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# PROJECTIVE NORMALITY OF ALGEBRAIC CURVES AND ITS APPLICATION TO SURFACES

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## Abstract

Let  $L$  be a very ample line bundle on a smooth curve  $C$  of genus  $g$  with  $(3g + 3)/2 < \deg L \leq 2g - 5$ . Then  $L$  is normally generated if  $\deg L > \max\{2g + 2 - 4h^1(C, L), 2g - (g - 1)/6 - 2h^1(C, L)\}$ . Let  $C$  be a triple covering of genus  $p$  curve  $C'$  with  $C \xrightarrow{\phi} C'$  and  $D$  a divisor on  $C'$  with  $4p < \deg D < (g - 1)/6 - 2p$ . Then  $K_C(-\phi^*D)$  becomes a very ample line bundle which is normally generated. As an application, we characterize some smooth projective surfaces.

## 1. Introduction

We work over the algebraically closed field of characteristic zero. Specially the base field is the complex numbers in considering the classification of surfaces. A smooth irreducible algebraic variety  $V$  in  $\mathbb{P}^r$  is said to be projectively normal if the natural morphisms  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \rightarrow H^0(V, \mathcal{O}_V(m))$  are surjective for every non-negative integer  $m$ . Let  $C$  be a smooth irreducible algebraic curve of genus  $g$ . We say that a base point free line bundle  $L$  on  $C$  is normally generated if  $C$  has a projectively normal embedding via its associated morphism  $\phi_L: C \rightarrow \mathbb{P}(H^0(C, L))$ .

Any line bundle of degree at least  $2g + 1$  on a smooth curve of genus  $g$  is normally generated but a line bundle of degree at most  $2g$  might fail to be normally generated ([8], [9], [10]). Green and Lazarsfeld showed a sufficient condition for  $L$  to be normally generated as follows ([5], Theorem 1): If  $L$  is a very ample line bundle on  $C$  with  $\deg L \geq 2g + 1 - 2h^1(C, L) - \text{Cliff}(C)$  (and hence  $h^1(C, L) \leq 1$ ), then  $L$  is normally generated. Using this, we show that a line bundle  $L$  on  $C$  with  $(3g + 3)/2 < \deg L \leq 2g - 5$  is normally generated for  $\deg L > \max\{2g + 2 - 4h^1(C, L), 2g - (g - 1)/6 - 2h^1(C, L)\}$ . As a corollary, if  $C$  is a triple covering of a genus  $p$  curve  $C'$  with  $C \xrightarrow{\phi} C'$  then it has a very ample  $K_C(-\phi^*D)$  which is normally generated for any divisor  $D$  on  $C'$  with  $4p < \deg D < (g - 1)/6 - 2p$ . It is a kind of generalization of the result that  $K_C(-rg_3^1)$  on a trigonal curve  $C$  is normally generated for  $3r \leq g/2 - 1$  ([7]).

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As an application to nondegenerate smooth surface  $S \subset \mathbb{P}^r$  of degree  $2\Delta - e$  with  $g(H) = \Delta + f$ ,  $\max\{e/2, 6e - \Delta\} < f - 1 < (\Delta - 2e - 6)/3$  for some  $e, f \in \mathbb{Z}_{\geq 1}$ , we obtain that  $S$  is projectively normal with  $p_g = f$  and  $-2f - e + 2 \leq K_S^2 \leq (2f + e - 2)^2/(2\Delta - e)$  if its general hyperplane section  $H$  is linearly normal, where  $\Delta := \deg S - r + 1$ . We derive this application using the methods in Akahori's paper ([2]).

We follow most notations in [1], [4], [6]. Let  $C$  be a smooth irreducible projective curve of genus  $g \geq 2$ . The Clifford index of  $C$  is taken to be  $\text{Cliff}(C) = \min\{\text{Cliff}(L) \mid h^0(C, L) \geq 2, h^1(C, L) \geq 2\}$ , where  $\text{Cliff}(L) = \deg L - 2(h^0(C, L) - 1)$  for a line bundle  $L$  on  $C$ . By abuse of notation, we sometimes use a divisor  $D$  on a smooth variety  $V$  instead of  $\mathcal{O}_V(D)$ . We also denote  $H^i(V, \mathcal{O}_V(D))$  by  $H^i(V, D)$  and  $h^0(V, L) - 1$  by  $r(L)$  for a line bundle  $L$  on  $V$ . We denote  $K_V$  the canonical line bundle on a smooth variety  $V$ .

## 2. Main results

Any line bundle of degree at least  $2g + 1$  on a smooth curve of genus  $g$  is normally generated. If the degree is at most  $2g$ , then there are curves which have a non normally generated line bundle of given degree ([8], [9], [10]). In this section, we investigate the normal generation of a line bundle with given degree on a smooth curve under some condition about the speciality of the line bundle.

**Theorem 2.1.** *Let  $L$  be a very ample line bundle on a smooth curve  $C$  of genus  $g$  with  $(3g+3)/2 < \deg L \leq 2g - 5$ . Then  $L$  is normally generated if  $\deg L > \max\{2g + 2 - 4h^1(C, L), 2g - (g - 1)/6 - 2h^1(C, L)\}$ .*

*Proof.* We have  $h^1(C, L) \geq 2$ , since  $2g - 5 \geq \deg L > 2g + 2 - 4h^1(C, L)$ . Suppose  $L$  is not normally generated. Then there exists a line bundle  $A \simeq L(-R)$ ,  $R > 0$ , such that (i)  $\text{Cliff}(A) \leq \text{Cliff}(L)$ , (ii)  $\deg A \geq (g - 1)/2$ , (iii)  $h^0(C, A) \geq 2$  and  $h^1(C, A) \geq h^1(C, L) + 2$  by the proof of Theorem 3 in [5]. Assume  $\deg K_C L^{-1} = 3$ . Then  $|K_C L^{-1}| = g_3^1$ . On the other hand,  $L = K_C(-g_3^1)$  is normally generated. So we may assume  $\deg K_C L^{-1} \geq 4$  and then  $r(K_C L^{-1}) \geq 2$  since  $\deg L > 2g + 2 - 4h^1(C, L)$ . Let  $B_1$  (resp.  $B_2$ ) be the base locus of  $K_C L^{-1}$  (resp.  $K_C A^{-1}$ ). And let  $N_1 := K_C L^{-1}(-B_1)$ ,  $N_2 := K_C A^{-1}(-B_2)$ . Then  $N_1 \preceq N_2$  since  $A \cong L(-R)$ ,  $R > 0$  and  $h^1(C, A) \geq h^1(C, L) + 2$ . Hence we have the following diagram,

$$\begin{array}{ccc} C & \xrightarrow{\phi_{N_2}} & C_2 \\ & \searrow \phi_{N_1} & \downarrow \pi: \text{projection} \\ & & C_1 \end{array}$$

where  $C_i = \phi_{N_i}(C)$ .

If we set  $m_i := \deg \phi_{N_i}$ ,  $i = 1, 2$ , then we have  $m_2 | m_1$ . If  $N_1$  is birationally very ample, then by Lemma 9 in [8] and  $\deg K_C L^{-1} < (g - 1)/2$  we have  $r(N_1) \leq [(\deg N_1 -$

1)/5]. It is a contradiction to  $\deg L > 2g + 2 - 4h^1(C, L)$  that is equivalent to  $\deg K_C L^{-1} < 4(h^0(C, K_C L^{-1}) - 1)$ . Therefore  $N_1$  is not birationally very ample, and then we have  $m_1 \leq 3$  since  $\deg K_C L^{-1} < 4(h^0(C, K_C L^{-1}) - 1)$ .

Let  $H_1$  be a hyperplane section of  $C_1$ . If  $|H_1|$  on a smooth model of  $C_1$  is special, then  $r(N_1) \leq (\deg N_1)/4$ , which is absurd. Thus  $|H_1|$  is nonspecial. If  $m_1 = 2$ , then

$$r(K_C L^{-1}(-B_1 + P + Q)) \geq r(K_C L^{-1}(-B_1)) + 1$$

for any pairs  $(P, Q)$  such that  $\phi_{N_1}(P) = \phi_{N_1}(Q)$  since  $|H_1|$  is nonspecial. Therefore we have  $r(L(-P - Q)) \geq r(L) - 1$  for  $(P, Q)$  such that  $\phi_{N_1}(P) = \phi_{N_1}(Q)$ , which contradicts that  $L$  is very ample. Therefore we get  $m_1 = 3$ . Suppose  $B_1$  is nonzero. Set  $P \leq B_1$  for some  $P \in C$ . Consider  $Q, R$  in  $C$  such that  $\phi_{N_1}(P) = \phi_{N_1}(Q) = \phi_{N_1}(R) = P'$  for some  $P' \in C_1$ . Since  $|H_1|$  is nonspecial, we have

$$\begin{aligned} r(K_C L^{-1}(Q + R)) &\geq r(N_1(P + Q + R)) = r(H_1 + P') \\ &= r(H_1) + 1 = r(K_C L^{-1}) + 1 \end{aligned}$$

which is a contradiction to the very ampleness of  $L$ . Hence  $K_C L^{-1}$  is base point free, i.e.,  $K_C L^{-1} = N_1$ . On the other hand, we have  $m_2 = 1$  or  $3$  for  $m_2 | m_1$ . Since  $K_C A^{-1}(-B_2) = N_2 \not\geq N_1 = K_C L^{-1}$ , we may set  $N_1 = N_2(-G)$  for some  $G > 0$ .

Assume  $m_2 = 1$ , i.e.  $K_C A^{-1}(-B_2) = N_2$  is birationally very ample. On the other hand we have  $r(N_2) \geq r(N_1) + (\deg G)/2$ , since  $N_2(-G) \cong N_1$  and  $\text{Cliff}(N_2) \leq \text{Cliff}(A) \leq \text{Cliff}(L) = \text{Cliff}(N_1)$ . In case  $\deg N_2 \geq g$  we have  $r(N_2) \leq (2 \deg N_2 - g + 1)/3$  by Castelnuovo's genus bound and hence

$$\text{Cliff}(L) \geq \text{Cliff}(N_2) \geq \deg N_2 - \frac{4 \deg N_2 - 2g + 2}{3} = \frac{2g - 2 - \deg N_2}{3} \geq \frac{g - 1}{6},$$

since  $N_2 = K_C A^{-1}(-B_2)$  and  $\deg A \geq (g - 1)/2$ . If we observe that the condition  $\deg L > 2g - (g - 1)/6 - 2h^1(C, L)$  is equivalent to  $\text{Cliff}(K_C L^{-1}) < (g - 1)/6$ , then we meet an absurdity. Thus we have  $\deg N_2 \leq g - 1$ , and then Castelnuovo's genus bound produces  $\deg N_2 \geq 3r(N_2) - 2$ . Note that the Castelnuovo number  $\pi(d, r)$  has the property  $\pi(d, r) \leq \pi(d - 2, r - 1)$  for  $d \geq 3r - 2$  and  $r \geq 3$ , where  $\pi(d, r) = (m(m - 1)/2)(r - 1) + m\epsilon$ ,  $d - 1 = m(r - 1) + \epsilon$ ,  $0 \leq \epsilon \leq r - 2$  (Lemma 6, [8]). Hence

$$\pi(\deg N_2, r(N_2)) \leq \cdots \leq \pi\left(\deg N_2 - \deg G, r(N_2) - \frac{\deg G}{2}\right) \leq \pi(\deg N_1, r(N_1)),$$

because of  $2 \leq r(N_1) \leq r(N_2) - (\deg G)/2$ . Since  $r(N_1) \geq (\deg N_1)/4$  and  $\deg N_1 < (g - 1)/2$ , we can induce a strict inequality  $\pi(\deg N_1, r(N_1)) < g$  as only the number regardless of birational embedding from the proof of Lemma 9 in [8]. It is absurd. Hence  $m_2 = 3$ , which yields  $C_1 \cong C_2$ .

Let  $H_2$  be a hyperplane section of  $C_2$ . If  $|H_2|$  on a smooth model of  $C_2$  is special, then  $r(N_2) \leq (\deg N_2)/6$ . Thus the condition  $\deg K_C L^{-1} < 4(h^0(C, K_C L^{-1}) - 1)$  yields the following inequalities:

$$\frac{2 \deg N_2}{3} \leq \text{Cliff}(N_2) \leq \text{Cliff}(N_1) \leq \frac{\deg N_1}{2},$$

which contradicts to  $N_1 \prec N_2$ . Accordingly  $|H_2|$  is also nonspecial.

Now we have  $r(N_i) = (\deg N_i)/3 - p$ ,  $i = 1, 2$  where  $p$  is the genus of a smooth model of  $C_1 \cong C_2$ . Therefore

$$\frac{\deg N_1}{3} + 2p = \text{Cliff}(N_1) \geq \text{Cliff}(N_2) = \frac{\deg N_2}{3} + 2p$$

which is a contradiction that  $\deg N_1 < \deg N_2$ . This contradiction comes from the assumption that  $L$  is not normally generated, thus the result follows.  $\square$

Using the above theorem, we obtain the following corollary under the same assumption:

**Corollary 2.2.** *Let  $C$  be a triple covering of a genus  $p$  curve  $C'$  with  $C \xrightarrow{\phi} C'$  and  $D$  a divisor on  $C'$  with  $4p < \deg D < (g-1)/6 - 2p$ . Then  $K_C(-\phi^*D)$  becomes a very ample line bundle which is normally generated.*

*Proof.* Set  $d := \deg D$  and  $L := K_C(-\phi^*D)$ . Suppose  $L$  is not base point free, then there is a  $P \in C$  such that  $|K_C L^{-1}(P)| = g_{3d+1}^{r+1}$ . Note that  $g_{3d+1}^{r+1}$  cannot be composed with  $\phi$  by degree reason. Therefore we have  $g \leq 6d+3p$  due to the Castelnuovo-Severi inequality. Hence it cannot occur by the condition  $d < (g-1)/6 - 2p$ . Suppose  $L$  is not very ample, then there are  $P, Q \in C$  such that  $|K_C L^{-1}(P+Q)| = g_{3d+2}^{r+1}$ . By the same method as above, we get a similar contradiction. Thus  $L$  is very ample. The condition  $d < (g-1)/6 - 2p$  produces  $\text{Cliff}(K_C L^{-1}) = d + 2p < (g-1)/6$  since  $\deg K_C L^{-1} = 3d$  and  $h^0(C, K_C L^{-1}) = h^0(C', D) = d - p + 1$ . Whence  $\deg L > 2g - (g-1)/6 - 2h^1(C, L)$  is satisfied. The condition  $4p < d$  induces  $\deg K_C L^{-1} > 4(h^0(C, K_C L^{-1}) - 1)$ , i.e.,  $\deg L > 2g + 2 - 4h^1(C, L)$ . Consequently  $L$  is normally generated by Theorem 2.1.  $\square$

**REMARK 2.3.** In fact, we have a similar result in [8] for trigonal curve  $C: K_C(-rg_3^1)$  is normally generated if  $3r < g/2 - 1$  ([7]). Thus our result could be considered as a generalization which deals with triple coverings under the some condition.

Let  $S \subseteq \mathbb{P}^r$  be a nondegenerate smooth surface and  $H$  a smooth hyperplane section of  $S$ . If  $H$  is projectively normal and  $h^1(H, \mathcal{O}_H(2)) = 0$ , then  $q = h^1(S, \mathcal{O}_S) =$

0,  $p_g = h^2(S, \mathcal{O}_S) = h^1(H, \mathcal{O}_H(1))$  and  $h^1(S, \mathcal{O}_S(t)) = 0$  for all nonnegative integer  $t$  ([2], Lemma 2.1, Lemma 3.1). Using Theorem 2.1, we can characterize smooth projective surfaces with the wider range of degrees and sectional genera. Recall the definition of  $\Delta$ -genus given by  $\Delta := \deg S - r + 1$ .

**Theorem 2.4.** *Let  $S \subset \mathbb{P}^r$  be a nondegenerate smooth surface of degree  $2\Delta - e$  with  $g(H) = \Delta + f$ ,  $\max\{e/2, 6e - \Delta\} < f - 1 < (\Delta - 2e - 6)/3$  for some  $e, f \in \mathbb{Z}_{\geq 1}$  and its general hyperplane section  $H$  is linearly normal. Then  $S$  is projectively normal with  $p_g = f$  and  $-2f - e + 2 \leq K_S^2 \leq (2f + e - 2)^2/(2\Delta - e)$ .*

*Proof.* From the linear normality of  $H$ , we get  $h^0(H, \mathcal{O}_H(1)) = r$  and hence

$$\begin{aligned} h^1(H, \mathcal{O}_H(1)) &= -\deg \mathcal{O}_H(1) - 1 + g(H) + h^0(H, \mathcal{O}_H(1)) \\ &= -2\Delta + e - 1 + g(H) + h^0(H, \mathcal{O}_H(1)) \\ &= g(H) - \Delta = f. \end{aligned}$$

Therefore we have  $h^1(H, \mathcal{O}_H(1)) > \deg((K_H \otimes \mathcal{O}_H(-1))/4) + 1$  since  $f > e/2 + 1$  and  $\deg \mathcal{O}_H(1) = 2\Delta - e = 2g(H) - 2 - (2f + e - 2)$ . Thus  $\mathcal{O}_H(1)$  satisfies  $\deg \mathcal{O}_H(1) > 2g(H) + 2 - 4h^1(H, \mathcal{O}_H(1))$ . The condition  $f - 1 > 6e - \Delta$  implies  $\deg \mathcal{O}_H(1) > 2g - (g - 1)/6 - 2h^1(H, \mathcal{O}_H(1))$ . Also the condition  $f - 1 < (\Delta - 2e - 6)/3$  yields  $\deg \mathcal{O}_H(1) > (3g + 3)/2$ . Hence  $\mathcal{O}_H(1)$  is normally generated by Theorem 2.1, and thus its general hyperplane section  $H$  is projectively normal since it is linearly normal. Therefore  $S$  is projectively normal with  $q = 0$ ,  $p_g = h^0(S, K_S) = h^1(H, \mathcal{O}_H(1)) = f > 1$  since  $h^1(H, \mathcal{O}_H(2)) = 0$  from  $\deg \mathcal{O}_H(1) > (3g + 3)/2$ .

If we consider the adjunction formula then  $K_S.H = 2f + e - 2$  and  $0 \rightarrow K_S \rightarrow K_S + H \rightarrow K_H \rightarrow 0$ . Thus we have  $0 \rightarrow H^0(S, K_S) \rightarrow H^0(S, K_S + H) \rightarrow H^0(H, K_H) \rightarrow 0$ , since  $H^1(S, K_S) = q = 0$ . Assume  $|K_S + H|$  has a fixed component  $B$ . Set  $p \in B \cap H$ , then  $p$  becomes a base point of  $|K_H|$  since  $H^0(S, K_S + H) \rightarrow H^0(H, K_H)$  is surjective, which cannot occur. Therefore  $K_S + H$  is free from fixed components. Thus for any irreducible curve  $C$  in  $S$ , we can choose effective  $D \in |H + K_S|$  such that  $D$  does not contain  $C$  and then  $D.C \geq 0$ , which implies  $H + K_S$  is nef. Hence we get  $K_S.(H + K_S) \geq 0$  and then

$$K_S^2 \geq -K_S.H = -2f - e + 2.$$

Thus  $-2f - e + 2 \leq K_S^2 \leq (2f + e - 2)^2/(2\Delta - e)$  by the Hodge index theorem  $K_S^2 H^2 \leq (K_S.H)^2$ . Hence the theorem is proved.  $\square$

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