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A NOTE ON MULTIPLY TRANSITIVE GROUPS

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The purpose of the present note is to prove the following theorem and to give some applications of it.

Theorem. *Let H be a transitive group on $\Gamma=\{1, 2, \cdots, n\}$ other than S_n and A_n, and assume H_1, the stabilizer of a letter 1, leaves only one letter 1 invariant. If H can be successively extended to 2-, 3-, $\cdots$, $(t+1)$-fold transitive groups, G^2, G^3, $\cdots$, $G^{t+1}=G$, then the centralizer of H in G is trivial and the outer automorphism group of H contains a subgroup isomorphic to S_t, the symmetric group on t letters.*

NOTATION. For a subgroup H of G, the normalizer (or centralizer) of H in G will be denoted by $N_G(H)$ (or $C_G(H)$). If G is a permutation group on Ω and a subset X of G fixes a subset Γ of Ω, then X induces a set of permutations on Γ, which is denoted by X^Γ.

To prove the theorem, we need the following

Lemma. *Let G be a permutation group on Ω, and H a subgroup of G which is transitive on a subset Γ of Ω. Then $C_G(H)$ is semi-regular or identity on Γ.*

Proof. Let c be an element of $C_G(H)$ and assume c fixes a letter α in Γ. Then $\alpha^h \in I(c)$ for every $h \in H$. Since H is transitive on Γ, $I(c) \supset \Gamma$. Namely $c^\Gamma=1$.

Proof of Theorem. Let H satisfy the assumption of the theorem and G be a t-times successive transitive extension of H operating $(t+1)$-fold transitively on $\Omega=\Gamma \cup \Delta$, where Δ is the set of new letters $\{1', 2', \cdots, t'\}$. We remark first that G does not contain an element whose degree is less than $t+1$. Here by the degree of an element x we mean the number of letters moved by x. In fact, if G contains such an element, G must contain the alternating group A^Ω by the t-fold transitivity of G.

Now let c be an element of $C_G(G_{1',2',\dots,t'})=C_G(H)$. Then $c^\Gamma=1$ or c^Γ is semi-regular by the above lemma. But since c centralizes H_α for $\alpha \in \Gamma$ and H_α^Γ fixes α only, c must fix α. Hence $c^\Gamma=1$. Then $c=1$ by the above remark.

The second part of the theorem is an easy consequence of the first part. By a lemma of Witt ([6], Th 9.4) we have $N_G(H)^{\Delta} \cong N_G(H)/H \cong S_t$.

On the other hand, $N_G(H)/C_G(H)H = N_G(H)/H$ is isomorphic to a subgroup of the outer automorphism group of H. Thus we have the assertion.

Now Nagao [4] proved that the stabilizer G_{1234} in a 4-fold transitive group G fixes exactly four letters unless G is S_5, A_6 or M_{11}. Hence we have

Corollary 1. *Let G be a non trivial t-fold transitive group with* $t \geq 4$. *Then the outer automorphism group of the stabilizer* $G_{1,2,\cdots,t-1}$ *contains* S_{t-1} *except the case* $G = M_{11}$ *with* $t=4$ *and* $G = M_{12}$ *with* $t=5$.

By Burnside's theorem, a minimal normal subgroup of a doubly transitive group is primitive simple or elementary abelian ([2], §154). Suzuki [5] proved that a doubly transitive group whose minimal normal subgroup is elementary abelian does not admit a twice successive transitive extension unless it is S_2, S_3, A_4, S_4 or M_9. If a doubly transitive group H has a non trivial 2 core, then by the theorem of Feit-Thompson, H has a minimal normal subgroup which is elementary abelian. Therefore H does not admit a twice successive transitive extention unless $H = S_3$ or M_9.

On the other hand, to 2 core free doubly transitive groups we can apply the following theorems of Brauer and Glauberman.

Theorem. (Brauer [1], Th. 5) *If G is 2 core free and a Sylow 2 subgroup S of G is elementary abelian of order at most eight, then the outer automorphism group of G is solvable unless* $|G| = 8$.

Theorem. (Glauberman [3], Th. 4) *If G is 2 core free and a Sylow 2 subgroup S of G satisfies any of the following conditions, then the outer automorphism group of G is solvable.*

(a) *Aut (S) is solvable.*

(b) *S can be generated by two elements.*

(c) *S can be generated by three elements and* $N_G(S)/C_G(S)$ *is not a 2 group.*

Thus by combining with our theorem we have

Corollary 2. *If H is a non trivial doubly transitive group and a Sylow 2 subgroup of H satisfies one of the above conditions, then H does not admit a five times successive transitive extention.*

REMARK. The author knows no simple group whose outer automorphism group contains S_4. Therefore from Corollary 1 we have that any simple group known at present can not be a stabilizer of four letters in a 5-fold transitive group unless $H = A_n$.

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