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ON SOME MAXIMAL GALOIS COVERINGS
OVER AFFINE AND PROJECTIVE PLANES

TETSUO NAKANO and KEN-ICHI TAMAI

(Received March 26, 1995)

Introduction

In Namba [3; Chapter 1], various examples of Galois coverings over affine and projective planes are constructed. Among them, the Galois coverings over $C^2$ with branch locus $B_3 := \{(v,w) \in C^2| v^3 = w^2\}$ are studied in detail ([3; pp.43–52]), and as an application, the existence or non-existence of some maximal Galois coverings over $P^2$ with branch locus $\overline{B_3} \cup l_{\infty}$ is shown, where $\overline{B_3}$ is the projective closure of $B_3$ and $l_{\infty}$ is the infinite line ([3; Proposition 1.3.27, 1.3.29]).

In this note, we extend his results to the Galois coverings over $C^2$ with branch locus $B_q := \{(v,w) \in C^2| v^q = w^2\}$, where $q$ is a positive odd integer, under the condition that the maximal Galois group $G(C^2,eB_q)$ of $(C^2,eB_q)$ is finite. It turns out that we have five cases in all, three cases of which appear in [3; p.43]. As an application, we determine when there exists the maximal Galois coverings over $P^2$ with branch locus $\overline{B_q} \cup l_{\infty}$, and also describe the explicit structure of $G(P^2,e\overline{B_q} + ml_{\infty})$ in these cases.

This note is organized as follows. In Section 1, we review some general facts from the Galois theory of branched coverings. We begin Section 2 with giving a simple presentation of $G(C^2,eB_q)$ in Proposition 2.1, from which we can construct the maximal abelian coverings of $(C^2,eB_q)$ easily (Proposition 2.3). Using this presentation, we determine when $G(C^2,eB_q)$ is finite in Theorem 2.4 according to Coxeter-Moser [1]. Then we give an explicit structure of $G(C^2,eB_q)$ in the cases where $G(C^2,eB_q)$ is finite, which is our main result (Theorem 2.6). When $G(C^2,eB_q)$ is infinite, we give a sufficient condition for $G(C^2,eB_q)$ to be unsolvable (Corollary 2.10). In Section 3, we describe the explicit structure of the maximal Galois group $G(P^2,e\overline{B_q} + ml_{\infty})$ and determine when the maximal Galois covering of $(P^2,e\overline{B_q} + ml_{\infty})$ exists in the cases where $G(C^2,eB_q)$ is finite (Proposition 3.1, Corollary 3.3).

We note that the isomorphisms given in Theorem 2.6 are more or less known in abstract form (cf. Coxeter-Moser [1;6.7], Namba [3;p.50]) and our contribution is the explicit description of these isomorphisms, which is essentially used in Section 3.

The case where $q$ is even seems more complicated, and will be studied in the forthcoming paper under the same title. We note that part of this note is taken
from the master thesis Tamai [6].

**Notations.** (1) For a group $G$ and $g_i \in G$ ($1 \leq i \leq n$), we denote by $\langle g_1, g_2, \ldots, g_n \rangle$ the subgroup of $G$ generated by $g_i$'s and by $N\langle g_1, g_2, \ldots, g_n \rangle$ the smallest normal subgroup containing $g_i$'s. $\mathbb{Z}[G]$ denotes the group ring of $G$ over the ring of integers $\mathbb{Z}$. For a ring $A$, $M_n(A)$ is the set of all square matrices of degree $n$ whose entries belong to $A$, and $I_n \in M_n(A)$ denotes the identity matrix.

(2) Given two groups $H$, $N$, we denote by $N \rtimes H$ the semi-direct product of $H$ and $N$. Here, $\alpha : H \to \text{Aut}(N)$ is a homomorphism and the product is defined by $(h_1, n_1)(h_2, n_2) := (h_1 h_2, n_1 n_2)$, where $h_i \in H$, $n_i \in N$, $n_1^2 = \alpha(h_2)(n_1)$.

(3) For a pair $(M, D)$ of a complex manifold $M$ and an effective divisor $D$ on $M$, we denote by $G(M, D)$ the maximal Galois group of $(M, D)$ and by $\pi : X(M, D) \to M$ the maximal Galois covering of $(M, D)$ if it exists (see Section 1 for the definitions).

1. Summary of the Galois theory of branched coverings

In this section, we summarize some general facts on the Galois theory of branched coverings in the category of complex analytic spaces, following Namba [3; Chapter 1].

Let $M$ be a complex manifold and $D = \sum e_i D_i$ ($e_i > 0$) an effective divisor on $M$, where $D_i$ is an irreducible component of $D$. We set $B := C_1 \cup D_2 \cup \cdots \cup D_x$. Let $\pi : X \to M$ be a branched covering over $M$, where $X$ is a normal irreducible reduced complex analytic space, $R_\pi \subset X$ the ramification locus of $\pi$, and $B_\pi \subset M$ the branch locus of $\pi$. For an irreducible hypersurface $C \subset X$, we denote by $e_C(\pi)$ the ramification index of $\pi$ at $C$ (cf. Namba [3; p.10]). We say that $\pi$ branches at $D$ (resp. at most at $D$) if the following three conditions are satisfied:

2. $R_\pi = \pi^{-1}(B)$ (resp. $R_\pi < \pi^{-1}(B)$).
3. For any irreducible hypersurface $C \subset X$ such that $\pi(C) = D_j$, $e_C(\pi) = e_j$ (resp. $e_C(\pi) < e_j$).

We fix a base point $p_0 \in M - B$ and take a point $p_j \in D_j - \text{Sing}(B)$, where $\text{Sing}(B)$ is the singular locus of $B$. Take a local coordinate system $(z_1, z_2, \ldots, z_n)$ defined on a neighborhood $U$ of $p_j$ such that (1) $p_j$ corresponds to the origin, (2) $B \cap U$ is given by $z_n = 0$. Take a loop $\delta_j$ around $D_j$ in $U$ defined by $\{(0,0,\ldots,0, e^{2\pi i t}/n) : 0 \leq t \leq 1\}$, where $\epsilon > 0$ is sufficiently small, and take a path $\omega_j$ in $M - B$ from $p_0$ to $q_j = (0,0,\ldots,\epsilon) \in U$. We define $\gamma_j = \omega_j^{-1} \delta_j \omega_j$, which is a loop around $D_j$ starting from $p_0$.

Let $J$ be the smallest normal subgroup $N\langle \gamma_1^{e_1}, \ldots, \gamma_s^{e_s} \rangle$ in $\pi_1(M - B, p_0)$ containing $\gamma_j^e$ ($1 \leq j \leq s$), which is determined independent of the choices of $\gamma_j$'s (we confuse a loop $\gamma_j$ with its homotopy class in $\pi_1(M - B, p_0)$). We set $G(M, D) := \pi_1(M - B, p_0)/J$. 

and call this the maximal Galois group of \((M,D)\) in this note. Then we have a Galois correspondence of the following type:

**Theorem 1.1** (cf. Namba [3; Theorem 1.3.9]). (1) There is a bijective map \(\Phi\) from the set \(\{f: X \to M | f\) is a finite Galois covering which branches at most at \(D\}\) \(\simeq\) to the set \(\{K \subset G(M,D) | K\) is a normal subgroup of finite index\}, where \(\simeq\) means the isomorphism between branched coverings over \(M\). \(\Phi(f)\) is defined by \(\Phi(f) = f^*(\pi_1(X - f^{-1}(B), q_0)) \mod J\), where \(q_0 \in X - f^{-1}(B)\) is a base point over \(p_0\) and \(f^*: \pi_1(X - f^{-1}(B), q_0) \to \pi_1(M - B, p_0)\) is the injective homomorphism induced by \(f\).

(2) This correspondence \(\Phi\) satisfies the following properties:

(a) \(G_f \simeq G(M,D)/\Phi(f)\), where \(G_f\) denotes the covering transformation group of \(f\).

(b) \(f_1\) dominates \(f_2\) if and only if \(\Phi(f_1) \subseteq \Phi(f_2)\). Here we say that a branched covering \(f_1: X_1 \to M\) dominates another covering \(f_2: X_2 \to M\) if there exists a surjective holomorphic map \(g: X_1 \to X_2\) such that \(f_2 \circ g = f_1\).

(3) \(f\) branches at \(D\) if and only if the order of \([\gamma_j]\) is \(e_j\) \((1 \leq j \leq s)\), where \([\gamma_j]\) \(\in G(M,D)/\Phi(f)\) denotes the coset containing \(\gamma_f\).

We call the universal Galois covering among the branched coverings which branch at most at \(D\) the maximal Galois covering of \((M,D)\) if it exists. More precisely,

**DEFINITION 1.2.** Let \(\pi: X \to M\) be a Galois covering which branches at \(D\). We say that \(\pi\) is the maximal Galois covering of \((M,D)\) if \(\pi\) dominates any branched covering which branches at most at \(D\).

Our next task is to give a criterion for the existence of the maximal Galois covering in terms of \(G(M,D)\). Consider a point \(p \in \text{Sing}(B)\) and take a sufficiently small neighborhood \(W\) of \(p\) in \(M\) which is an open ball with respect to a local coordinate system with center \(p\). Let \(i_*: \pi_1(W - (W \cap B), p_0') \to \pi_1(M - B, p_0)\) be the homomorphism induced by the inclusion \(i: W - (W \cap B) \to M - B\), where \(p_0' \in W - (W \cap B)\) is a base point, and let \(g: \pi_1(M - B, p_0) \to G(M,D)\) be the natural surjection. We have a composition map \(g \circ i_*: \pi_1(W - (W \cap B), p_0') \to G(M,D)\). Consider the following condition on a subgroup \(K \subset G(M,D)\):

**CONDITION 1.3.** For any point \(p \in \text{Sing}(B)\), \((g \circ i_*)^{-1}(K)\) is of finite index in \(\pi_1(W - (W \cap B), p_0')\).

We set \(\tilde{K} := \cap K\), where \(K\) runs over all the subgroups of \(G(M,D)\) satisfying Condition 1.3. \(\tilde{K}\) is a normal subgroup of \(G(M,D)\).

**Theorem 1.4** (cf. Namba [3; Theorem 1.3.10]). There exists the maximal
Galois covering $\pi: X(M,D) \to M$ of $(M,D)$ if and only if the following two conditions are satisfied:

1. $\bar{K}$ satisfies Condition 1.3.

2. $\text{ord}([\gamma_j]) = e_j$ $(1 \leq j \leq s)$, where $[\gamma_j] \in G(M,D)/\bar{K}$ means the coset containing $\gamma_j$.

In this case, we have the following:

(a) $G_\pi \cong G(M,D)/\bar{K}$.

(b) $X(M,D)$ is simply-connected.

Assume that $G(M,D)$ is finite. Then we have the following corollary to Theorem 1.4, since Condition 1.3 is satisfied for any subgroup of $G$ and hence $\bar{K} = \{1\}$ in this case.

**Corollary 1.5.** If $G(M,D)$ is finite, then there exists the maximal Galois covering $\pi: X(M,D) \to M$ of $(M,D)$ if and only if $\text{ord}([\gamma_j]) = e_j$ $(1 \leq j \leq s)$, where $[\gamma_j] \in G(M,D)$ is the coset containing $\gamma_j$. In this case, $G_\pi \cong G(M,D)$.

**2. Calculation of $G(C^2, eB_q)$**

For an integer $q > 0$, we set $B_q = \{(v,w) \in C^2 | w^2 = v^q\}$. Suppose that $q$ is odd. Let $\gamma$ be a loop around $B_q$ in $C^2 - B_q$ as in Section 1 and we define $G(e; q) := G(C^2, eB_q) = \pi_1(C^2 - B_q, p_0)/N\langle \gamma \rangle$. Suppose that $q = 2r$ is even. We set $B_q^1 = \{(v,w) \in C^2 | w = v^r\}$ and $B_q^2 = \{(v,w) \in C^2 | w = -v^r\}$ so that $B_q = B_q^1 \cup B_q^2$. Let $\gamma_1$ be a loop around $B_q^1$ ($i = 1, 2$) and we set $G(e_1, e_2; q) := G(C^2, e_1B_q^1 + e_2B_q^2) = \pi_1(C^2 - B_q, p_0)/N\langle \gamma_1, \gamma_2 \rangle$. Then $G(e; q)$ and $G(e_1, e_2; q)$ have the following simple presentations.

**Proposition 2.1.**

\[
G(e; q) \cong \langle a, b | a^q b = b a^q b \rangle \quad \text{if } q \text{ is odd.}
\]

\[
G(e_1, e_2; q) \cong \langle a, b | a^{e_1} b^{e_2} = b^{e_2} a^{e_1} b^{e_2} a^{e_1} \rangle \quad \text{if } q \text{ is even.}
\]

**Proof.** Put $S^3 := \{(v,w) \in C^2 | |v|^2 + |w|^2 = 1\}$, $T^2 := \{(v,w) \in C^2 | |v| = |w| = \frac{1}{\sqrt{2}}\}$, and $k(2,q) := \left\{ \left( \frac{1}{\sqrt{2}} e^{4\pi i s}, \frac{1}{\sqrt{2}} e^{2\pi i q s} \right) \in C^2 | 0 \leq s \leq 1 \right\}$ ($i = \sqrt{-1}$). Then $k(2,q) \subset T^2$ is the torus knot (or link) of type $(2,q)$. Let $C(k(2,q)) = \{(v,tw) \in C^2 | t \geq 0, (v,w) \in k(2,q)\}$ be the cone over $k(2,q)$. Since $(C^2, B_q)$ is homeomorphic to $(C^2, C(k(2,q)))$, it follows that $\pi_1(C^2 - B_q)$ is isomorphic to $\pi_1(S^3 - k(2,q))$ (we omit the base point of the fundamental group). Now, we take the Wirtinger generators $x_1, x_2, \cdots, x_q$ as in the
figure above and obtain the Wirtinger presentation of the knot (or link) group $\pi_1(S^3-k(2,q))$ as follows (cf. Stillwell [5;4.2.3]):

$$\pi_1(S^3-k(2,q)) \cong \langle x_1, x_2, \ldots, x_q | x_1 x_q = x_2 x_1 = x_3 x_2 = \cdots = x_q x_1 x_2, x_q x_1 x_2 \rangle.$$  

From this presentation, we eliminate $x_3, \ldots, x_q$ and get the following presentations:

$$\pi_1(S^3-k(2,q)) \cong \begin{cases} 
\langle x_1, x_2 | x_1 x_2 \cdots x_2 x_1 = x_2 x_1 \cdots x_2, x_2 \rangle & (q: \text{odd}) \\
\langle x_1, x_2 | x_1 x_2 \cdots x_1 x_2 = x_2 x_1 \cdots x_2, x_2 \rangle & (q: \text{even}) 
\end{cases}$$

If $q$ is odd, then we can take $x_1$ as $\gamma$. If $q$ is even, we can take $x_i$ as $\gamma_i$ ($i=1,2$). Hence $\pi_1(C^2-B_q)/\langle \gamma \rangle$ (or $\pi_1(C^2-B_q)/\langle \gamma_1 \gamma_2 \rangle$) has the desired presentation.

\textsc{Remark 2.2.} The groups that have the same presentations as in Proposition 2.1 appear in Coxeter-Moser [1; 6.7], in which $G(e^e q)$ is denoted by $e_i [q] e_2$ ($q: \text{even}$) and $G(e q)$ by $e [q] e$ ($q: \text{odd}$). These groups also occur in the theory of regular complex polygons.

We recall that an abelian covering $\pi : X \to M$ of a complex manifold $M$ which branches at $D$ is called maximal if $\pi$ dominates any abelian covering of $M$ which branches at most at $D$. The maximal abelian covering of $C^2$ which branches at $e B_q$ ($q: \text{odd}$) or $e B_q$ ($q: \text{even}$) can be obtained easily as follows.

\textbf{Proposition 2.3.} (1) Assume that $q$ is odd. Set $X := \{(u,v,w) \in \mathbb{C}^3 | u^e + v^q - w^2 \}$
and define \( \pi : X \to C^2 \) by \( \pi((u,v,w)) = (v,w) \). Then \( \pi : X \to C^2 \) is the maximal abelian covering of \( C^2 \) which branches at \( eB^q \). \( G_\pi \) is isomorphic to \( \mathbb{Z}/e\mathbb{Z} \).

(2) Assume that \( q = 2r \) is even. For a pair of positive integers \( e_1, e_2 \), set \( Y = \{ (u_1, u_2, v) \in C^3 \mid u_1^r + 2v - u_2^r = 0 \} \) and define \( \rho : Y \to C^2 \) by \( \rho((u_1, u_2, v)) = (v, u_1^r + v) \). Then \( \rho : Y \to C^2 \) is the maximal abelian covering of \( C^2 \) which branches at \( e_1B_4^1 + e_2B_4^2 \). \( G_\rho \) is isomorphic to \( \mathbb{Z}/e_1\mathbb{Z} \oplus \mathbb{Z}/e_2\mathbb{Z} \).

Proof. (1) \( X \) is a normal irreducible surface, and it is easy to see that \( \pi : X \to C^2 \) is a cyclic covering of degree \( e \) which branches at \( eB^q \). We show that \( \pi : X \to C^2 \) is the maximal abelian covering which branches at \( eB^q \). Let \( \mu : Z \to C^2 \) be any abelian covering which branches at most at \( eB^q \). By Theorem 1.1, it is enough to show \( \Phi(\pi) \subseteq \Phi(\mu) \). For a group \( G \), we set \( G^{ab} = G/G' \), where \( G' \) is the commutator subgroup of \( G \).

Since \( G(e;q)_{ab} \simeq \langle a, b \mid a^e = 1, ab \cdots aba = ba \cdots bab, ab = ba \rangle \simeq \mathbb{Z}/e\mathbb{Z} \),

the index of \( G(e;q) \) in \( G(e;q)_{ab} \) is \( e \). Since \( G(e;q)_{ab} / \Phi(\pi) \simeq G_\pi \simeq \mathbb{Z}/e\mathbb{Z} \), we conclude \( \Phi(\pi) = G(e;q)' \).

Now, \( G(e;q)_{ab} / \Phi(\mu) \simeq G_\mu \) is abelian and hence \( \Phi(\mu) \supseteq G(e;q)' = \Phi(\pi) \).

(2) Set \( X_1 = \{ (u_1, v, w) \in C^3 \mid u_1^r = w - v \} \), \( X_2 = \{ (u_2, v, w) \in C^2 \mid u_2^r = w + v \} \), and define \( \pi_i : X_i \to C^2 \) by \( \pi_i((u_i, v, w)) = (v, w) \) (\( i = 1, 2 \)). \( \pi_i \) is a cyclic covering of degree \( e_i \) over \( C^2 \). We form the fibred product \( X_1 \times X_2 \) of \( \pi_1 \) and \( \pi_2 \), which is isomorphic to \( Y \) over \( C^2 \). \( Y \) is an abelian covering over \( C^2 \) which branches at \( e_1B_4^1 + e_2B_4^2 \) with Galois group isomorphic to \( \mathbb{Z}/e_1\mathbb{Z} \oplus \mathbb{Z}/e_2\mathbb{Z} \). On the other hand, it is easy to see \( G(e_1, e_2; q^{ab}) \simeq \mathbb{Z}/e_1\mathbb{Z} \oplus \mathbb{Z}/e_2\mathbb{Z} \), from which it follows that \( Y \) is the maximal abelian covering which branches at \( e_1B_4^1 + e_2B_4^2 \) as in (1).

In the rest of this note, we are concerned with the explicit structure of \( G(e;q) \), assuming \( q \) is odd. The following theorem is essentially due to Coxeter-Moser [1].

**Theorem 2.4.** Let \( e \geq 2 \) and \( q \geq 3 \) be integers with \( q \) odd. Then \( G(e;q) \) is a finite group if and only if \( e = 2 \) or \( (e, q) = (3,3), (4,3), (5,3), (3,5) \).

We need a lemma for the proof of Theorem 2.4.

**Lemma 2.5** (cf. Coxeter-Moser[1:p.79]). \( G(2, e; 2q) \) contains a subgroup of index 2 which is isomorphic to \( G(e;q) \).

Proof of Lemma (2.5). By adjoining a new letter \( c := aba \) to \( G(2, e; 2q) \)
Let \( H := \langle b, c \rangle \) be a subgroup of \( G = G(2, e; 2q) \) generated by \( b \) and \( c \). Then the index \([G:H]\) of \( H \) in \( G \) is 2. Indeed, any word \( w = w(a(b,c)) \in G \) can be rewritten as \( w = \varphi(b,c) \) or \( a\varphi(b,c) \), where \( \varphi(b,c) \) is word not containing \( a \), since \( a^2 = 1 \), \( ab = ca \) and \( ba = ac \). Hence we have \([G:H] \leq 2\). Assume \( a \in H \). We have \( G(2,e;2q) \cong F(a,b,c) / N \), where \( F(a,b,c) \) is a free group generated by \{a, b, c\} and \( Y = \{b^q, bcb^{-1}c^{-1}b^{-1}c^{-1}, a^2, abac^{-1}\} \) is the relation set. Then \( a = \eta(b,c)\lambda \) in \( F(a,b,c) \), where \( \lambda \) is a finite product of conjugates of words or inverses of words in \( Y \). This gives a contradiction since the sum of the exponents of \( a \) in \( \lambda \) is even. Therefore \( a \notin H \) and \([G:H] = 2\).

Next, we calculate a presentation of \( H \) according to Johnson [2; Chapter 9]. Let \( X = \{a,b,c\} \) be a generator set of \( G \) and \( U = \{1, a\} \) a Schreier transversal for \( H \) in \( G \). The \( B\bar{R} \)-table is given as follows:

**Table 1**

<table>
<thead>
<tr>
<th></th>
<th>b</th>
<th>c</th>
<th>a</th>
<th>( b^q )</th>
<th>( bcb\ldots b^{-1}c^{-1} )</th>
<th>( a^2 )</th>
<th>( abac^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b</td>
<td>c</td>
<td>1</td>
<td>( b^q )</td>
<td>( bcb\ldots b^{-1}c^{-1} )</td>
<td>( a^2 )</td>
<td>( abac^{-1} )</td>
</tr>
<tr>
<td>a</td>
<td>( ab^{-1}a )</td>
<td>( ac^{-1} )</td>
<td>( a^2 )</td>
<td>( ab^{-1}a )</td>
<td>( ab\ldots c^{-1}a^{-1} )</td>
<td>( a^2 )</td>
<td>( a^2bac^{-1}b^{-1} )</td>
</tr>
</tbody>
</table>

Here the rows are indexed by \( U \) and the columns by \( (X,Y) \). In the left-hand half of the table, the \((u,x)-entry\) is \( uxu^{-1} \), where \( u \in U \) is the element which belongs to the same coset modulo \( H \) as \( uX \), and in the right-hand half of the table, the \((u,y)-entry\) is \( yu^{-1} \). Hence the \( B\bar{S} \)-table is given as follows:

**Table 2**

<table>
<thead>
<tr>
<th></th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( d_3 )</th>
<th>( d_4 )</th>
<th>( d_5 )</th>
<th>( d_6 )</th>
<th>( d_7 )</th>
<th>( d_8 )</th>
</tr>
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<tbody>
<tr>
<td>( d_1 )</td>
<td>( d_3 )</td>
<td>( d_4 )</td>
<td>( d_5 )</td>
<td>( d_6 )</td>
<td>( d_7 )</td>
<td>( d_8 )</td>
<td>( d_9 )</td>
<td>( d_{10} )</td>
</tr>
</tbody>
</table>

Here \( d_1 = b \), \( d_2 = ab^{-1} \), \( d_3 = c \) etc., and the elements in the right-hand half of the \( B\bar{S} \)-table are those in the \( B\bar{R} \)-table rewritten in terms of \( d_i \)'s. It follows that \( H \) is presented as \( \langle d_i \mid 1 \leq i \leq 5 \rangle \) eight relations in the \( B\bar{S} \)-table). By eliminating \( d_2 \), \( d_4 \) and \( d_5 \), we have \( H \cong \langle d_1, d_2, d_3 \rangle \) where \( d_1d_3 = d_5 = 1 \), \( d_1d_3d_1 \ldots d_1 = d_3d_1d_3 \ldots d_3 \).
Since $Q^{-1}d_1Q = d_3$, where $Q = d_1d_3d_1 \cdots d_3d_1d_3 \cdots d_3$, we conclude $H \cong G(e; q)$.

Proof of Theorem 2.4. Assume that $G(e; q)$ is finite. By Lemma 2.5, $G(2, e; 2q)$ is also finite. We recall that the polyhedral group $P(x, y, z) := \langle a, b | a^x = b^y = (ab)^z = 1 \rangle$ $(x, y, z \geq 2)$ is finite if and only if $(x, y, z) = (2, 2, 2), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ and their permutations (cf. Coxeter-Moser [1; 6.4]). Since $P(2, e; q)$ is a homomorphic image of $G(2, e; 2q)$, we have $(e, q) = (2, q), (3, 3), (4, 3), (5, 3), (3, 5)$, where $q$ is odd and $\geq 3$. The converse part of the proof follows from Theorem 2.6 below, or Coxeter-Moser [1; p. 79].

The following theorem is the main ingredient of this note. We note that the isomorphisms $\varphi_2$, $\varphi_4$, and $\varphi_5$ for $q = 3$ in the theorem are given in Namba [3; p. 50] in abstract form, and $\varphi_2$ is found in Coxeter-Moser [1; p. 78]. We make a detailed calculation of group presentations for this theorem since it gives the explicit form of $\varphi_i$, which is essentially used in section 3.

We denote by $C_n$ the cyclic group of order $n$, by $D_{2q} := \langle x, y | x^q = y^2 = 1, y^{-1}xy = x^{-1} \rangle$ the dihedral group of order $2q$, by $Q_8 := \langle x, y | x^2 = y^2, x^4 = 1, y^{-1}xy = x^{-1} \rangle$ the quaternion group of order 8, and by $SL(2, \mathbb{Z}_5)$ the special linear group of degree 2 whose entries belong to $\mathbb{Z}_5 = \mathbb{Z}/n\mathbb{Z}$.

**Theorem 2.6.** We have the following isomorphisms:

1. $\varphi_1 : G(2; q) \cong D_{2q}$; $\varphi_1(a) = y$, $\varphi_1(b) = y^{-1}x$.
2. $\varphi_2 : G(3; 3) \cong SL(2, \mathbb{Z}_3)$; $\varphi_2(a) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\varphi_2(b) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$
3. $\varphi_3 : G(4; 3) \cong (Q_8 \times C_3) \times C_3$; $\varphi_3(a) = s \cdot t \cdot 1$, $\varphi_3(b) = s \cdot t \cdot x$ ($C_3 = \langle t \rangle$, $c_4 = \langle s \rangle$).
4. $\psi_3 : G(4; 3) \cong SL(2, \mathbb{Z}_3) \times C_4$; $\psi_3(a) = s \cdot I_2$, $\psi_3(b) = s \cdot \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ ($C_4 = \langle s \rangle$). $\alpha$, $\beta$, $\gamma$ are described in the proof.
5. $\varphi_4 : G(5; 3) \cong SL(2, \mathbb{Z}_5) \times C_5$; $\varphi_4(a) = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} w^3$, $\varphi_4(b) = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} w^3$ ($C_5 = \langle w \rangle$).
6. $\varphi_5 : G(3; 5) \cong SL(2, \mathbb{Z}_3) \times C_5$; $\varphi_5(a) = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} w^2$, $\varphi_5(b) = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} w^2$ ($C_3 = \langle w \rangle$).

Proof. In this proof, we set $G := G(e; q)$ for short. (1) is clear from the definition of $D_{2q}$.

Set $s = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \in SL(2, \mathbb{Z}_3)$. Then we have $s^3 = 1$, $sts = tst$. Since $\{s, t\}$
generates $\text{SL}(2,\mathbb{Z}_3)$ we have a surjective homomorphism $\varphi_2 : G \to \text{SL}(2,\mathbb{Z}_3)$ defined by $\varphi_2(a) = s, \varphi_2(b) = t$. Since $\text{ord}(\text{SL}(2,\mathbb{Z}_3)) = 24$, it is enough to show $\text{ord}(G) = 24$.

Consider the following exact sequence

$$1 \to G' \to G \to C_3 \to 1,$$

where $f(a) = f(b) = w, C_3 = \langle w \rangle$, and $G'$ is the commutator subgroup of $G$. We take $U := \{1, a, a^2\}$ as a Schreier transversal for $G'$ in $G$. Then the $B\hat{R}$ and $B\hat{S}$ tables for $G'$ are given as follows:

### Table 3

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>$abab^{-1}a^{-1}b^{-1}$</th>
<th>$a^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$ba^{-1}$</td>
<td>$abab^{-1}a^{-1}b^{-1}$</td>
<td>$a^3$</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>$aba^{-2}$</td>
<td>$a^2bab^{-1}a^{-1}b^{-1}a^{-1}$</td>
<td>$a^3$</td>
</tr>
<tr>
<td>$a^2$</td>
<td>$a^3$</td>
<td>$a^2b$</td>
<td>$a^2bab^{-1}a^{-1}b^{-1}a^{-2}$</td>
<td>$a^3$</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th></th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$c_3c_1c_4^{-1}c_2^{-1}$</td>
<td>$c_1$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$c_4c_2^{-1}c_1^{-1}c_3^{-1}$</td>
<td>$c_1$</td>
</tr>
</tbody>
</table>

Hence $G' \simeq \langle c_i \mid 1 \leq i \leq 4 \rangle /$ six relations in the $B\hat{S}$-table $\simeq \langle c_2, c_3 \mid c_3 = c_2c_3c_2, c_2 = c_3c_2c_3 \rangle$, which is isomorphic to $\mathbb{Q}_8$ by the correspondence $c_2 \to x, c_3 \to y$. Thus $\text{ord}(G') = 8$ so that $\text{ord}(G) = 24$.

(3) We take a Schreier transversal $U := \{1, a, a^2, a^3\}$ for $G'$ in $G$. The $B\hat{R}$- and $B\hat{S}$-tables are given as follows:

### Table 5

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>$abab^{-1}a^{-1}b^{-1}$</th>
<th>$a^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$ba^{-1}$</td>
<td>$abab^{-1}a^{-1}b^{-1}$</td>
<td>$a^4$</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>$aba^{-2}$</td>
<td>$a^2bab^{-1}a^{-1}b^{-1}a^{-1}$</td>
<td>$a^4$</td>
</tr>
<tr>
<td>$a^2$</td>
<td>1</td>
<td>$a^2ba^{-3}$</td>
<td>$a^3bab^{-1}a^{-1}b^{-1}a^{-2}$</td>
<td>$a^4$</td>
</tr>
<tr>
<td>$a^3$</td>
<td>$a^4$</td>
<td>$a^3b$</td>
<td>$a^2bab^{-1}a^{-1}b^{-1}a^{-3}$</td>
<td>$a^4$</td>
</tr>
</tbody>
</table>
Hence $G' \simeq \langle c_i \mid 1 \leq i \leq 5 \rangle$ eight relations in the $B\delta$-table $\simeq \langle c_2, c_4 \mid c_2 = c_2 c_2 c_4, c_4 = c_2 c_4 c_2 \rangle$. Next, we take $U := \{1, c_2, c_2^2\}$ as a Schreier transversal for $(G')$ in $G'$ and the $B\delta$- and $B\delta$-tables are given as follows: (we set $c_2 = p, c_4 = q$ in the $B\delta$-table)

### Table 6

<table>
<thead>
<tr>
<th></th>
<th>$c_2$</th>
<th>$c_3 c_2^{-1} c_2^{-1}$</th>
<th>$c_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Hence $G' \simeq \langle d_i \mid 1 \leq i \leq 4 \rangle$ six relations in the $B\delta$-table $\simeq \langle d_3, d_4 \mid d_3^2 d_4^2 = 1, d_3 = d_4 d_3 d_4 \rangle$, which is isomorphic to $Q_8$ by the correspondence $d_3 \rightarrow x, d_4 \rightarrow y$. We have $c_2^3 = d_2^2 \in (G') \simeq Q_8$ and hence $\text{ord}(c_2) = 6$. Set $H := \langle c_2^2 \rangle \simeq C_3$. Then $G' = (G') \times H \simeq Q_8 \times C_3$, where $\alpha : C_3 = \langle t \rangle \rightarrow \text{Aut}(Q_8)$ is given by $\alpha(t)(x) = x't = yx^2$ and $\alpha(t)(y) = y' = yx$ since $c_2^2 d_3 c_2^2 = d_4^{-1}$ and $c_2^2 d_4 c_2^2 = d_3 d_4^{-1}$. If we set $L := \langle \alpha \rangle \subset G$, then $L \simeq C_4$ and $G = G' \times L \simeq (Q_8 \times C_3) \times C_4$, where $\beta : C_4 = \langle s \rangle \rightarrow \text{Aut}(Q_8 \times C_3)$ is given by $t' = t s^2, x^t = x, y^t = y x$. $a \in G$ corresponds to $s \cdot 1 \cdot 1 \in (Q_8 \times C_3) \times C_4$, and $b$ to $s \cdot t \cdot x$ under this isomorphism.

Next, we show $SL(2, Z_3) \simeq Q_8 \times C_3$. Set $X := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ Y := \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \ T := \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \in SL(2, Z_3)$. Then we have $Y^2 = X^2, X^4 = 1, Y^{-1} X Y = X^{-1}$ and $\{X, Y\}$ generates
a subgroup $N$ isomorphic to $Q_8$. We also have $\operatorname{ord}(T) = 3$ and set $M := \langle T \rangle$. Then $SL(2, Z_3) = N \rtimes M \simeq Q_8 \rtimes C_3$ since $T^{-1}XT = YX^2$, $T^{-1}YT = YX$. Hence $G \simeq SL(2, Z_3) \rtimes C_4$, where $\gamma : C_4 = \langle s \rangle \to \operatorname{Aut}(SL(2, Z_3))$ is given by $X^s = X$, $Y^s = YX$, $T^s = T^2 YX^3$. $a \in G$ corresponds to $s \cdot I_2$ and $b$ to $s \cdot \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$.

(4) We have $G^{ab} \simeq C_5$ and take $U = \{1, a, a^2, a^3, a^4\}$ as a Schreier transversal for $G'$ in $G$. Then the $BR$- and $BS$-tables are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(ab^{-1}a^{-1}b^{-1})</th>
<th>(a^5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(ba^{-1})</td>
<td>(ab^{-1}a^{-1}b^{-1})</td>
<td>(a^5)</td>
</tr>
<tr>
<td>(a)</td>
<td>1</td>
<td>(aba^{-2})</td>
<td>(a^2bab^{-1}a^{-1}b^{-1}a^{-1})</td>
<td>(a^5)</td>
</tr>
<tr>
<td>(a^2)</td>
<td>1</td>
<td>(a^2ba^{-3})</td>
<td>(a^3bab^{-1}a^{-1}b^{-1}a^{-2})</td>
<td>(a^5)</td>
</tr>
<tr>
<td>(a^3)</td>
<td>1</td>
<td>(a^3ba^{-4})</td>
<td>(a^4bab^{-1}a^{-1}b^{-1}a^{-3})</td>
<td>(a^5)</td>
</tr>
<tr>
<td>(a^4)</td>
<td>(a^5)</td>
<td>(a^4b)</td>
<td>(a^5bab^{-1}a^{-1}b^{-1}a^{-4})</td>
<td>(a^5)</td>
</tr>
</tbody>
</table>

Hence $G' \simeq \langle c_i \mid (1 \leq i \leq 6) \rangle$ ten relations in the $BS$-table $\simeq \langle c_2, c_4 | c_4c_2^2c_4 = c_2c_4c_2, c_4^{-1}c_2^{-1}c_4 = c_2^{-1}c_4c_2 \rangle$. Now, if we set $d_2 = \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix}$, $d_4 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} \in SL(2, Z_3)$, then we have $d_4d_2d_4 = d_2d_4d_2$ and $d_4^{-1}d_2^{-1}d_4 = d_2^{-1}d_4d_2$. Since $\{d_2, d_4\}$ generates $SL(2, Z_3)$, we have a surjective homomorphism $F : G' \to SL(2, Z_3)$ defined by $F(c_i) = d_i (i = 2, 4)$. By applying the same argument as in the case (5) in Remark 2.7 below, we find $G' \simeq SL(2, Z_3)$ and hence $F$ is an isomorphism.

Next, we show $G \simeq SL(2, Z_3) \times C_5$. Set $x := aba \in G$. Then $x^{10} \in G'$ and we have $x^{10} = (ab)^{15} = (c_3c_5c_1c_2c_4c_3) = (c_3^2c_2)^3$ which corresponds to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\in SL(2, Z_3)$. Hence we have $\operatorname{ord}(x) = 20$. If we set $w := x^4$, then $\operatorname{ord}(w) = 5$ and $w$
belongs to the center of $G$. Since $\{w| 0 \leq i \leq 4\}$ is a transversal for $G'$ in $G$, we conclude $G \simeq \text{SL}(2,\mathbb{Z}_3) \times C_5$. We have $G' \ni w_2^2a = (c_2 c_2 c_2 c_2^2 c_4$ which corresponds to \[
abla \]
 in $\text{SL}(2,\mathbb{Z}_3)$. Hence $a \in G$ corresponds to $\left[ \begin{array}{cc} 4 & 1 \\ 1 & 3 \end{array} \right] \in \text{SL}(2,\mathbb{Z}_3) \times C_5$, where $C_5 = \langle w \rangle$. Similarly, $b \in G$ corresponds to $\left[ \begin{array}{cc} 1 & 4 \\ 0 & 1 \end{array} \right] \in \text{SL}(2,\mathbb{Z}_3) \times C_5$.

(5) Similar to (4). We give some data for convenience. The $B\mathcal{R}$- and $B\mathcal{S}$-tables for $G'$ in $G$ are given as follows:

Table 11

|  $a$ |  $b$ |  $abab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$ |  $a^3$ |  $b^3$
|-----|-----|---------------------------------|-----|-----|
|  1  |  1  |  $abab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$ |  $a^3$ |  $b^3$
|  $a$ |  1  |  $a^2babab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$ |  $a^3$ |  $ab^2a^{-1}$
|  $a^2$ |  $a^3$ |  $a^2b$ |  $a^3$ |  $a^2b^3a^{-2}$

Table 12

- $c_2$
- $c_3$
- $c_1$
- $c_4$

\[
\begin{array}{cc}
   c_2 c_1 c_2 c_3 c_3^{-1} c_4 c_2^{-1} c_2 & c_1 \\
   c_4 c_3 c_2 c_3 c_2^{-1} c_3^{-1} c_4^{-1} c_1 & c_1 \\
   c_1 c_2 c_4 c_2^{-1} c_3^{-1} c_3^{-1} c_4^{-1} c_1 & c_1 \\
   c_4 c_2 c_3 & c_4 c_2 c_3
\end{array}
\]

$G' \simeq \langle c_1 (1 \leq i \leq 4) \rangle$ nine relations in the $B\mathcal{S}$-table

$\simeq \langle c_2, c_3 | c_2 c_2 c_2 = c_3^2, c_3 c_2 c_3 = c_2^2 c_2 c_2 \rangle

\simeq \text{SL}(2,\mathbb{Z}_3),$

where $c_2$ corresponds to $\left[ \begin{array}{cc} 4 & 4 \\ 0 & 4 \end{array} \right]$ and $c_3$ to $\left[ \begin{array}{cc} 4 & 0 \\ 4 & 4 \end{array} \right]$. We have $G = G' \times \langle w \rangle \simeq \text{SL}(2,\mathbb{Z}_3)$

$\times C_5$, where $w = (ababa)^4 \in G$ (ord$(ababa) = 12)$. $a \in G$ corresponds to $\left[ \begin{array}{cc} 0 & 2 \\ 2 & 4 \end{array} \right]$, $w^2 \in \text{SL}(2,\mathbb{Z}_3)$ and $b \in G$ to $\left[ \begin{array}{cc} 3 & 4 \\ 3 & 1 \end{array} \right], w^2$.

Remark 2.7. By Corollary 1.5, there exists the maximal Galois covering $\pi : X(C^2, eB_q) \rightarrow C^2$ in the five cases above. We have $X(C^2, eB_q) \simeq C^2$ in these cases. Since the cases (2),(3),(4) are studied in Namba [3: p.50], we briefly discuss the remaining two cases. In the case (1), the maximal Galois covering $\pi_1 : C^2 \rightarrow C^2$
of \((C^2, 2B_q)\) is given by \((v, w) = \pi_1((s, t)) = (st, (1/2)(s^q + t^q))\). Indeed, if we set \(N = \{(x, y, z) \in C^3 | xy = z^q\}\) and \(M = \{(u, v, w) \in C^3 | w^2 - u^2 = v^q\}\), then \(\pi_1\) is decomposed as \(\pi_1 = f \circ g \circ h\), where \(C^2 \rightarrow N \Rightarrow M \rightarrow C^2\), and \(f, g, h\) are defined as follows:
\[
h((s, t)) = (s^q, t^q, st), \quad g((x, y, z)) = ((1/2)(x - y), z, (1/2)(x + y)), \quad \text{and} \quad f((u, v, w)) = (v, w).
\]
Since \(h\) is unramified outside \((0, 0, 0) \in N\) and \(f\) branches at \(2B_q\), we conclude that \(\pi_1\) is a covering which branches at \(2B_q\). It is easy to see that \(G_{\pi_1}\) is isomorphic to \(D_{2q}\) and generated by \(\sigma, \tau \in \text{Aut}(C^2)\), where \(\sigma((s, t)) = (\zeta s, \zeta^{-1} t) (\zeta = e^{2\pi /q})\) and \(\tau((s, t)) = (t, s)\). Hence we conclude that \(\pi_1\) is the maximal Galois covering of \((C^2, 2B_q)\). In the case \(5\), let \(G : C^2 \rightarrow C^2 / SL(2, \mathbb{Z}_5) \simeq L = \{(u, v, w) \in C^2 | u^3 + v^5 - w^2 = 0\}\) be the quotient map giving the binary icosahedral kleinian singularity (cf. Pinkham [4]), and define \(F : L \rightarrow C^2\) by \(F(u, v, w) = (v, w)\). Then \(\pi_5 = F \circ G : C^2 \rightarrow C^2\) is a covering which branches at \(3B_5\). Since \(C^2 - \pi_5^{-1}\) \((0, 0) = C^2 - \{(0, 0)\}\) is simply-connected, \(\pi_5\) is the maximal Galois covering of \((C^2, 3B_5)\) by Namba [3; Corollary 1.3.12]. We also find that the maximal Galois group \(G(C^2, 3B_5) \simeq G_{\pi_5}\) is an extension of \(SL(2, \mathbb{Z}_5)\) by \(C_3\) by this geometric argument without group-theoretic computation.

In the case where \(G(e, q)\) is an infinite group, to determine whether \(G(e, q)\) is solvable or not is a fundamental problem in the Galois theory of branched coverings. As for this, we have the following result:

**Theorem 2.8.** \(G(e, q)' = (G(e, q)')'\) if and only if \(e\) is odd and \(\text{GCD}(e, q) = 1\).

**Proof.** We set \(G = G(e, q)\) and \(G(e, q)' = N\) for short. According to Johnson [2; Chapter 12], let \(F(X)\) be a free group generated by \(X = \{S, T\}\) and \(J = \left[ \begin{array}{c} e^S \\ e^S \\ e^T \\ e^T \end{array} \right] \in M_2(\mathbb{Z}[F(X)]\) be the jacobian of \(G\), where \(w = S^T S^T \cdots S^T T^T S^{-1} T^{-1} \in F(X)\) and \(\frac{\partial}{\partial S}, \frac{\partial}{\partial T}\) are the Fox derivations. Let \(F(X) \rightarrow G \rightarrow G / N = C_e = \langle s \rangle\) are the natural surjections and we denote by the same symbol the map \(M_2(\mathbb{Z}[F(X)]) \rightarrow M_2(\mathbb{Z}[\phi]) \rightarrow M_2(\mathbb{Z}[C_e])\) induced by them. We set \(D = \mu \circ \psi \circ \phi(J) \in M_{2e}(\mathbb{Z}),\) where \(\mu : \mathbb{Z}[C_e] \rightarrow M_2(\mathbb{Z})\) (or \(\mu : M_2(\mathbb{Z}[C_e]) \rightarrow M_{2e}(\mathbb{Z})\)) is the blowing-up map (cf. Johnson [2;12.1]).

Now, we have
\[
\psi \circ \phi(J) = \left[ \begin{array}{cc} e + s + s^2 + \cdots + s^{e-1} & 0 \\ s^2 + s^4 + \cdots + s^{(q-1)} & s^{-1} + s^{-3} + \cdots + s^{-q} \\ -s^{-1} - s^{-3} - \cdots - s^{-(q-1)} & -s^{-2} - s^{-4} - \cdots - s^{-(q-1)} \end{array} \right].
\]
which is equivalent to
\[
L := \begin{bmatrix}
  e + s + s^2 + \cdots + s^{e-1} & 0 \\
  s^{-1} + s^{-3} + \cdots + s^{-q} - s^{-2} - s^{-4} - \cdots - s^{-(q-1)} & 0
\end{bmatrix}.
\]
Here we say that two matrices are equivalent if they can be transformed to each other by elementary transformations.

We have \( \mu \left( \sum_{i=0}^{e-1} s^i \right) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in M_e(Z) \), which is equivalent to
\[
\begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{bmatrix}.
\]
Since \( N / N' \times \mathbb{Z}^{e-1} \cong \mathbb{Z}^{2e} / \text{Im} \ \mu(L) \) (Johnson [2; Proposition 1, p. 161]), we conclude \( N = N' \) if and only if \( e \) is odd and \( \text{GCD}(e, q) = 1 \) by the following lemma.

**Lemma 2.9.** For an odd integer \( q \geq 3 \), set \( z := \sum_{i=1}^{q} (-1)^i s^{-i} \in \mathbb{Z}[^C_e] \), \( C_e = \langle s \rangle \). Then \( z \) is a unit in \( \mathbb{Z}[^C_e] \) if and only if \( e \) is odd and \( \text{GCD}(e, q) = 1 \).

**Proof.** We identify \( \mathbb{Z}[^C_e] \) with \( A := \mathbb{Z}[x]/(x^e - 1) \), where \( s^{-1} \) corresponds to the coset \( x \) of \( x \). Since \( x \) is a unit in \( A \), we show \( \bar{z} = \sum_{i=0}^{q-1} (-1)^i x^i \in A \) is a unit if and only if \( e \) is odd and \( \text{GCD}(e, q) = 1 \). Now, assume \( e \) is even. If there exist \( F(x), G(x) \in \mathbb{Z}[x] \) such that \( \left( \sum_{i=0}^{q-1} (-1)^i x^i \right) P(x) + (x^e - 1) G(x) = 1 \) in \( \mathbb{Z}[x] \), then we have \( qF(-1) = 1 \) by setting \( x = -1 \), contradiction. Assume that \( e \) is odd and \( \text{GCD}(e, q) = 1 \). By replacing \( x \) with \( -x \), we may assume that \( \bar{z} = \sum_{i=0}^{q-1} x^i \in B : = \mathbb{Z}[x]/(x^e + 1) \).

Since \( \text{GCD}(2e, q) = 1 \), there exist \( P(x), Q(x) \in \mathbb{Z}[x] \) such that \( (x^e - 1) P(x) + (x^e - 1) Q(x) = x - 1 \) by the Euclidean division algorithm. Thus \( \left( \sum_{i=0}^{q-1} x^i \right) P(x) + (x^e + 1) \left( \sum_{i=0}^{q-1} x^i \right) Q(x) = 1 \) and hence \( \bar{z} \in B \) is a unit. Assume that \( e \) is odd and \( \text{GCD}(e, q) = d > 1 \). Since \( \sum_{i=0}^{d-1} x^i \) divides \( \sum_{i=0}^{q-1} x^i \) in \( B \), it is enough to show \( \sum_{i=0}^{d-1} x^i \) is a non-unit in \( B \). Suppose
that there exist \( f(x), g(x) \in \mathbb{Z}[x] \) such that

\[
f(x)(x^e + 1) + g(x) \left( \sum_{i=0}^{d-1} x^i \right) = 1 \cdots (1).
\]

We have \( x^e + 1 = \left( \sum_{i=0}^{d-1} x^i \right) h(x) + 2 \) for some \( h(x) \in \mathbb{Z}[x] \) by a direct division. By substituting this to (1), we obtain

\[
(h(x)f(x) + g(x)) \left( \sum_{i=0}^{d-1} x^i \right) + 2f(x) = 1.
\]

Then \( \sum_{i=0}^{d-1} x^i \) is a unit in \( \mathbb{Z}_2[x] \), contradiction.

**Corollary 2.10.** If \((e,q)\) is not one of those in Theorem 2.4 and \( \gcd(e,q) = 1 \) with \( e \) odd, then \( G(e,q) \) is an infinite unsolvable group.

3. Calculation of \( G(P^2, eB_q + ml) \)

Let \( G[e,m;q] := G(P^2, eB_q + ml) \) be the maximal Galois group of \( (P^2, eB_q + ml) \), where \( B_q = \{(x_0, x_1, x_2) \in P^2 | x_0 = x_1^2 x_2^{-2}\} \) and \( l_\infty = \{(x_0, x_1, x_2) \in P^2 | x_2 = 0\} \) (the infinite line). Let \( \delta \) be a loop around \( l_\infty \) in \( P^2 - (B_q \cup l_\infty) \) and \([\delta] \in \pi_1(P^2 - (B_q \cup l_\infty)) = \pi_1(C^2 - B_q) \) the homotopy class of \( \delta \). It is easy to see \([\delta^{-1}] = \left[ \begin{array}{c} x_1 x_2 x_1 \cdots x_1 \end{array} \right] \)

\[ q \]

\[ a^e = c^m = 1, \quad aba\cdots a = bab\cdots b = c^{-1} \] \( \simeq G(e;q) / N < Q^m \) \( (Q := aba\cdots a = bab\cdots b ) \).

The explicit structure of \( G[e,m;q] \) for the \((e,q)\)'s given in Theorem 2.4 is as follows:

**Proposition 3.1.**

1. \( G[2,m;q] \simeq \) \[
\begin{cases} 
D_{24} & \text{if } m \text{ is even} \\
\{1\} & \text{if } m \text{ is odd}
\end{cases}
\]

2. \( G[3,m;3] \simeq \) \[
\begin{cases} 
C_3 & \text{if } m \equiv 1,3 \pmod{4} \\
PSL(2,\mathbb{Z}_3) & \text{if } m \equiv 2 \pmod{4} \\
SL(2,\mathbb{Z}_3) & \text{if } m \equiv 0 \pmod{4}
\end{cases}
\]
G[4,m;3] \simeq \begin{cases} 
\{1\} & \text{if } m \equiv 1,3,5,7 \pmod{8} \\
(Q_{\alpha,C_3}) \triangleright C_4 / \langle s^2 \cdot 1 \cdot x \rangle & \text{if } m \equiv 2,6 \pmod{8} \\
((C_2 \times C_2) \triangleright C_3) \triangleright C_4 & \text{if } m \equiv 4 \pmod{8} \\
G(4;3) & \text{if } m \equiv 0 \pmod{8} 
\end{cases}

G[5,m;3] \simeq \begin{cases} 
\{1\} & \text{if } m \equiv 1,5,7,11 \pmod{12} \\
PSL(2,Z_5) & \text{if } m \equiv 2,10 \pmod{12} \\
SL(2,Z_5) & \text{if } m \equiv 3,9 \pmod{12} \\
SL(2,Z_5) \times C_3 & \text{if } m \equiv 4,8 \pmod{12} \\
SL(2,Z_5) \times C_3 & \text{if } m \equiv 0 \pmod{12} 
\end{cases}

G[3,m;5] \simeq \begin{cases} 
\{1\} & \text{if } m \equiv 1,3,7,9,11,13,17,19 \pmod{20} \\
PSL(2,Z_5) & \text{if } m \equiv 2,6,14,18 \pmod{20} \\
SL(2,Z_5) & \text{if } m \equiv 4,8,12,16 \pmod{20} \\
SL(2,Z_5) \times C_5 & \text{if } m \equiv 10 \pmod{20} \\
SL(2,Z_5) \times C_5 & \text{if } m \equiv 0 \pmod{20} 
\end{cases}

Proof. (1) We have $G[2,m;q] \simeq \langle b, d | b^2 = d^4 = 1, b^{-1} db = d^{-1} \rangle / N\langle x^m \rangle$, where $x = bd^{q-1}/2$, by setting $d = ab$. Since $\text{ord}(x) = 2$, we have $G[2,m;q] \simeq D_{2q}$ if $m$ is even. If $m$ is odd, then $N\langle x^m \rangle = D_{2q}$ and hence $G[2,m;q] = \{1\}$.

(2) We have $\text{ord}(Q) = 4$ since $\varphi_2(Q) = \varphi_2(aba) = \begin{bmatrix} 0 & 1 \\
2 & 0 \end{bmatrix} \in SL(2,Z_5)$ (cf. Theorem 2.6). If $m \equiv 1,3 \pmod{4}$, then $G[2,m;q] \simeq \langle a, b | a^3 = 1, ab = ba = 1 \rangle \cong G(3;3) / N\langle Q^2 \rangle \simeq SL(2,Z_5) \times \{ \pm I_2 \} \simeq PSL(2,Z_3)$. If $m \equiv 2 \pmod{4}$, then $G[3,m;q] \simeq G(3;3) \simeq SL(2,Z_3)$.

(3) We have $\varphi_3(Q) = s^3 \cdot r^2 \cdot y \in (Q_{\alpha,C_3}) \triangleright C_4$, and hence $\text{ord}(Q) = 8$. If $m \equiv 1,3,5,7 \pmod{8}$, then $G[4,m;3] \simeq \langle a,b | a^4 = aba = bab = 1 \rangle \cong \{1\}$. If $m \equiv 2,6 \pmod{8}$, then $\varphi_3(Q^5) = s^2 \cdot 1 \cdot x$ and hence $G[4,m;3] \simeq (Q_{\alpha,C_3}) \triangleright C_4 / \langle s^2 \cdot 1 \cdot x \rangle$. If $m \equiv 4 \pmod{8}$, then $\varphi_3(Q^4) = 1 \cdot 1 \cdot x^2$ and hence $G[4,m;3] \simeq ((C_2 \times C_2) \triangleright C_3) \triangleright C_4$, where $\bar{\alpha}$, $\bar{\beta}$ are the homomorphisms induced by $\alpha, \beta$. If $m \equiv 0 \pmod{8}$, then $G[4,m;3] \simeq G(4;3)$.

(4) We have $\varphi_4(Q) = \begin{bmatrix} 3 & 0 \\
1 & 2 \end{bmatrix} \in SL(2,Z_5) \times C_3$ so that $\text{ord}(Q) = 20$. If $m \equiv 1,3,7,9,11,13,17,19 \pmod{20}$, then $G[5,m;3] \simeq \langle a, b | a^5 = 1, ab = bab = 1 \rangle \cong \langle a | a^5 = a^{-3} = 1 \rangle \cong \{1\}$. If $m \equiv 2,6,14,18 \pmod{20}$, then $G[5,m;3] \simeq SL(2,Z_5) \times C_5 / \{ \pm I_2 \} \times C_5 \simeq \{1\}$.
If $m \equiv 4, 8, 12, 16 \pmod{20}$, then $G[5,m;3] \simeq SL(2, \mathbb{Z}_5) \times C_5 / C_5 \simeq SL(2, \mathbb{Z}_5)$. If $m \equiv 5, 15 \pmod{20}$, then we have $\varphi_4(Q^5) = \left( \begin{array}{cc} 3 & 0 \\ 1 & 2 \end{array} \right)$, $1$. Hence $G[5,m;3] \simeq C_5$ since $\langle \begin{array}{cc} 3 & 0 \\ 1 & 2 \end{array} \rangle = SL(2, \mathbb{Z}_5)$. If $m \equiv 10 \pmod{20}$, then we have $\varphi_4(Q^{10}) = \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right)$, $1$ so that $G[5,m;3] \simeq PSL(2, \mathbb{Z}_5) \times C_5$. If $m \equiv 0 \pmod{20}$, then $G[5,m;3] \simeq PSL(2, \mathbb{Z}_5) \times C_5$.

\[(5)\] We have $\varphi_5(Q) = \left( \begin{array}{cc} 2 & 3 \\ 0 & 3 \end{array} \right)$ and the rest is similar to (4).

**Remark 3.2.** In the case (3), $G[4,m;3]$ ($m \equiv 2, 6 \pmod{8}$) is isomorphic to the symmetric group $S_4$ of degree 4 (cf. Namba [3;p.50]).

**Corollary 3.3.** Suppose that $(e,q)$ is one of those given in Theorem 2.4 and let $D = eB_q + ml_\alpha$. Then there exists the maximal covering $\pi : X(P^2, D) \to P^2$ if and only if

1. $m = 2$ in the case where $(e,q) = (2,q)$ ($q$ odd),
2. $m = 1, 2, 4$ in the case where $(e,q) = (3,3)$,
3. $m = 2, 4, 8$ in the case where $(e,q) = (4,3)$,
4. $m = 2, 4, 5, 10, 20$ in the case where $(e,q) = (5,3)$,
5. $m = 2, 3, 4, 6, 12$ in the case where $(e,q) = (3,5)$.

In these cases, the Galois group $G_\pi$ of $\pi$ is isomorphic to $G[e,m;q]$.

**Proof.** Since $\text{ord}(a) = 2$ and $\text{ord}(Q) = 2$ in $G[2,m;q] \simeq D_{2q}$ ($m$: even), (1) follows from Corollary 1.5. By calculating $\text{ord}(a)$ and $\text{ord}(Q)$ using Theorem 2.6 and Proposition 3.1, the other assertions follow from Corollary 1.5 similarly. □

Let $S(e,m;q) = X(P^2, eB_q + ml_\alpha)$ be the maximal Galois covering for the $(e,m;q)$'s given in Corollary 3.3. Then $S(e,m;q)$ is a normal projective irreducible rational surface since it is a compactification of $C^2 / \text{finite group}$. One of the singularities of $S(e,m;q)$ lies over $(0,0,1)$, which is a quotient singularity. To determine the structure of $S(e,m;q)$ (especially the singularities lying over $(0,1,0)$) will be an interesting problem, which will be discussed elsewhere. We give a list of $S(e,m;q)$ and $G[e,m;q]$ for convenience.

<table>
<thead>
<tr>
<th>$S(e,m;q)$</th>
<th>$S(2,2q)$</th>
<th>$S(3,1;3)$</th>
<th>$S(3,2;3)$</th>
<th>$S(3,4;3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G[e,m;q]$</td>
<td>$D_{2q}$</td>
<td>$C_3$</td>
<td>$PSL(2, \mathbb{Z}_3)$</td>
<td>$SL(2, \mathbb{Z}_3)$</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|}
\hline
S(4,2;3) & S(4,4;3) & S(4,8;3) \\
\hline
S_4 & ((C_2 \times C_2)\rtimes C_3)\rtimes C_4 & (Q_8 \rtimes C_3)\rtimes C_4 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
S(5,2;3) & S(5,4;3) & S(5,5;3) & S(5,10;3) & S(5,20;3) \\
\hline
PSL(2,\mathbb{Z}_5) & SL(2,\mathbb{Z}_5) & C_5 & PSL(2,\mathbb{Z}_5) \times C_5 & SL(2,\mathbb{Z}_5) \times C_5 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
S(3,2;5) & S(3,3;5) & S(3,4;5) & S(3,6;5) & S(3,12;5) \\
\hline
PSL(2,\mathbb{Z}_5) & C_3 & SL(2,\mathbb{Z}_5) & PSL(2,\mathbb{Z}_5) \times C_3 & SL(2,\mathbb{Z}_5) \times C_3 \\
\hline
\end{array}
\]

References


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