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ON COMMUTATORS IN REAL SEMISIMPLE LIE GROUPS

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1. Introduction

Every element of the commutator subgroup G' of a group G is a finite product of commutators. We shall say that G has property (C) if every element of G' is a commutator in G. For finite groups which do not have property (C) see [11], [13, p. 258], and [14].

Ore [19] raised the question whether all finite non-abelian simple groups G have property (C). If the character table of G is known then it is easy to check which conjugacy classes of G consist of commutators, see [14]. Ito [15] showed that the alternating groups A_n $(n\geq 5)$ have property (C), see also [19]. For other results on Ore's question consult [11] and the references therein.

Shoda [22] showed that if k is an algebraically closed field then the group $SL_n(k)$ has property (C). From the results of Thompson [23] it follows that $SL_n(k)$ has property (C) if k is an arbitrary field except in the case when $n\equiv 2$ (mod 4), k contains a primitive n-th root of unity, and the equation $x^2+y^2=-1$ has no solution in k. In the exceptional case only the central elements of order n are not commutators.

Goto [10] showed that connected compact topological group G whose commutator subgroup is dense in G has property (C). In particular, all connected compact semisimple Lie groups have property (C), see also [2, p. 33]. Pasiencier and Wang [20] proved the same result for connected complex semisimple Lie groups. Their result has been extended to connected semisimple algebraic groups over algebraically closed fields by Ree [21].

Note that $-1\in SL_2(\boldsymbol{R})$ is not a commutator of $SL_2(\boldsymbol{R})$ by Thompson's results. More generally, we show in Proposition 1 that if $\varepsilon=\exp(2\pi i/n)$, $n=p+q$, $p\geq q\geq 1$, then $\varepsilon\in SU(p,q)$ is not a commutator in $SU(p,q)$. This includes the previous case because $SU(1,1)\cong SL_2(\boldsymbol{R})$. Thus there exist connected almost simple real Lie groups which do not possess property (C).

Isaacs [14] observes that no example seems to be known of a non-abelian simple group which possesses a non-commutator. We think that the simple

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group of [9] should be examined in this respect. Here we raise and discuss the following conjecture.

Conjecture A. Every simple real Lie group G has property (C). (By definition a simple real Lie group is connected and centerless.)

It is known that this conjecture holds in the following cases:

(i) if G is compact [6],

(ii) if G has complex structure [20],

(iii) if $G=PSL_n(\boldsymbol{R})$, see [23, Theorem 3].

Kursov [17] claimed that the quaternionic special linear group $SL_n(\boldsymbol{H})$ has property (C). His proof of this (and other results) contains two mistakes on which we will comment later. We shall give a different proof of his claim.

Let G be a connected semisimple real Lie group and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the Cartan decomposition of its Lie algebra $\mathfrak{g}$. If $\mathfrak{k}$ is semisimple we show that every element of G is a product of two commutators. We also show that the groups $Sp(p, q)$, $p\geq q\geq 1$, and the identity components of the groups $SO(p, 1)$ for $p\equiv 0$ or $3 \pmod 4$ have property (C).

2. Results and proofs

As observed in the introduction, the result of Pasiencier and Wang about complex semisimple groups does not extend to the real case. In our first proposition we show that the groups $SU(p, q)$, $p\geq q\geq 1$, do not have property (C). (All connected semisimple Lie groups are perfect, i.e., they coincide with their commutator subgroups.)

Proposition 1. *Let $\varepsilon=\exp(2\pi i/n)$ and let p, q be positive integers such that $n=p+q$ and $p\geq q\geq 1$. Then $\varepsilon\in SU(p, q)$ is not a commutator in $SU(p, q)$.*

Proof. Assume that $\varepsilon=aba^{-1}b^{-1}$ where $a, b\in SU(p, q)$. Then $aba^{-1}=\varepsilon b$, and consequently the spectrum of b has the form $\{\lambda\varepsilon^k\colon 0\leq k<n\}$. Let $J=I_p\oplus(-I_q)$ and let $f(x, y)=x^*Jy$ for column vectors $x, y\in\boldsymbol{C}^n$. Let v_k be an eigenvector of b for eigen-value $\lambda\varepsilon^k$, $0\leq k<n$. Since the spectrum of b has the above form, it follows easily that $|\lambda|=1$ and the numbers $\alpha_k=f(v_k, v_k)$, $0\leq k<n$, are real and nonzero, see for instance [7]. Precisely p of these numbers are positive and the remaining q of them are negative. Since a is an isometry of $(\boldsymbol{C}^n, f)$ which carries the subspace spanned by v_k to the subspace spanned by v_{k+1} (indices mod n), we have a contradiction. QED.

In view of the above proposition, it is of interest to find the least integer n such that every element of a real connected semisimple Lie group G can be expressed as a product of n commutators. It is not known (to this author) whether this statement is true with $n=2$. Our next result shows that it is true when the maximal compact subgroup of the adjoint group of G is semisimple.

Theorem 2. *Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of a connected semisimple real Lie group G. If $\mathfrak{k}$ is semisimple then every element of G is a product of two commutators.*

Proof. The connected Lie subgroup K which corresponds to $\mathfrak{k}$ is a maximal compact subgroup of G and the map $K\times\mathfrak{p}\to G$, defined by $(y, X)\to y \exp X$ is a diffeomorphism [4, p. 168]. Hence if $x\in G$ we can write $x=y \exp X$ where $y\in K$ and $X\in\mathfrak{p}$. By a result of Goto [10] y is a commutator in K. It remains to show that $\exp X$ is a commutator in G. Let $\mathfrak{a}$ be a Cartan subspace (i.e. a maximal abelian subspace) of $\mathfrak{p}$ containing X. The Weyl group W of $(\mathfrak{g}, \mathfrak{k})$ is the quotient group $N_K(\mathfrak{a})/C_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$, resp. $C_K(\mathfrak{a})$, is the normalizer, resp. centralizer, of $\mathfrak{a}$ in K for the adjoint action Ad of K on $\mathfrak{g}$. It is well known [12, p. 289] that W acts faithfully on $\mathfrak{a}$ as a finite reflection group. We shall now use an argument of Goto [10]. If c is a Coxeter element of W then $c-1$ is invertible [1, Théorème 1, p. 119] and hence we can write $X=(c-1)Y$ for some $Y\in\mathfrak{a}$. If $z\in N_K(\mathfrak{a})$ is chosen so that $c=Ad\, z|_{\mathfrak{a}}$ then

$$\begin{aligned}\exp X &= \exp(c-1)Y = \exp c(Y)\cdot\exp(-Y)\\ &= \exp(\mathrm{Ad}\, z)(Y)\cdot\exp(-Y)\\ &= z(\exp Y)z^{-1}\cdot\exp(-Y)\,. \qquad\qquad \text{QED.}\end{aligned}$$

Next we consider the case of the special linear group over the real quaternions $\boldsymbol{H}$. We shall use some elementary facts of linear algebra over division rings for which we recommend the reader to consult the references [3] and [16]. By $1, i, j, k$ we denote the standard basis of $\boldsymbol{H}$ over $\boldsymbol{R}$. Every matrix $A\in GL_n(\boldsymbol{H})$ is conjugate to an upper triangular matrix B. The diagonal entries of B are the eigenvalues of A and they are unique up to conjugacy in $\boldsymbol{H}^*$. The special linear group $SL_n(\boldsymbol{H})$ consists of all matrices $A\in GL_n(\boldsymbol{H})$ for which the product of all n eigenvalues has norm 1. Of course this is equivalent to the requirement that the Dieudonné's determinant of A is equal to 1. $SL_n(\boldsymbol{H})$ is a connected closed Lie subgroup of $GL_n(\boldsymbol{H})$ of co-dimension 1. This group is almost simple (and perfect) with center of order 2. Kursov [17] stated that every element of $SL_n(\boldsymbol{H})$ is a commutator in that group. But the proof of his Lemma 2 contains two mistakes, which we were not able to fix.

First, he claims that if A is a diagonal matrix, say $A=\mathrm{diag}(\alpha_1, \cdots, \alpha_n)$, then its characteristic matrix tI_n-A has only one invariant factor (by t we denote an indeterminate which commutes with quaternions). For the definition of invariant factors we refer to [3] or [16]. A counterexample is provided by the quaternionic matrix $A=\mathrm{diag}(i, j, k)$. Indeed, by performing elementary row and column transformations, we get

$$\begin{pmatrix} t-j & 0 \\ 0 & t-k\end{pmatrix}\to\begin{pmatrix} t-j & t-k \\ 0 & t-k\end{pmatrix}\to\begin{pmatrix} t-j & j-k \\ 0 & t-k\end{pmatrix}\to\begin{pmatrix} t-j & j-k \\ 0 & -2(t-k)\end{pmatrix}$$

$$\to\begin{pmatrix} t-i & 1 \\ 0 & (t-k)(j-k) \end{pmatrix}\to\begin{pmatrix} 1 & 0 \\ 0 & (t-k)(j-k)(t-j) \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & (t-k)(t+k)(j-k) \end{pmatrix}$$
$$\to\begin{pmatrix} 1 & 0 \\ 0 & (t-k)(t+k) \end{pmatrix},$$

and consequently A has two (non-constant) invariant factors $t-i$ and $(t-k)(t+k)=t^2+1$.

Second, he claims that, if $R=\boldsymbol{H}[t]$ and if the quaternions $\beta_1, \cdots, \beta_n$ are distinct, we have an isomorphism of right R-modules

$$R/(t-\beta_1)\cdots(t-\beta_n)R \cong R/(t-\beta_1)R\oplus\cdots\oplus R/(t-\beta_n)R\,.$$

A counterexample: Let $n=3$, and let the β's be the quaternionic units i, j, k. Then the module on the right hand side is annihilated by t^2+1 while the one on the left hand side is not.

Kursov's paper is quoted in [8], but the above mistakes apparently have not been noticed so far. Hence the main result of [17], which deals with general division rings remains in doubt. We shall now prove Kursov's claim about $SL_n(\boldsymbol{H})$.

Theorem 3. *Every element of $SL_n(\boldsymbol{H})$ is a commutator.*

Proof. For $n=1$ this follows from [10]. We show first that a Jordan block $J_m(\lambda)$, of size m, with eigenvalue $\lambda\in\boldsymbol{C}$, $|\lambda|=1$, is a commutator in $SL_m(\boldsymbol{H})$. If $\lambda\neq-1$ choose $\mu\in\boldsymbol{C}$ such that $\lambda\mu^2=-1$. If $X=jJ_m(\mu)$ and $Y=kI_m$ then since $\bar{\mu}\neq\mu$ we have

$$XYX^{-1}Y^{-1}=-J_m(\bar{\mu})\,J_m(\mu)^{-1}\sim J_m(-\bar{\mu}^2)=J_m(\lambda)\,,$$

where $\sim$ is the similarity relation. In the case $\lambda=-1$ we take $X=jJ_m(i)$ and $Y=jI_m$. Then

$$XYX^{-1}Y^{-1}=J_m(-i)\,J_m(i)^{-1}\sim J_m(-1)=J_m(\lambda)\,.$$

Next let $X\in SL_n(\boldsymbol{H})$ be arbitrary. By making use of the Jordan canonical form, see [16], and in view of the previous remark we may assume that X has the form

$$X=J_{m_1}(\lambda_1)\oplus\cdots\oplus J_{m_s}(\lambda_s),\ m_1+\cdots+m_s=n\ ;$$

where $\lambda_r\in\boldsymbol{C}$ and $|\lambda_r|\neq1$, $1\leq r\leq s$.

Let $(\mu_1, \cdots, \mu_n)$ be the main diagonal of X and let

$$Y=\nu\ \mathrm{diag}(1, \mu_1, \mu_1\mu_2, \cdots, \mu_1\mu_2\cdots\mu_{n-1})\,,$$

where $\nu\in\boldsymbol{C}$ is chosen so that $\nu^2\mu_1\mu_2\cdots\mu_n=1$. Since $X\in SL_n(\boldsymbol{H})$, we have $|\nu|=1$. The main diagonal of XY is

$$\nu\cdot(\mu_1, \mu_1\mu_2, \cdots, \mu_1\mu_2\cdots\mu_n)\,.$$

Since $\mu_r=\lambda_1$ for $1\leq r\leq m_1$ and $|\lambda_1|\neq 1$, the first m_1 diagonal entries of XY are distinct. Similarly the next m_2 diagonal entries of XY are distinct, etc. These facts and the obvious block-diagonal structure of XY imply that XY is similar to the diagonal matrix Z having the same diagonal entries as XY.

Since $\nu\mu_1\mu_2\cdots\mu_n=\nu^{-1}=\bar{\nu}$, we have $Z\sim Y$. Hence $XY\sim Z\sim Y$, and so X is a commutator in $GL_n(\boldsymbol{H})$, and consequently also in $SL_n(\boldsymbol{H})$. QED.

In the last theorem we show that two infinite series of real semisimple Lie groups have property (C). If G is a Lie group then by G_0 we denote the identity component of G.

Theorem 4. *If G is one of the groups $Sp(p, q)$, $p\geq q\geq 1$, or the groups $SO(p, 1)_0$, $p\equiv 0$ or 3 (mod 4), then every element of G is a commutator.*

Proof. Let $a\in G$ be arbitrary. Let $\mathfrak{g}$ be the Lie algebra of G and exp: $\mathfrak{g}\to G$ the exponential map of G. We need the following two facts. First, for the groups mentioned in the theorem the exponential map is surjective, see [5] and [18]. Hence we can write $a=\exp X$ for some $X\in\mathfrak{g}$. Second, for the same groups and for any $X\in\mathfrak{g}$ there exists an inner automorphism α of $\mathfrak{g}$ such that $\alpha(X)=-X$, see [6]. Thus, if Ad is the adjoint representation of G, we have $\alpha=\mathrm{Ad}\, b$ for some $b\in G$. Hence, we have

$$\begin{aligned} a &= \exp X = \exp(X/2)\cdot\exp(X/2) = \exp(\alpha(-X/2))\cdot\exp(X/2) \\ &= \exp(\mathrm{Ad}\, b\,(-X/2))\cdot\exp(X/2) = b\,\exp(-X/2)\cdot b^{-1}\cdot\exp(X/2)\,, \end{aligned}$$

i.e., a is a commutator in G. QED.

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