqq k$, then $MU\langle n+1\rangle_j(X)$ is so for the same j.*

Proof. First we assume that $MU\langle n\rangle_j(X)$ is torsion free abelian for each $j \leqq k$. Consider the following commutative diagram

$$\begin{array}{ccccccc} \to MU\langle n+1\rangle_{j-2n-2}(X) & \xrightarrow{\cdot x_{n+1}} & MU\langle n+1\rangle_j(X) & \to & MU\langle n\rangle_j(X) & \to & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to MU\langle n+1\rangle_{j-2n-2}(X)\otimes Q & \to & MU\langle n+1\rangle_j(X)\otimes Q & \to & MU\langle n\rangle_j(X)\otimes Q & \to & 0 \end{array}$$

where $j \leqq k$. The upper row is exact by (I. 1. 3) and the bottom row is exact by virtue of Dold's theorem. By a routine discussion involving an induction on degree j we can show that

$$MU\langle n+1\rangle_j(X) \to MU\langle n+1\rangle_j(X)\otimes Q$$

is a monomorphism, i.e., $MU\langle n+1\rangle_j(X)$ is torsion free abelian for $j \leqq k$. And there exists a short exact sequence

$$0 \to MU\langle n+1\rangle_{j-2n-2}(X) \to MU\langle n+1\rangle_j(X) \to MU\langle n\rangle_j(X) \to 0.$$

Then, the assumption that $MU\langle n\rangle_j(X)$ is free abelian for each $j \leq k$ implies that $MU\langle n+1\rangle_j(X)$ is so for the same j.

By Lemma 1 and iterated applications of Lemmas 2 and I. 3 we have

Proposition 3. *Let X be a connective CW-spectrum and $0 \leq n < m \leq \infty$. If $MU\langle n\rangle_j(X)$ is (torsion) free as a Z-module for each $j \leq k$, then*
i) *$MU\langle m\rangle_j(X)$ is (torsion) free as a Z-module for each $j \leq k$, and*
ii) *$\mu\langle n-1\rangle$: $MU_i(X) \to MU\langle n-1\rangle_i(X)$ is an epimorphism for each $i \leq k+2n+1$.*

Corollary 4. *Let X be a connective CW-spectrum and $W \to X \subset Y$ a partial connective MU_*-resolution of X of length 1. If $\mu\langle n\rangle$: $MU_j(X) \to MU\langle n\rangle_j(X)$ is an epimorphism for each $j \leq k$, then $\mu\langle n-1\rangle$: $MU_i(Y) \to MU\langle n-1\rangle_i(Y)$ is an epimorphism for $i \leq k+2n+1$.*

Proof. Obviously

$$0 \to MU\langle n\rangle_j(Y) \to MU\langle n\rangle_{j-1}(W) \to MU\langle n\rangle_{j-1}(X) \to 0$$

is exact for $j \leq k$. So $MU\langle n\rangle_j(Y)$ is free abelian for $j \leq k$. The required result follows from Proposition 3.

2. *Proof of Theorem* 1. We prove in the order: III)$\to$II)$\to$I)$\to$III)$\to$III)'$\to$I).

"II)$\to$I)" and "III)$\to$III)'" are trivial and "III)$\to$II)" has already been established in Corollary 1. 9.

I)$\to$III): By induction on n. The n=0 case is true by Theorem I. 4.

Assume that $\mu\langle n\rangle$: $MU_*(X) \to MU\langle n\rangle_*(X)$ is an epimorphism, $n \geqq 1$. A partial connective MU_*-resolution $W \to X \subset Y$ of X yields the following commutative diagram

$$\begin{array}{l} 0 \to \mathrm{Tor}_{1,*}^{MU_*}(MU\langle n\rangle_*, MU_*(X)) \to MU\langle n\rangle_* \underset{MU_*}{\otimes} MU_{*+1}(Y) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \to MU\langle n\rangle_{*+1}(Y) \\ \qquad\qquad \to MU\langle n\rangle_* \underset{MU_*}{\otimes} MU_*(W) \to MU\langle n\rangle \underset{MU_*}{\otimes} MU_*(X) \to 0 \\ \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad \to \quad MU\langle n\rangle_*(W) \quad \to \quad MU\langle n\rangle_*(X) \to 0 \end{array}$$

with exact rows. Since $MU_*(Y) \to MU\langle n-1\rangle_*(Y)$ is an epimorphism by Corollary 4, the induction hypothesis shows that $\mathrm{Tor}_{p,*}^{MU_*}(MU\langle n-1\rangle_*, MU_*(Y))=0$ for all $p \geqq 1$. Consider the exact sequence

$$\begin{array}{l} \mathrm{Tor}_{p,j-2n}^{MU_*}(MU\langle n\rangle_*, MU_*(Y)) \to \mathrm{Tor}_{p,j}^{MU_*}(MU\langle n\rangle_*, MU_*(Y)) \\ \qquad\qquad\qquad\qquad \to \mathrm{Tor}_{p,j}^{MU_*}(MU\langle n-1\rangle_*, MU_*(Y)) \end{array}$$

induced by the exact sequence $0 \to MU\langle n\rangle_* \xrightarrow{\cdot x_n} MU\langle n\rangle_* \to MU\langle n-1\rangle_* \to 0$. By an induction on degree j we can see

$$\mathrm{Tor}_{p,*}^{MU_*}(MU\langle n\rangle_*, MU_*(Y))=0 \quad \text{for all } p \geqq 1.$$

Therefore the left vertical map in the above diagram is an isomorphism because III) implies II). So it follows immediately that

$$\mathrm{Tor}_{1,*}^{MU_*}(MU\langle n\rangle_*, MU_*(X))=0$$

Moreover we have

$$\mathrm{Tor}_{p+1,*}^{MU_*}(MU\langle n\rangle_*, MU_*(X))\cong\mathrm{Tor}_{p,*}^{MU_*}(MU\langle n\rangle_*, MU_*(Y))=0$$

for all $p\geqq 1$.

III)'→I): By an induction on degree k we show that $\mu\langle n\rangle_k$: $MU_k(X)\to MU\langle n\rangle_k(X)$ is an epimorphism. First, remark that $MU_l(X)=MU\langle n\rangle_l(X)=0$ for sufficiently small l. Now we assume that $\mu\langle n\rangle_j$ is an epimorphism for each $j\leq k-1$. From a partial connective MU_*-resolution $W\to X\subset Y$ of X we obtain the following commutative diagram

$$\begin{array}{l}
0=\mathrm{Tor}_{1,k-1}^{MU_*}(MU\langle n\rangle_*, MU_*(X))\to(MU\langle n\rangle_*\underset{MU_*}{\otimes}MU_*(Y))_k \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow \\
\qquad MU\langle n\rangle_k(W)\to MU\langle n\rangle_k(X)\to MU\langle n\rangle_k(Y) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to(MU\langle n\rangle_*\underset{MU_*}{\otimes}MU_*(W))_{k-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to MU\langle n\rangle_{k-1}(W)
\end{array}$$

with exact rows. The right vertical map is an isomorphism by Proposition 1. 5 and the left one is an epimorphism by Corollary 4. By chasing the above diagram we see immediately that $MU\langle n\rangle_k(W)\to MU\langle n\rangle_k(X)$ is an epimorphism. Hence we get that

$$\mu\langle n\rangle_k\colon MU_k(X)\to MU\langle n\rangle_k(X)$$

is an epimorphism.

Proof of Theorem 2. We prove by an induction on n that I) implies 0). The $n=0$ case is true by Theorem 1. 4.

Assuming that $\mu\langle n\rangle: MU_*(X)\to MU\langle n\rangle_*(X)$ is an epimorphism, $n\geqq 1$, $\mu\langle n-1\rangle: MU_*(Y)\to MU\langle n-1\rangle_*(Y)$ is an epimorphism because of Corollary 4. So we see by the induction hypothesis that hom $\dim_{MU_*} MU_*(Y)\leq n$. This implies

$$\text{hom}\ \dim_{MU_*} MU_*(X)\leq n+1.$$

3. Here we discuss another condition that $\mu\langle n\rangle$ is an epimorphism.

Lemma 5. *Let X be a connective CW-spectrum and $0\leq n\leq m<\infty$. If $\mu\langle n\rangle: MU_*(X)\to MU\langle n\rangle_*(X)$ is an epimorphism, then X admits a connective $MU\langle m\rangle_*$-resolution of length $n+1$.*

Proof. By induction on n. The $n=0$ case is true by (I. 3. 2).

Next, assume that $\mu\langle n\rangle: MU_*(X)\to MU\langle n\rangle_*(X)$ is an epimorphism, $n\geqq 1$. By Lemma I. 4 $\mu\langle m\rangle: MU_*(X)\to MU\langle m\rangle_*(X)$ are epimorphisms for all $m\geqq n$. This implies that a partial connective MU_*-resolution $W\to X\subset Y$ of X of length 1 forms a partial connective $MU\langle m\rangle_*$-resolution of X of length 1. On the other hand, $\mu\langle n-1\rangle: MU_*(Y)\to MU\langle n-1\rangle_*(Y)$ is an epimorphism by Corollary 4. The induction hypothesis insists that Y admits a connective $MU\langle m\rangle_*$-resolution of length n. Consequently X satisfies the required property.

As is easily seen, Lemma 5 implies

Proposition 6. *Let X be a connective CW-spectrum and $0\leq n<\infty$. $\mu\langle n\rangle$: $MU_*(X)\to MU\langle n\rangle_*(X)$ is an epimorphism if and only if X admits a connective $MU\langle n\rangle_*$-resolution.*

Finally we restrict our interest to the special cases $n=0$, 1.

Proposition 7. *Let X be a connective CW-spectrum and $0\leq n<\infty$. The following conditions are equivalent:*

i) hom $\dim_{MU_*} MU_*(X)\leq 1$;

ii)$_n$ *X admits a connective $MU\langle n\rangle_*$-resolution of length* 1;

iii) *X admits a connective H_*-resolution.*

Proof. "i)$\to$ii)$_n$" and "iii)$\to$i)" follow from Theorem I. 4, Lemma 5 and Proposition 6.

ii)$_n\to$iii): Let $W\to X\subset Y$ be a (partial) connective $MU\langle n\rangle_*$-resolution of X of length 1. Remark that $H_*(Y)$ is free abelian. Since $H_*(X; Q)\to H_*(Y; Q)$ is a zero map, we see immediately that $H_*(X)\to H_*(Y)$ is a zero map. This means that $W\to X\subset Y$ is a (partial) connective H_*-resolution of X of length 1.

Proposition 8. *Let X be a connective CW-spectrum and $1\leq n<\infty$. The following conditions are equivalent*:

i) hom $\dim_{MU_*} MU_*(X)\leq 2$;

ii)$_n$ *X admits a connective $MU_{Td}\langle n\rangle_*$-resolution of length* 2;

iii) *X admits a connective k_*-resolution.*

Proof. "i)$\to$ii)$_n$" and "iii)$\to$i)" follow from Theorem I. 7, Lemma 5 and Proposition 6.

ii)$_n\to$iii): Let $\{X_k, W_k\}_{k\geqq 0}$ be a connective $MU_{Td}\langle n\rangle_*$-resolution of X of length 2. Note that X_1 admits a connective $MU_{Td}\langle n\rangle_*$-resolution of length 1. Proposition 7 insists that hom $\dim_{MU_*} MU_*(X_1)\leq 1$ and X_1 admits a connective k_*-resolution of length 1. Since $MU_{Td}\langle n\rangle_*(X_1)$ is free abelian, $MU_*(X_1)$ is so by Proposition 3. Now we get the following commutative diagram

$$\begin{array}{ccccccc} 0\to MU_*(X_1) & \to & MU_*(X_1)\otimes Q & \to & MU_*(X_1; Q/Z) & \to 0 \\ \downarrow \zeta_1 & & \downarrow \zeta_1' & & \downarrow \zeta_1'' & \\ \to k_*(X_1) & \to & k_*(X_1)\otimes Q & \to & k_*(X_1; Q/Z) & \to \end{array}$$

with exact rows (cf., [13]). We have

$$\mathrm{Tor}^{MU_*}_{p+1,*}(MU_*(X_1; Q/Z), Z)\cong \mathrm{Tor}^{MU_*}_{p,*}(MU_*(X_1), Z)=0$$

for all $p\geqq 1$ because $\mathrm{Tor}^{MU_*}_{p,*}(MU_*(X_1)\otimes Q, Z)=0$ for $p\geqq 1$. This means that the right vertical map ζ_1'' is an epimorphism by Theorem I. 7. Hence $k_*(X_1)$ is torsion free abelian. Then, from the triviality of the map $k_*(X)\otimes Q\to k_*(X_1)\otimes Q$ we can easily see that

$$0\to k_{*+1}(X_1)\to k_*(W_0)\to k_*(X)\to 0$$

is exact. Thus $W_0\to X\subset X_1$ is a partial connective k_*-resolution of X of length 1. Consequently X admits a connective k_*-resolution of length 2.

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References

[1]–[12] are listed at the end of paper [I].

[13] Z. Yosimura: *A note on complex K-theory of infinite CW-complexes*, J. Math. Soc. Japan **26** (1974), to appear.

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