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AN ANALYTIC METHOD IN PROBABILISTIC COMBINATORICS

EUGENIJUS MANSTAVIČIUS

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Abstract

We deal with the value distribution problem for the linear combinations of multiplicities of the cycle lengths of a random permutation. To examine the characteristic functions, we derive asymptotic formulas for ratios of the Taylor coefficients of the relevant generating series. The proposed version of analytic method does not require any analytic continuation of these series outside the convergence disk.

1. Introduction

We are concerned with the value distribution problem of mappings defined on random permutations. For this purpose, one can apply the probabilistic approach developed by R. Arratia, A.D. Barbour and S. Tavaré [1] which is similar to Kubilius' method [17] in probabilistic number theory. Another possibility is to apply the Fourier transforms and to explore relevant asymptotic formulas for the Taylor coefficients of analytic functions. In this direction, the most popular transfer method cultivated by P. Flajolet and A. Odlyzko [9] requires analytic continuation of the generating series outside the convergence disk. Therefore it loses in generality. So far, the most promising method remains the approach extending the Halász' [12] ideas. The first attempt to go along this path was made in our paper [18]. Later that was continued in [20] and [21]. Recently [4], to examine distributions with respect to the Ewens probability on the symmetric group, jointly with G.J. Babu and V. Zacharovas we proposed a simpler version. We now proceed these investigations. Finally, we note that an application of the Voronoi summation formulas is also possible (see the forthcoming paper [25]).

Let $\mathbf{S}_n$ be the symmetric group and $\sigma \in \mathbf{S}_n$ be a permutation having $k_j(\sigma) \geq 0$ cycles of length j, $1 \leq j \leq n$. The *structure vector* is defined as $\bar{k}(\sigma) := (k_1(\sigma), \ldots, k_n(\sigma))$. If $\ell(\bar{k}) := 1k_1 + \cdots + nk_n$, where $\bar{k} := (k_1, \ldots, k_n) \in \mathbf{Z}^{+^n}$, then we have the relation

$$\ell(\bar{k}(\sigma)) = n. \tag{1}$$

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Moreover, if $\ell(\bar{k}) = n$, then the set $\{\sigma \in \mathbf{S}_n\colon \bar{k}(\sigma) = \bar{k}\}$ agrees with the class of conjugate permutations in $\mathbf{S}_n$. Set

$$\nu_n(\,\cdots) = (n!)^{-1}\#\{\sigma \in \mathbf{S}_n :\ \cdots\}$$

for the uniform probability measure on $\mathbf{S}_n$. If ξ_j, $j \geq 1$, are independent Poisson random variables (r.vs) given on some probability space $\{\Omega, \mathcal{F}, P\}$, $\mathbf{E}\xi_j = 1/j$, and $\bar{\xi} := (\xi_1, \ldots, \xi_n)$, then [1]

$$\nu_n(\bar{k}(\sigma) = \bar{k}) = \mathbf{1}\{\ell(\bar{k}) = n\} \prod_{j=1}^{n} \frac{1}{j^{k_j} k_j!} = P(\bar{\xi} = \bar{k} \mid \ell(\bar{\xi}) = n).$$

Moreover,

$$(k_1(\sigma), \ldots, k_n(\sigma), 0, \ldots) \overset{\nu_n}{\Rightarrow} (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots)$$

in the sense of convergence of the finite dimensional distributions. Here and in what follows we assume that $n \to \infty$. Despite that, in dealing with the asymptotic value distribution of the linear combinations

$$h_n(\sigma) := a_{n1}k_1(\sigma) + \cdots + a_{nn}k_n(\sigma), \quad a_{nj} \in \mathbf{R}, \tag{2}$$

called *completely additive* (shortly, *additive*) functions, we face a lot of obstacles. The main reason is dependence of the summands arising from relation (1). So far we lack a general theory. The probabilistic number theory is a bit ahead in this regard (see [6] or [17]). Following its tradition, in our case the main problem can be formulated as follows:

Under what conditions the frequencies $V_n(x; h_n, \alpha) := \nu_n(h_n(\sigma) - \alpha(n) < x)$ with some $\alpha(n) \in \mathbf{R}$ weakly converge to a limit distribution function?

Only in the case of degenerated limit law we have the final answer. To give some impression, we just formulate this result. Set $x^* = \min\{1, |x|\}\operatorname{sign} x$.

Theorem 1 ([23]). *Let $h_n(\sigma)$ be defined in* (2). *The frequencies $V_n(x; h_n, \alpha)$ weakly converge to the degenerated at the point $x = 0$ distribution function if and only if*

$$\sum_{j \leq n} \frac{(a_{nj} - \lambda j)^{*2}}{j} = o(1)$$

and

$$\alpha(n) = n\lambda + \sum_{j \leq n} \frac{(a_{nj} - \lambda j)^{*}}{j} + o(1)$$

for some sequence $\lambda = \lambda_n \in \mathbf{R}$.

After the appearance of V.L. Goncharov's paper [11] the value distribution of particular functions on $\mathbf{S}_n$ was examined by P. Erdős and P. Turán [7]. Under the extra condition $a_{nj} = 0$ for $r < j \le n$, where $r \log r = o(n)$, meaning that $h_n(\sigma)$ is supported only by short cycles, a solution of the main problem was given by V.L. Kolchin and V.P. Chistyakov [16] (see also [15], Section 1.10). Theorem 6 of the paper [2] extended this result under the condition $r = o(n)$ only. R. Arratia and S. Tavaré also observed that the approximation of $h_n(\sigma)$ by an appropriate sum of independent r.vs does not hold if $r \neq o(n)$. That showed the limits of their probabilistic method. Our analytic approach [18] had some advantages in proving general limit theorems, especially for additive functions supported by the long cycles.

The authors of [1] and previous papers have demonstrated the importance of the weighted probabilities in $\mathbf{S}_n$. If $\theta > 0$ is fixed and $w(\sigma) = k_1(\sigma) + \cdots + k_n(\sigma)$ denotes the number of cycles, then

$$\nu_{n,\theta}(\{\sigma\}) := \theta^{w(\sigma)} \left(\sum_{\sigma \in \mathbf{S}_n} \theta^{w(\sigma)} \right)^{-1} = \theta^{w(\sigma)} \theta_{(n)}^{-1},$$

where $\sigma \in \mathbf{S}_n$ and $\theta_{(n)} := \theta(\theta+1)\cdots(\theta+n-1)$, also defines a probability measure on $\mathbf{S}_n$. Identifying the class of conjugate permutations $\{\sigma \in \mathbf{S}_n \colon \bar{k}(\sigma) = \bar{k}\}$ with the partition $n = 1k_1 + \cdots + nk_n$, say κ, we induce the Ewens probability

$$\nu_{n,\theta}(\bar{k}(\sigma) = \bar{k}) =\colon P(\{\kappa\}) = \frac{n!}{\theta_{(n)}} \prod_{j=1}^{n} \left(\frac{\theta}{j} \right)^{k_j} \frac{1}{k_j!}$$

on the set of partitions. Since its introduction into the models of mathematical genetics [8], this probability proved to be useful in many other applied statistical problems (see, for instance, [14]). Generalizing we can define the following extension of $\nu_{n,\theta}$.

By definition, a *completely multiplicative* (shortly, *multiplicative*) function $g \colon \mathbf{S}_n \to \mathbf{C}$ has a decomposition

$$g(\sigma) = \prod_{j=1}^{n} g_j^{k_j(\sigma)},$$

where $\{g_j\}$ is a complex sequence with the property $g_j \not\equiv 0$ and $0^0 := 1$. Now setting

$$\nu_n^{(d)}(\{\sigma\}) = d(\sigma) \left(\sum_{\sigma \in \mathbf{S}_n} d(\sigma) \right)^{-1}, \quad \sigma \in \mathbf{S}_n,$$

where $d \colon \mathbf{S}_n \to \mathbf{R}^+$ is a multiplicative function, maybe, depending on n and defined via $\{d_j := d_{nj}\}$, we define a probability measure on $\mathbf{S}_n$. For motivation, we can refer to

[5], where this measure appears in models of the equilibrium state of some reversible coagulation-fragmentation processes.

In this paper, we examine the asymptotic distribution of $h_n(\sigma)$ with respect to $\nu_n^{(d)}$ assuming the following condition

$$0 < \theta^- \le d_j \le \theta^+ < \infty \tag{3}$$

with some constants θ^- and θ^+. If $\theta^- = \theta^+$, we have the case of Ewens probability studied extensively in [1] and in the series of author's papers written jointly with G.J. Babu (see the references in [4]).

2. Results

Theorem 1 indicates that the function $h_n(\sigma)$ defined in (2) can have the additive component $\lambda\ell(\bar{k}(\sigma)) = \lambda n$ with some $\lambda = \lambda_n \in \mathbf{R}$. Taking this into account, we set $a_{nj}(\lambda) = a_{nj} - \lambda j$. Let Y_j be independent Poisson r.vs, $\mathbf{E}Y_j = d_j/j$ where $j \ge 1$.

Theorem 2. *Assume condition* (3). *Let* $h_n\colon \mathbf{S}_n \to \mathbf{R}$ *be a sequence of additive functions such that, for some* $\lambda = \lambda_n \in \mathbf{R}$,

$$\sum_{j \le n} \frac{a_{nj}(\lambda)^{*2}}{j} \ll 1, \tag{4}$$

and

$$\sum_{\varepsilon n \le j \le n} \frac{a_{nj}(\lambda)^{*2}}{j} = o(1) \tag{5}$$

for each $0 < \varepsilon < 1$.

The following assertions are equivalent:

(i) *the sequence of distribution functions*

$$V_n(x) := \nu_n^{(d)}\big(h_n(\sigma) - \alpha(n) < x\big)$$

weakly converges to a limit distribution;

(ii) *the sequence of distribution functions*

$$P\Bigg(\sum_{j \le n} a_{nj}(\lambda)Y_j - (\alpha(n) - n\lambda) < x\Bigg)$$

weakly converges to a limit distribution;

(iii) *there exists a nondecreasing bounded function* $\Psi(u)$ *defined on* $\bar{\mathbf{R}}$ *such that*

$$\Psi_n(u) := \sum_{\substack{j\le n\\ a_{nj}(\lambda)<u}} \frac{d_j a_{nj}(\lambda)^{*2}}{j}$$

weakly converges to $\Psi(u)$, $\Psi_n(\pm\infty)\to\Psi(\pm\infty)$, *and*

$$\alpha(n) = n\lambda_n + \sum_{j\le n} \frac{d_j a_{nj}(\lambda)^*}{j} + \alpha + o(1) \tag{6}$$

for some constant $\alpha\in\mathbf{R}$.

If the condition (iii) *is satisfied, the limit distribution for the sequences in* (i) *or* (ii) *is the same and its characteristic function has the form*

$$\exp\left\{-it\alpha + \int_{\mathbf{R}} (e^{itu} - 1 - itu^*)u^{*^{-2}}\, d\Psi(u)\right\},\quad t\in\mathbf{R}.$$

The class of limit distributions agrees with the family of infinitely divisible distributions.

Condition (4) follows from (iii) and, thus, from (ii). Theorem 2 of [23] shows that it also holds if the sequence of distribution functions $\nu_n(h_n(\sigma)-\alpha(n)<x)$ is relatively compact. Using similar argument one can extend that for the weighted distributions $V_n(x)$. So, in some parts of Theorem 2, the only extra condition is (5). It allows to truncate the additive functions up to the short cycles. Sometimes, as in Theorem 1, it is necessary or implied by other conditions. We now reckon two such cases.

Corollary 3. *In the previous notation, let*

$$\Psi_n(u) \to \begin{cases} 1 & \textit{if} \quad u>0,\\ 0 & \textit{if} \quad u<0 \end{cases} \tag{7}$$

for some sequence λ_n *of real numbers and let* $\alpha(n)$ *be given by* (6) *with* $\alpha=0$. *Then*

$$V_n(x) \to \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2}\, du. \tag{8}$$

Observe that (7) assures (4) and (5), thus, this corollary concerns the additive functions supported by short cycles only. In [3] by constructing an additive function on long cycles, we have shown that even for $d(\sigma)\equiv 1$ the Lindeberg type condition (7) is not necessary for the relation (8).

To demonstrate a possibility to derive necessary and sufficient convergence conditions, we present the following result.

Theorem 4. *Let condition* (3) *be satisfied and suppose that the functions* $d(\sigma)$ *and* $h_n(\sigma) = h(\sigma)$ *do not depend on* n (*so do also* d_j *and* $a_{nj} =: a_j$) *and* $\alpha(n) \in \mathbf{R}$. *The distribution functions* $\nu_n^{(d)}(h(\sigma) - \alpha(n) < x)$ *weakly converge to a limit law if and only if the series*

$$\sum_{j=1}^{\infty} \frac{d_j (a_j - \lambda j)^{*^2}}{j} \tag{9}$$

converges for some fixed $\lambda \in \mathbf{R}$ *and*

$$\alpha(n) = n\lambda + \sum_{j \le n} \frac{d_j (a_j - \lambda j)^*}{j} + \alpha_1 + o(1), \tag{10}$$

where $\alpha_1 \in \mathbf{R}$ *is a constant.*

Theorem 4 contains the following analog of the Kolmogorov three series theorem.

Corollary 5. *Let condition* (3) *be satisfied. In the notation of* Theorem 4, *the distribution functions* $\nu_n^{(d)}(h(\sigma) < x)$ *weakly converge to a limit law if and only if the series*

$$\sum_{j=1}^{\infty} \frac{d_j a_j^{*^2}}{j}$$

and

$$\sum_{j=1}^{\infty} \frac{d_j a_j^*}{j}$$

converge.

Most probably, the probabilistic method [1], can be refined to prove our results provided that the truncation assumption (5) is satisfied. So far, that was possible only in the case when $d_j \sim \theta$ as $j \to \infty$. Showing another advantage of our analytic approach, as in [18] or [4], one could examine the case when (5) is not satisfied.

3. Quotients of the Taylor coefficients

In this section we derive some asymptotic formulas for the quotients of the Taylor coefficients using an analytic approach which is considerably simpler than that proposed in our paper [21]. Partial cases have been examined in the paper [4].

Let $\{d_j\}$, $j \geq 1$ be a sequence satisfying condition (3). Set

$$D(z) = \sum_{n\geq 0} D_n z^n := \exp\left\{\sum_{j\geq 1} \frac{d_j z^j}{j}\right\}.$$

For a sequence of complex numbers $\{b_j\}$, $j \geq 1$ depending on n and, maybe, on other parameters, $|b_j| \leq 1$ define

$$M(z) = \sum_{n\geq 0} M_n z^n := \exp\left\{\sum_{j\geq 1} \frac{d_j b_j z^j}{j}\right\}.$$

We explore the asymptotic behavior of the ratio M_n/D_n as $n \to \infty$. The goal is to obtain the uniform remainder term estimates. Note that recently G. Freiman and B.L. Granovsky [10] obtained an asymptotic formula for M_n if in our notation $b_j = 1$ and $d_j \asymp j^a$ with $a > 0$. Their method does not work for complex valued b_j.

Without loss of generality, we can take $d_j = b_j = 0$ if $j > n$. Check that differentiating $D(z)$ and comparing the coefficients in the equality obtained we derive the recurrence relation

$$D_n = \frac{1}{n} \sum_{j=1}^{n} d_j D_{n-j}. \tag{11}$$

By virtue of (3) this further leads (see, for instance, [21], Lemma 3.1) to

$$\theta^- c(\theta^+) \exp\left\{\sum_{j\leq n} \frac{d_j - 1}{j}\right\} \leq D_n \leq e\theta^+ \exp\left\{\sum_{j\leq n} \frac{d_j - 1}{j}\right\}, \tag{12}$$

where $c(\theta^+) > 0$ is a constant and $n \geq 1$. Moreover, trivially $|M_n| \leq D_n$.

Proposition 6. *Assume condition* (3). *If*

$$\sum_{j\leq n} \frac{d_j(1 - \Re b_j)}{j} \leq L < \infty, \tag{13}$$

then

$$M_n = \frac{1}{2\pi i n} \int_{1-iK}^{1+iK} \exp\left\{w + \sum_{j\leq n} \frac{d_j b_j}{j} e^{-wj/n}\right\} dw + O(D_n(K^{-c} + n^{-1/2}))$$

for each $2 \leq K \leq n$ *with some positive constant* $c = c(\theta^-)$. *The constant in* $O(\,\cdot\,)$ *depends at most on* L, θ^-, *and* θ^+.

As a corollary, we obtain a result, proved in [21]. Now, for a positive sequence $\mu_n = o(1)$, we assume the additional condition

$$\frac{1}{n}\sum_{j\le n} d_j|1-b_j| \le \mu_n = o(1). \tag{14}$$

Proposition 7. *Under the conditions of* Proposition (6) *and* (14), *the following asymptotic formula holds*

$$\frac{M_n}{D_n} = \exp\left\{\sum_{j\le n}\frac{d_j(b_j-1)}{j}\right\} + O(\mu_n^{c_1} + n^{-c_2}).$$

The constant in $O(\,\cdot\,)$ depends at most on L, θ^-, and θ^+ while $c_1 = c_1(\theta^-,\theta^+) > 0$ and $c_2 = c_2(\theta^-,\theta^+) > 0$.

In the sequel, for brevity, we use the symbol $\ll$ in the place of $O(\,\cdot\,)$. We will need the following estimate obtained in [21].

Proposition 8. *Let condition* (3) *be satisfied. Then*

$$\frac{M_n}{D_n} \ll \exp\left\{-c_3\min_{|\tau|\le\pi}\sum_{j\le n}\frac{d_j(1-\Re(b_je^{-i\tau j}))}{j}\right\},$$

where $c_3 = c_3(\theta^-,\theta^+) > 0$ is a constant.

To prove Proposition 6, we use Cauchy's formula

$$M_n = \frac{1}{2\pi i}\left(\int_{\Delta_0} + \int_{\Delta}\right)\frac{M(z)}{z^{n+1}}\,\mathrm{d}z =: J_0 + J,$$

where $\Delta_0 = \{z = re^{i\tau} : |\tau| \le K/n\}$, $\Delta = \{z = re^{i\tau} : K/n < |\tau| \le \pi\}$, $r = e^{-1/n}$, and $2 \le K \le n$. Check that the substitution $z = e^{-w/n}$ reduces J_0 to the main term of M_n in Proposition 6. Thus it remains to examine the integral J. The main role is played by the polynomial sequence

$$L(z) := \sum_{j\le n}\frac{d_j(b_j-1)}{j}z^j,$$

therefore we start with its estimates. For a parameter $0 < u \le 2$, we set

$$E(u) := \exp\left\{2 \sum_{\substack{j\le n \\ |b_j-1|>u}} \frac{d_j|b_j-1|}{j}\right\} \le \exp\left\{4u^{-1}\sum_{j\le n}\frac{d_j|b_j-1|^2}{j}\right\}$$
$$\le \exp\{-4u^{-1}\Re L(1)\},$$

since $|1-b_j|^2 \le 2(1-\Re b_j)$ for $|b_j| \le 1$. Denote $\theta(u) = 4u\theta^+/\pi$ and

$$l(u) = E(u)\exp\{\Re L(1)\}.$$

Note that, for brevity, the sequences index $n \ge 1$ is omitted in the notation.

Lemma 9. *Let $r = e^{-1/n}$, $z = re^{i\tau}$, and $|\tau| \le \pi$. Then, for arbitrary $0 < u \le 2$,*

$$\exp\{|L(z)-L(1)|\} = \exp\left\{\left|\sum_{j\le n}\frac{d_j(b_j-1)}{j}(z^j-1)\right|\right\} \ll E(u)\left|\frac{1-z}{1-r}\right|^{\theta(u)},$$

where the constant in $\ll$ depends only on u and θ^+.

Proof. We use the argument given in the paper [21]. We have

$$|1-e^{ix}| = \frac{4}{\pi}\sum_{m\in\mathbf{Z}}\frac{1}{1-4m^2}e^{imx}$$

for arbitrary $x \in \mathbf{R}$. Hence

$$\begin{aligned}|L(z)-L(1)| &\le \theta^+u\sum_{j\le n}\frac{|z^j-1|}{j} + \log E(u)\\ &\le \theta^+u\sum_{j\ge 1}\frac{r^j|e^{i\tau j}-1|}{j} + \theta^+u + \log E(u)\\ &\le \theta(u)\log\frac{|1-re^{i\tau}|}{1-r} + \theta^+u + \log E(u)\\ &\quad + \theta(u){\sum_{m\in\mathbf{Z}\setminus\{0\}}}'\frac{1}{4m^2-1}\log\frac{|1-re^{im\tau}|}{|1-re^{i\tau}|},\end{aligned}\tag{15}$$

where the dash denotes that the summation is restricted to those $m \ne 0$ for which $|1-re^{im\tau}| > |1-re^{i\tau}|$. For such m, we also have $|1-re^{im\tau}|/|1-re^{i\tau}| \ll m$ with an absolute constant in $\ll$. So, the last sum in (15) is bounded. For some bounded quantity $C(\theta^+, u)$, we obtain

$$|L(z) - L(1)| \le \theta(u) \log \frac{|1 - re^{i\tau}|}{1 - r} + \log E(u) + C(\theta^+, u).$$

The lemma is proved. □

Lemma 10. *Let $r = e^{-1/n}$, $z = re^{i\tau}$, $|\tau| \le \pi$, and $0 < u \le 2$ be arbitrary. In the notation above, we have*

$$M(z) \ll nD_n l(u) \left| \frac{1 - z}{1 - r} \right|^{\theta(u) - \theta^-}.$$

Proof. It suffices to use the identity

$$M(z) = D(1) \exp\{L(1)\} \exp\{L(z) - L(1)\} \frac{D(z)}{D(1)},$$

(12), the estimate

$$\begin{aligned} \frac{|D(z)|}{D(1)} &\le \exp\left\{ \sum_{j \le n} \frac{d_j r^j}{j} (\cos \tau j - 1) \right\} \\ &\ll \exp\left\{ \sum_{j \ge 1} \frac{d_j r^j}{j} (\cos \tau j - 1) \right\} \le \left| \frac{1 - r}{1 - z} \right|^{\theta^-}, \end{aligned}$$

and Lemma 9. The lemma is proved. □

Lemma 11. *Let $r = e^{-1/n}$, $2 \le K \le n$, and $0 \le j \le n$. If condition* (3) *is satisfied, then*

$$I_K(j) := \int\limits_{K/n < |\tau| \le \pi} M(re^{i\tau}) e^{-ij\tau} \, \mathrm{d}\tau \ll D_j \log K + D_n + \sum_{3j/2 < m \le n} \frac{D_m}{m}.$$

Proof. Integrating the power series by parts, we obtain

$$\begin{aligned} I_K(j) &= \sum_{m=0}^{\infty} M_m r^m \int\limits_{K/n < |\tau| \le \pi} e^{i(m-j)\tau} \, \mathrm{d}\tau \\ &= M_j r^j \left(2\pi - 2\frac{K}{n} \right) - 2 \sum_{m \ge 0, m \ne j} M_m r^m \frac{\sin((K/n)(m - j))}{m - j}. \end{aligned}$$

From (12) we have $D_j \ll D_m \ll D_j$ if $|m-j| \le j/2$, therefore

$$I_K(j) \ll D_j + D_j \sum_{1\le|m-j|\le j/2} \left|\frac{\sin((K/n)(m-j))}{m-j}\right| + \sum_{|m-j|>j/2} \frac{D_m}{|m-j|} r^m. \tag{16}$$

Further we apply the estimates

$$\sum_{\substack{1\le|m-j|\le j/2\\ m\ne j}} \left|\frac{\sin((K/n)(m-j))}{m-j}\right| \le \sum_{1\le|m-j|\le n/K} \frac{K}{n} + \sum_{n/K\le|m-j|\le j/2} \frac{1}{|m-j|}$$
$$\ll 1 + \log\left(2 + \frac{j}{2}\frac{K}{n}\right) \ll \log K$$

and

$$\sum_{|m-j|>j/2} \frac{D_m}{|m-j|} r^m \ll \frac{1}{j+1} \sum_{0\le m<j/2} D_m + \sum_{3j/2<m\le n} \frac{D_m}{m} + \frac{1}{n}\sum_{m>n} D_m r^m$$
$$\ll D_j + \sum_{3j/2<m\le n} \frac{D_m}{m} + \frac{D(1)}{n}.$$

In the last step we have used (3) and (11). The estimate $D(1) \ll nD_n$ following from (12) yields the desired result. Lemma 11 is proved. □

Lemma 12. *Let $\varepsilon \in [2/n, 1/2]$ be arbitrary. If $\theta(u) < \theta^-$, then*

$$J \ll (\varepsilon^{\theta^-} + \varepsilon \log(\varepsilon^{-1}))D_n \log K + D_n l(u) K^{\theta(u)-\theta^-} \varepsilon^{-1/2}.$$

Proof. Since by Lemma 10

$$\max_{z\in\Delta} |M(z)| \ll nD_n l(u) K^{\theta(u)-\theta^-}, \tag{17}$$

integrating by parts, we obtain

$$\begin{aligned} J &= \frac{1}{2\pi i n} \int_\Delta \frac{M(z)}{z^n} \left(\sum_{j\le n} d_j b_j z^{j-1}\right) \mathrm{d}z + O(D_n l(u) K^{\theta(u)-\theta^-}) \\ &= \frac{1}{2\pi n} \sum_{j\le n} d_j b_j r^{j-n} \int_{K/n\le|\tau|\le\pi} M(re^{i\tau}) e^{i\tau(j-n)} \,\mathrm{d}\tau + O(D_n l(u) K^{\theta(u)-\theta^-}) \\ &\ll \frac{1}{n} \sum_{0\le j\le T} I_K(j) + \frac{1}{n} \sum_{T<j\le n} \left| \int_{K/n\le|\tau|\le\pi} M(re^{i\tau}) e^{-ij\tau} \,\mathrm{d}\tau \right| + D_n l(u) K^{\theta(u)-\theta^-} \\ &=: \frac{1}{n} \sum_{0\le j\le T} I_K(j) + I + D_n l(u) K^{\theta(u)-\theta^-}, \end{aligned} \tag{18}$$

where $T=[\varepsilon n]$ and $I_K(j)$ have been defined in Lemma 11. This lemma yields

$$\frac{1}{n}\sum_{0\le j\le T} I_K(j) \ll \frac{\log K}{n}\sum_{0\le j\le T} D_j + \frac{TD_n}{n} + \frac{1}{n}\sum_{0\le j\le T}\sum_{3j/2<m\le n}\frac{D_m}{m}.$$

Since by virtue of (3), (11), and (12), we have

$$D_m \ll D_n\left(\frac{n}{m}\right)^{1-\theta^-}, \quad T\le m\le n$$

from the last estimate, using (11) again, we obtain

$$\begin{aligned}\frac{1}{n}\sum_{0\le j\le T} I_K(j) &\ll (\varepsilon^{\theta^-}+\varepsilon)D_n\log K + \frac{T}{n}\sum_{3T/2\le m\le n}\frac{D_m}{m} + \frac{1}{n}\sum_{m\le 3T/2} D_m \\ &\ll (\varepsilon^{\theta^-}+\varepsilon)D_n\log K + \varepsilon D_n n^{1-\theta^-}\sum_{3T/2\le m\le n} m^{\theta^- -2} \\ &\ll (\varepsilon^{\theta^-}+\varepsilon\log(\varepsilon^{-1}))D_n\log K.\end{aligned}\tag{19}$$

To estimate the term I in (18), we again use integration by parts and (17). Further applying Cauchy's inequality, we obtain

$$\begin{aligned} I &\ll \frac{1}{n}\sum_{T<j\le n}\frac{1}{j}\left|\int_{K/n\le|\tau|\le\pi} M'(re^{i\tau})e^{-i(j-1)\tau}\,\mathrm{d}\tau\right| + \frac{1}{n}\sum_{T<j\le n}\frac{1}{j}\max_{z\in\Delta}|M(z)| \\ &\ll \frac{1}{n\sqrt{T}}\left(\sum_{1\le j\le n}\left|\int_{K/n\le|\tau|\le\pi} M'(re^{i\tau})e^{-i(j-1)\tau}\,\mathrm{d}\tau\right|^2\right)^{1/2} \\ &\quad + D_n l(u)K^{\theta(u)-\theta^-}\log(\varepsilon^{-1}).\end{aligned}$$

The integrals under the last sum are just the Fourier coefficients of an appropriate function therefore, via Parseval's identity, we further have

$$\begin{aligned} I &\ll \frac{1}{n\sqrt{T}}\left(\int_{K/n\le|\tau|\le\pi}|M'(re^{i\tau})|^2\,\mathrm{d}\tau\right)^{1/2} + D_n l(u)K^{\theta(u)-\theta^-}\log(\varepsilon^{-1}) \\ &= \frac{1}{n\sqrt{T}}\left(\int_{K/n\le|\tau|\le\pi}|M(re^{i\tau})|^2\left|\sum_{j\le n} d_j b_j r^j e^{i\tau j}\right|^2\mathrm{d}\tau\right)^{1/2} \\ &\quad + D_n l(u)K^{\theta(u)-\theta^-}\log(\varepsilon^{-1}).\end{aligned}$$

By Lemma 10 and Parseval's identity again, we obtain

$$
\begin{aligned}
I &\ll D_n l(u) K^{\theta(u)-\theta^-} (n\varepsilon)^{-1/2} \left(\int_{K/n \le |\tau| \le \pi} \left| \sum_{j \le n} d_j b_j r^j e^{i\tau j} \right|^2 \mathrm{d}\tau \right)^{1/2} \\
&\quad + D_n l(u) K^{\theta(u)-\theta^-} \log(\varepsilon^{-1}) \\
&\ll D_n l(u) K^{\theta(u)-\theta^-} \varepsilon^{-1/2}.
\end{aligned}
$$

Inserting this and (19) into (18) we complete the proof of Lemma 12. □

Proof of Proposition 6. As we have mentioned $M_n = J_0 + J$ and J_0 gives the main term. It remains to apply Lemma 12. Fix u to assure $\theta(u) = 4u\theta^+/\pi \le \theta^-/2$. The condition of Proposition 6 implies $l(u) \ll 1$. If $\theta^- < 2$, and $K \ge K_0(\theta^-)$ is sufficiently large, choosing $\varepsilon = K^{-\theta^-/2}$ we obtain

$$
J \ll K^{-((\theta^-)^2/2) \wedge (\theta^-/4)} D_n .
$$

The same choice of ε is possible and the last estimate holds if $\theta^- \ge 2$ and $2/n \le K^{-\theta^-/2}$. For $(n/2)^{2/\theta^-} \le K \le n$, we can take $\varepsilon = n^{-2/3}$ to get even better estimate than we need $J \ll (\log^2 n) n^{-2/3} D_n$.

The proposition is proved. □

Proof of Proposition 7. As a corollary, from Proposition 6 we have

$$
D_n = \frac{1}{2\pi i n} \int_{1-iK}^{1+iK} e^w D(e^{-w/n})\, dw + O(D_n(K^{-c} + n^{-1/2})) \tag{20}
$$

and

$$
\begin{aligned}
M_n &= \exp\left\{ \sum_{j \le n} \frac{d_j(b_j - 1)}{j} \right\} \\
&\quad \times \frac{1}{2\pi i n} \int_{1-iK}^{1+iK} e^w D(e^{-w/n}) \exp\left\{ \sum_{j \le n} \frac{d_j(b_j - 1)}{j} (e^{-wj/n} - 1) \right\} dw \\
&\quad + O(D_n(K^{-c} + n^{-1/2})).
\end{aligned}
$$

In the previous notation, the last sum under the exponential function is just $L(e^{-w/n}) - L(1)$ therefore using Lemma 9 and the trivial estimate

$$
|L(e^{-w/n}) - L(1)| = \left| \sum_{j \le n} \frac{d_j(b_j - 1)}{j} (e^{-wj/n} - 1) \right| \le |w| \mu_n ,
$$

following from (14), we obtain

$$\exp\{L(e^{-w/n}) - L(1)\} - 1 \ll |w|\mu_n \left|\frac{1-e^{-w/n}}{1-r}\right|^{\theta(1)} \ll |w|^{1+\theta(1)}\mu_n.$$

Here the constant in $\ll$ depends at most on θ^+ and L. By Lemma 10 we also have $D(e^{-w/n}) \ll nD_n$ with the additional dependence on θ^-. Inserting the last estimates into the integral expression of M_n, we derive

$$M_n = \exp\left\{\sum_{j\le n} \frac{d_j(b_j-1)}{j}\right\} \frac{1}{2\pi i n} \int_{1-iK}^{1+iK} e^w D(e^{-w/n})\, dw$$
$$+ O(D_n\mu_n K^{2+\theta(1)}) + O(D_n(K^{-c} + n^{-1/2})).$$

Applying now (20) and choosing $K = (\min(\mu_n^{-1}, n))^{c_1}$ with sufficiently small positive constant c_1 depending at most on θ^- and θ^+, we complete the proof of Proposition 7. □

We end this section with the observation that Propositions 6 and 7 hold under condition $l(u) \ll 1$ for some $0 < u \le \pi\theta^-/8\theta^+$ which is weaker than (13).

4. Proofs of Theorems and Corollaries

The main probabilistic ingredient is the following lemma.

Lemma 13. *Assume that a sequence of characteristic functions $\varphi_n(t)$ has the following representation*

$$\varphi_n(t) = \exp\left\{-it\gamma_n + \int_{\mathbf{R}} (e^{itu} - 1 - itu^*)u^{*^{-2}}\, d\Psi_n(u)\right\}, \quad t \in \mathbf{R},$$

where $\gamma_n \in \mathbf{R}$ and $\Psi_n(u)$ is a nondecreasing bounded function defined on $\bar{\mathbf{R}}$. Then $\varphi_n(t)$ converges to a characteristic function if and only if there exist a constant $\gamma \in \mathbf{R}$ and a nondecreasing bounded function $\Psi(u)$ defined on $\bar{\mathbf{R}}$ such that $\gamma_n \to \gamma$, $\Psi_n(u)$ weakly converges to $\Psi(u)$, and $\Psi_n(\pm\infty) \to \Psi(\pm\infty)$ as $n \to \infty$.

Proof. See [24]. Check that we have slightly changed Lévy's canonical representation. Our form can reduced to the original one by substitution

$$u^{*^{-2}}\Psi_n(u) = u^{-2}(1+u^2)\tilde{\Psi}_n(u),$$

where $\tilde{\Psi}_n(u)$ is a nondecreasing bounded function defined on $\bar{\mathbf{R}}$. □

The so-called convergence of types of distributions controls the centralizing constants.

Lemma 14. *Let $F_n(x)$, $F(x)$, and $G(x)$ be distribution functions. If, for some α_n, $F_n(x)$ and $F_n(x+\alpha_n)$ weakly converge to $F(x)$ and $G(x)$ respectively, then there exists a constant $\alpha \in \mathbf{R}$ such that $\alpha_n \to \alpha$ and $F(x+\alpha) = G(x)$.*

Proof. See [24]. □

Proof of Theorem 2. Equivalence of (ii) and (iii) is well known. Actually, it follows from Lemmas 13 and 14.

Let conditions (4) and (5) be satisfied. We have $V_n(x) = \nu_n^{(d)}\big((h_n(\sigma) - \lambda\ell(\overline{k}(\sigma))) - (\alpha(n) - \lambda n) < x\big)$, thus without loss of generality, we can take $\lambda = 0$. We now apply Proposition 7 for

$$M_n = \frac{1}{n!}\sum_{\sigma\in\mathbf{S}_n} d(\sigma)e^{ith_n(\sigma)}, \quad t \in \mathbf{R}.$$

Check that

$$\sum_{j\le n}\frac{d_j}{j}(1-\cos ta_{nj}) \le (1+T^2)\sum_{j\le n}\frac{d_j a_{nj}^{*^2}}{j} \tag{21}$$

and

$$\begin{aligned}\mu_n &:= \frac{1}{n}\sum_{j\le n} d_j|1-e^{ita_{nj}}| \\ &\le 2\theta^+\varepsilon + \left(\sum_{\varepsilon n<j\le n}\frac{d_j}{j}\right)^{1/2}\left(2\sum_{\varepsilon n<j\le n}\frac{d_j}{j}(1-\cos ta_{nj})\right)^{1/2} \\ &\ll \varepsilon + \left(\log\left(\frac{1}{\varepsilon}\right)+1\right)^{1/2}\left(\sum_{\varepsilon n<j\le n}\frac{d_j a_{nj}^{*^2}}{j}\right)^{1/2}\end{aligned} \tag{22}$$

uniformly in $|t| \le T$ for arbitrary $T > 0$ and $0 < \varepsilon < 1$. Under conditions (4) and (5), sum (21) is bounded and $\mu_n = o(1)$. So from Proposition 7 we obtain an asymptotic formula for the characteristic function $\varphi_n(t)$ of $V_n(x)$. We have

$$\begin{aligned}\varphi_n(t) &= \exp\left\{-it\alpha(n) + \sum_{j\le n}\frac{d_j}{j}(1-e^{ita_{nj}})\right\} + o(1) \\ &= \exp\left\{-it\left(\alpha(n) - \sum_{j\le n}\frac{d_j a_{nj}^*}{j}\right) + \int_{\mathbf{R}}(e^{itu}-1-itu^*)u^{*^{-2}}\,d\Psi_n(u)\right\} + o(1)\end{aligned}$$

uniformly in $|t| \le T$. Now, equivalence of (i) and (iii) follows from Lemmas 13 and 14.

The last expression of $\varphi_n(t)$ shows that the limit law must be infinitely divisible. To show that the limiting distributions comprise the whole class, it suffices to apply Theorem 1 of A. Hildebrand [13].

Theorem 2 is proved. □

Proof of Theorem 4. Since convergence of (9) assures conditions (21) and, as in (22), the estimate $\mu_n = o(1)$, the sufficiency part follows from Theorem 2.

Assume now that $\nu_n^{(d)}(h(\sigma) - \alpha(n) < x)$ weakly converges to a distribution function. For the characteristic functions, this implies that

$$\frac{e^{-it\alpha(n)}}{D_n} \sum_{\sigma \in \mathbf{S}_n} d(\sigma) e^{ith(\sigma)} = \varphi(t) + o(1)$$

uniformly in $|t| \le T$ for each $T > 0$. Moreover, $|\varphi(t)| \ge 1/2$ in some neighborhood $|t| \le t_0$ with $0 < t_0 \le 1$. Hence and by Proposition 8 for every such t there exist a $\lambda(t) \in [-\pi, \pi]$ such that

$$\sum_{j=1}^{\infty} \frac{d_j(1 - \cos(ta_j - \lambda(t)j))}{j} < \infty. \tag{23}$$

By (3), the factors d_j can be omitted in the series (23). Combining this for t_1, t_2, and $t_1 + t_2$ from the interval $[-t_0, t_0]$ and using the inequality

$$1 - \cos(x + y) \le 2(1 - \cos x) + 2(1 - \cos y), \quad x, y \in \mathbf{R}, \tag{24}$$

we obtain

$$\sum_{j=1}^{\infty} \frac{1 - \cos((\lambda(t_1 + t_2) - \lambda(t_1) - \lambda(t_2))j)}{j} < \infty.$$

This is possible only in the case $\|(\lambda(t_1 + t_2) - \lambda(t_1) - \lambda(t_2))/2\pi\| = 0$, where $\|\cdot\|$ denotes the distance to the nearest integer. As it has been observed in [22], the last equality implies the linearity of the function $\lambda(t)$. So, we can write $\lambda(t) = \lambda t$ with a constant λ for $t \in [-t_0, t_0]$. Inserting this into (23) we see that the series

$$\sum_{j=1}^{\infty} \frac{d_j(1 - \cos(ta_j(\lambda)))}{j}.$$

converges if $|t| \le t_0$. Here $a_j(\lambda) := a_j - \lambda j$. Again by (24), the convergence region for the last series can be extended to $t \in \mathbf{R}$. Using the inequality $1 - \cos x \ge 2x^2/\pi$ for $|x| \le \pi$ and integration over the interval $[0, T]$ with an arbitrary $T > 0$, we establish that the convergence of the last series is equivalent to condition (9). Under it, using the proved sufficiency part of this theorem and Lemma 14, we see that the centralization sequence $\alpha(n)$ must have the form (10).

Theorem 4 is proved. □

Proof of Corollary 5. Sufficiency trivially follows from Theorem 4. If the limit law exists, by this theorem we obtain convergence of (9) and relation (10) for $\alpha(n) = 0$ with some constant $\lambda \in \mathbf{R}$. It implies

$$-\lambda = \frac{1}{n} \sum_{j \le n} \frac{d_j (a_j - \lambda j)^*}{j} + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right) = o(1)$$

as $n \to \infty$. Hence $\lambda = 0$. Moreover, by (10), we obtain

$$\sum_{j \le n} \frac{d_j a_j^*}{j} = -\alpha_1 + o(1).$$

This shows convergence of the remaining series in Corollary 5.

The corollary is proved. □

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Institute of Mathematics and Informatics
Akademijos str. 4
LT–08663 Vilnius
Lithuania
e-mail: eugenijus.manstavicius@mif.vu.lt