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## A NOTE ON MULTIPLY TRANSITIVE GROUPS

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The purpose of the present note is to prove the following theorem and to give some applications of it.

**Theorem.** *Let  $H$  be a transitive group on  $\Gamma = \{1, 2, \dots, n\}$  other than  $S_n$  and  $A_n$ , and assume  $H_1$ , the stabilizer of a letter 1, leaves only one letter 1 invariant. If  $H$  can be successively extended to 2-, 3-,  $\dots$ ,  $(t+1)$ -fold transitive groups,  $G^2, G^3, \dots, G^{t+1} = G$ , then the centralizer of  $H$  in  $G$  is trivial and the outer automorphism group of  $H$  contains a subgroup isomorphic to  $S_t$ , the symmetric group on  $t$  letters.*

NOTATION. For a subgroup  $H$  of  $G$ , the normalizer (or centralizer) of  $H$  in  $G$  will be denoted by  $N_G(H)$  (or  $C_G(H)$ ). If  $G$  is a permutation group on  $\Omega$  and a subset  $X$  of  $G$  fixes a subset  $\Gamma$  of  $\Omega$ , then  $X$  induces a set of permutations on  $\Gamma$ , which is denoted by  $X^\Gamma$ .

To prove the theorem, we need the following

**Lemma.** *Let  $G$  be a permutation group on  $\Omega$ , and  $H$  a subgroup of  $G$  which is transitive on a subset  $\Gamma$  of  $\Omega$ . Then  $C_G(H)$  is semi-regular or identity on  $\Gamma$ .*

Proof. Let  $c$  be an element of  $C_G(H)$  and assume  $c$  fixes a letter  $\alpha$  in  $\Gamma$ . Then  $\alpha^h \in I(c)$  for every  $h \in H$ . Since  $H$  is transitive on  $\Gamma$ ,  $I(c) \supset \Gamma$ . Namely  $c^\Gamma = 1$ .

Proof of Theorem. Let  $H$  satisfy the assumption of the theorem and  $G$  be a  $t$ -times successive transitive extension of  $H$  operating  $(t+1)$ -fold transitively on  $\Omega = \Gamma \cup \Delta$ , where  $\Delta$  is the set of new letters  $\{1', 2', \dots, t'\}$ . We remark first that  $G$  does not contain an element whose degree is less than  $t+1$ . Here by the degree of an element  $x$  we mean the number of letters moved by  $x$ . In fact, if  $G$  contains such an element,  $G$  must contain the alternating group  $A^\Omega$  by the  $t$ -fold transitivity of  $G$ .

Now let  $c$  be an element of  $C_G(G_{1', 2', \dots, t'}) = C_G(H)$ . Then  $c^\Gamma = 1$  or  $c^\Gamma$  is semi-regular by the above lemma. But since  $c$  centralizes  $H_\alpha$  for  $\alpha \in \Gamma$  and  $H_\alpha^\Gamma$  fixes  $\alpha$  only,  $c$  must fix  $\alpha$ . Hence  $c^\Gamma = 1$ . Then  $c = 1$  by the above remark.

The second part of the theorem is an easy consequence of the first part. By a lemma of Witt ([6], Th 9.4) we have  $N_G(H)^\Delta \cong N_G(H)/H \cong S_t$ .

On the other hand,  $N_G(H)/C_G(H)H = N_G(H)/H$  is isomorphic to a subgroup of the outer automorphism group of  $H$ . Thus we have the assertion.

Now Nagao [4] proved that the stabilizer  $G_{1234}$  in a 4-fold transitive group  $G$  fixes exactly four letters unless  $G$  is  $S_5, A_6$  or  $M_{11}$ . Hence we have

**Corollary 1.** *Let  $G$  be a non trivial  $t$ -fold transitive group with  $t \geq 4$ . Then the outer automorphism group of the stabilizer  $G_{1,2,\dots,t-1}$  contains  $S_{t-1}$  except the case  $G = M_{11}$  with  $t=4$  and  $G = M_{12}$  with  $t=5$ .*

By Burnside's theorem, a minimal normal subgroup of a doubly transitive group is primitive simple or elementary abelian ([2], §154). Suzuki [5] proved that a doubly transitive group whose minimal normal subgroup is elementary abelian does not admit a twice successive transitive extension unless it is  $S_2, S_3, A_4, S_4$  or  $M_9$ . If a doubly transitive group  $H$  has a non trivial 2 core, then by the theorem of Feit-Thompson,  $H$  has a minimal normal subgroup which is elementary abelian. Therefore  $H$  does not admit a twice successive transitive extension unless  $H = S_3$  or  $M_9$ .

On the other hand, to 2 core free doubly transitive groups we can apply the following theorems of Brauer and Glauberman.

**Theorem.** (Brauer [1], Th. 5) *If  $G$  is 2 core free and a Sylow 2 subgroup  $S$  of  $G$  is elementary abelian of order at most eight, then the outer automorphism group of  $G$  is solvable unless  $|G| = 8$ .*

**Theorem.** (Glauberman [3], Th. 4) *If  $G$  is 2 core free and a Sylow 2 subgroup  $S$  of  $G$  satisfies any of the following conditions, then the outer automorphism group of  $G$  is solvable.*

- (a) *Aut  $(S)$  is solvable.*
- (b)  *$S$  can be generated by two elements.*
- (c)  *$S$  can be generated by three elements and  $N_G(S)/C_G(S)$  is not a 2 group.*

Thus by combining with our theorem we have

**Corollary 2.** *If  $H$  is a non trivial doubly transitive group and a Sylow 2 subgroup of  $H$  satisfies one of the above conditions, then  $H$  does not admit a five times successive transitive extension.*

REMARK. The author knows no simple group whose outer automorphism group contains  $S_4$ . Therefore from Corollary 1 we have that any simple group known at present can not be a stabilizer of four letters in a 5-fold transitive group unless  $H = A_n$ .

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