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LINEAR RELATIONS BETWEEN THETA SERIES

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1. Introduction

For a positive integer $m \equiv 0 \pmod 8$, let $S_1, \ldots, S_{h(m)}$ be a complete set of representatives of $GL_m(\mathbf{Z})$-classes of positive definite, symmetric, even integral, unimodular matrices of size m. If n is a positive integer, then in the usual way one can attach to S_ν ($\nu = 1, \ldots, h(m)$) a theta series $\vartheta_{S_\nu}^{(n)}$ which is a modular form of weight $m/2$ and genus n on the Siegel modular group $\Gamma_n := Sp_n(\mathbf{Z}) \subset GL_{2n}(\mathbf{Z})$. For each ν, the sequence $(\vartheta_{S_\nu}^{(n)})_{n\geq 1}$ is a so-called stable system of Siegel modular forms, i.e. $\vartheta_{S_\nu}^{(n)}$ for each $n \geq 1$ is obtained from $\vartheta_{S_\nu}^{(n+1)}$ by applying the Siegel Φ-operator.

It is widely believed that for $m \geq 24$ the series $\vartheta_{S_\nu}^{(n)}$ ($\nu = 1, \ldots, h(m)$) are linearly independent if and only if $n \geq m/2$. The latter assertion was proved by Borcherds, Freitag and Weissauer [2] in the first case $m = 24$. Nothing so far seems to be known for $m > 24$. However, one of the authors [6] using a similar method as in [2] proved the partial result that for arbitrary $m \equiv 0 \pmod{24}$ the series $\vartheta_{S_\nu}^{((m/2)-1)}$ ($\nu = 1, \ldots, h(m)$) are linearly dependent.

The main purpose of this note is to show that on average as $m \to \infty$, $m \equiv 0 \pmod 8$ there are in fact at least $(1/2)[m/24]$ independent linear relations between the above series. For aprecise statement see the Theorem in Section 2.

The proof—quite different from the ideas of [2, 6]—combines three inputs: firstly Ikeda's lifting theorem [4], secondly the characterization of Hecke eigenforms lying in the space generated by the theta series in terms of the behavior of their standard L-functions at special points due to Böcherer [1] and Weissauer [7], and thirdly a result of Iwaniec and Sarnak [5] on the non-vanishing of central critical values of Hecke L-functions of elliptic cuspidal Hecke eigenforms.

At the end of the paper, in Section 3 we will make some further remarks on the relation between Ikeda's lifting theorem and theta series. We remark that the relevance of the Ikeda lift in connection with theta series was also noticed by Conrey and Farmer in [3].

2. Linear relations

For k an integer and n a positive integer we denote by $M_k(\Gamma_n)$ the space of Siegel modular forms of weight k with respect to Γ_n and by $S_k(\Gamma_n)$ the subspace of cusp forms.

If S is a positive definite, symmetric, even integral matrix of size m, we let

$$\vartheta_S^{(n)}(Z) = \sum_{G \in M_{m,n}(\mathbf{Z})} e^{\pi i \operatorname{tr}(S[G]Z)}$$

be the associated theta series in genus n. Here $S[G] := G'SG$ with G' the transpose of G and $Z \in \mathcal{H}_n$, the Siegel upper half-space of genus n. If S is unimodular (such an S exists if and only if $m \equiv 0 \pmod 8$), then $\vartheta_S^{(n)} \in M_{m/2}(\Gamma_n)$.

We shall prove

Theorem. *For $m \in \mathbf{N}$ with $m \equiv 0 \pmod 8$, let S_ν $(\nu = 1, \ldots, h(m))$ be a complete set of representatives of $GL_m(\mathbf{Z})$-classes of positive definite, symmetric, even integral, unimodular matrices of size m and let $\kappa(m)$ be the number of independent linear relations between the associated theta series $\vartheta_{S_\nu}^{((m/2)-1)}$ $(\nu = 1, \ldots, h(m))$ in genus $(m/2) - 1$. Then we have*

$$\liminf_{M \to \infty} \frac{1}{M/8} \sum_{24 \le m \le M,\ m \equiv 0 \pmod 8} \frac{\kappa(m)}{[m/24]} \ge \frac{1}{2}.$$

Proof. Since $\vartheta_{S_\nu}^{(m/2)} | \Phi = \vartheta_{S_\nu}^{((m/2)-1)}$, clearly $\kappa(m)$ is equal to the dimension of the intersection of $S_{m/2}(\Gamma_{m/2})$ and the space $\Theta(m, m/2)$ spanned by the functions $\vartheta_{S_\nu}^{(m/2)}$ $(\nu = 1, \ldots, h(m))$.

If $F \in S_{m/2}(\Gamma_n)$ is a Hecke eigenform and $n \ge m/2$, then by [1, 7] (see in particular Thm. 4.1 of [1]), F is in the space spanned by the $\vartheta_{S_\nu}^{(n)}$ $(\nu = 1, \ldots, h(m))$ if and only if the standard zeta function $L_{\mathrm{st}}(F, s)$ associated to F has a simple pole at $s = 1 + n - (m/2)$. In particular, it follows that a Hecke eigenform $F \in S_{m/2}(\Gamma_{m/2})$ is in $\Theta(m, m/2)$ if and only if $L_{\mathrm{st}}(F, s)$ has a simple pole at $s = 1$.

On the other hand, according to Ikeda's theorem [4] if $f \in S_{2k}(\Gamma_1)$ is a normalized Hecke eigenform, then for each $n \ge 1$ with $n \equiv k \pmod 2$ there is a Hecke eigenform $F \in S_{k+n}(\Gamma_{2n})$ such that

$$L_{\mathrm{st}}(F, s) = \zeta(s) \prod_{j=1}^{2n} L(f, s + k + n - j)$$

where $L(f, s)$ is the Hecke L-function of f. Moreover, if f_1 and f_2 are two different normalized eigenforms in $S_{2k}(\Gamma_1)$, then the associated liftings F_1 and F_2 belong to different Hecke eigenspaces, hence must be orthogonal. □

Note that $L(f, s) \neq 0$ for $\mathrm{Re}(s) > k + (1/2)$ due to the absolute convergence of the Euler product, hence by the functional equation also $L(f, s) \neq 0$ for $0 < \mathrm{Re}(s) < k - (1/2)$.

Specializing Ikeda's result to the case $k = n = m/4$ and combining with the previous result, we see that $\kappa(m)$ is greater or equal to the number of normalized Hecke eigenforms f in $S_{m/2}(\Gamma_1)$ such that $L(f, m/4) \neq 0$.

According to Thm. 7 of [5] one has

$$\liminf_{K\to\infty} \frac{1}{K/2} \sum_{6\leq k\leq K,\ k\equiv 0 \pmod 2} \frac{\#\{f \in \mathcal{H}_{2k} \mid L(f,k) \geq 1/(\log 2k)^2\}}{\#\mathcal{H}_{2k}} \geq \frac{1}{2},$$

where $\mathcal{H}_{2k}$ is the set of normalized Hecke eigenforms in $S_{2k}(\Gamma_1)$. Letting $k = m/4$ and using well-known dimension formulas, our result follows.

REMARK. One conjectures that $L(f, k) \neq 0$ for *all* $f \in \mathcal{H}_{2k}$ when $k \geq 6$ is even. For some numerical calculations in this respect, see [3].

3. Further remarks

We want to discuss very shortly the relevance of Ikeda's lifting map from Hecke eigenforms in $S_{2k}(\Gamma_1)$ to Hecke eigenforms in $S_{k+n}(\Gamma_{2n})$ in connection with theta series when $n > k$, $n + k \equiv 0 \pmod 4$. From what was pointed out in the proof of the Theorem in Section 2, we immediately see that in this case the lifted form F is not in the space $\Theta(2(n+k), 2n)$ generated by the relevant theta series (in fact, F then is orthogonal to $\Theta(2(n+k), 2n)$, cf. [1, p. 6, l. 16]). Thus we obtain a negative answer to the question raised in [1, XII (C)] about the equality of the spaces $M_{n+k}(\Gamma_{2n})$ and $\Theta(2(n+k), 2n)$.

We conclude with observing that in [2] and [4] we have two different, highly non-trivial methods of constructing cusp forms of the same weight and genus. If we assume that the conjecture mentioned above in the Remark at the end of Section 2 is true, then in both cases these forms are in $\Theta(4n, 2n)$. It would be interesting to know how the spaces spanned by these forms are related.

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