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A COCHAIN COMPLEX ASSOCIATED TO THE STEENROD ALGEBRA

In Memory of the late Professor José Adem

NOBUO SHIMADA

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0. Introduction

In [8], the author introduced an acyclic, free resolution of the ground ring $\mathbb{Z}$ of integers (resp. its localization $\mathbb{Z}(p)$ for a prime $p$) as the trivial module over the Landweber-Novikov algebra $S$ (resp. $S(p) = \mathbb{Z}(p) \otimes S$), which is considerably smaller than the bar resolution.

In this paper, the same method of construction is applied to the case of the mod $p$ Steenrod algebra $A$. The resulted resolution $X = A \otimes \hat{X} \rightarrow \mathbb{Z}/p$ has inductively defined differential $d$ and contracting homotopy $\sigma$, and is naturally embedded in the bar resolution $B(A)$ as a direct-summand subcomplex.

The apparent feature of this resolution is that it seems to be an immediate 'lift' of the May resolution [5], while the latter is a resolution over the associated graded algebra $E^0 A$ for the augmentation filtration on the Steenrod algebra. In fact, the corresponding filtration on $X$ leads to an equivalent of the May spectral sequence, of which $E^1 X$ is isomorphic to the May resolution and $E^r$-terms are the same as those of the May spectral sequence for $r \geq 2$.

In the case $p = 2$, the chain complex $\hat{X}$ will be given as a polynomial ring $P$, and the dual cochain complex $P^*$ has a non-associative product, which induces the usual associative product in its cohomology $H^*(A) = \text{Ext}^*_{\mathbb{Z}/2}(\mathbb{Z}/2, \mathbb{Z}/2)$, the $E_2$-term of the Adams spectral sequence [1,2].

May [5] studied extensively his spectral sequence and succeeded to obtain a great deal of information about $H^*(A)$ (See also, Tangora [10] and Novikov [7]).

It is hoped that the present work could be useful for calculating the differentials in the May spectral sequence and the ring structure of $H^*(A)$.

In this paper we shall restrict ourselves to the case $p = 2$. A parallel treatment for the odd prime case will be only suggested in the last section.
1. Notation and results

Let $A_*$ be the dual Hopf algebra ([6],[9]) of the mod 2 Steenrod algebra $A$. $A_*$ is given as the polynomial algebra $\mathbb{Z}/2[\xi_1, \xi_2, \ldots]$ over $\mathbb{Z}/2$ on indeterminates $\xi_i (i \geq 1)$ of degree $2^i - 1$, with comultiplication

$$\psi_{\xi_k} = \sum_{i=0}^{k} \xi_k^{2^i} \otimes \xi_i \quad (\xi_0 = 1).$$

Let $e_{i,k}=(\xi_i^k)^*$ denote the dual element of $\xi_i^k$ with respect to the monomial basis $\{\xi_1, \xi_2, \ldots, \xi_n\}$ of $A_*^*$.

**Lemma 1.1.** (i) The Steenrod algebra $A$ is multiplicatively generated by the set $\{e_{i,2^k}; i \geq 1, k \geq 0\}$, (ii) the set $\{1, e_{i_1,2^{k_1}} \cdots e_{i_n,2^{k_n}}; (i_1,k_1) < (i_2,k_2) < \cdots < (i_n,k_n)\}$ in the lexicographical order forms a $\mathbb{Z}/2$-basis of $A$, of which elements $e_{i,k} = e_{i,2^{k_1}} \cdots e_{i_n,2^{k_n}}$ are called admissible monomials.

Let $L$ denote the $\mathbb{Z}/2$-submodule of $A$ spanned by the set $\{e_{i,2^k}; i \geq 1, k \geq 0\}$, and $sL = \mathbb{Z}/2 \langle \{e_{i,2^k}\}; \; i \geq 1, k \geq 0 \rangle$, the suspension of $L$, with bideg $\langle e_{i,2^k} \rangle = (1,2^k(2^i - 1))$. Denote by $P = P(sL)$ the polynomial algebra (symmetric tensor algebra) on $sL$. We use the notation

$$\langle e_J \rangle = \langle e_{j_1,2^{l_1}}, \ldots, e_{j_s,2^{l_s}} \rangle = \langle e_{j_1,2^{l_1}} \rangle \otimes \cdots \otimes \langle e_{j_s,2^{l_s}} \rangle$$

with the index sequence

$$J : (j_1,l_1) \leq (j_2,l_2) \leq \cdots \leq (j_s,l_s),$$

in the lexicographical order and call it a canonical monomial in $P$.

**Theorem 1.2.** $X = A \otimes P$, with an inductively defined differential $d$ gives an acyclic $A$-free resolution of $\mathbb{Z}/2$.

**Proposition 1.3.** There exist natural $A$-linear chain maps $f : X \to B(A)$ and $g : B(A) \to X$, such that $g \circ f = \text{id}$ and $f(P) \subset \bar{B}(A) = \mathbb{Z}/2 \otimes_{A} B(A) \subset B(A)$.

**Proposition 1.4.** The chain complex $P$ with the induced differential $\bar{d} = \mathbb{Z}/2 \otimes_{A} d$ has a comultiplication $\Delta : P \to P \otimes P$ such that $(\bar{d} \otimes 1 + 1 \otimes \bar{d}) \Delta = \Delta \bar{d}$. This is not coassociative in general, but $(\Delta \otimes 1) \Delta$ and $(1 \otimes \Delta) \Delta$ are chain homotopic.

**Corollary 1.5.** The dual complex $P^*$ of $P$ with differential $\delta = \bar{\delta}$ has a non-associative product, therein $\delta$ is a derivation. This product induces the usual product in the cohomology $H^*(P^*, \delta) = H^*(A)$. 
2. Preliminary

The lemma 1.1 may be well-known ([6],[4]), but we will recall its proof, since the resolution (Theorem 1.2) stems from the lemma.

We shall take the dual basis \( \{ \xi^*_\omega \} \) of \( A \) (See §1). By definition the product of basis elements is given by

\[
\xi^*_\omega \cdot \xi^*_\sigma = \sum \left( \xi^*_\omega \otimes \xi^*_\sigma \right)(\psi \xi_\lambda) \cdot \xi^*_\tau.
\]

Define the height of \( \xi^*_\omega \) to be \( \Sigma k_i \), the sum of exponents in the monomial \( \xi_\omega = \xi_1^{k_1} \cdots \xi_n^{k_n} \). Then we have the equality

\[
(2.1) \quad \xi^*_\omega \cdot (\xi^*_\sigma)^* = \xi^*_\omega + \sum_{\sigma} \xi^*_\sigma,
\]

where \( \xi_\omega = \xi_1^{k_1} \cdots \xi_n^{k_n} \), \( \xi_\sigma = \xi_1^{k_1} \cdots \xi_n^{k_n} \), and the second summand in the right hand side is a sum of suitable basis elements of height \( h(\xi^*_\sigma) < h(\xi^*_\omega) \). In fact, \( \xi_\sigma \) are so chosen that \( \psi \xi_\sigma \) contain \( \xi_\omega \otimes \xi_\sigma^n \) as a summand, and such a \( \xi_\sigma \) must be of the form

\[
(2.2) \quad \xi_\sigma = \xi_1^{v_1} \cdots \xi_n^{v_n} \xi_{n+1}^{v_1} \xi_{n+2}^{v_2} \cdots \xi_{n+1}^{v_{n-1}},
\]

with \( \sum_{i=0}^{n-1} v_i = k_n \), \( u_i + 2^i v_i = k_i \) (for \( 1 \leq i \leq n-1 \)).

Then

\[
(\xi^*_\omega)^* = \sum_{i=1}^{n-1} u_i + \sum_{i=0}^{n-1} v_i = \sum_{i=1}^{n-1} k_i - 2^n \sum_{i=0}^{n-1} v_i < \sum_{i=0}^{n-1} k_i = h(\xi^*_\omega).
\]

Now by induction on height we conclude that any basis element \( \xi^*_\omega \) of \( A \) can be expressed by a sum of products of \( e_{i,k} = (\xi^*_i)^* \). But we can see easily that \( e_{i,k} \) with \( k, \not \text{a power of 2, is also decomposable into a sum of products of } e_{i,2^i} \). This proves (i) of Lemma 1.1.

Note further that

\[
(2.3) \quad (\xi^*_i)^* \cdot (\xi^*_j)^* = \binom{k+1}{k}(\xi^*_{i+k})^* + \Sigma \text{ terms of lower height}
\]

and

\[
(2.4) \quad [(\xi^*_i)^*,(\xi^*_j)^*] = \Sigma \text{ terms of lower height for } i \neq j.
\]

It follows then (ii) of Lemma 1.1.
Here are a few examples of (2.3) and (2.4):

\[ [e_{1,1}, e_{1,2}] = e_{2,1}, \quad [e_{1,1}, e_{2,1}] = 0 \]
\[ [e_{1,1}, e_{2,2}] = e_{3,1} = [e_{1,4}, e_{2,1}] \]
\[ [e_{1,2}, e_{2,2}] = e_{1,1} \cdot e_{3,1} \]
\[ e_{1,2} \cdot e_{1,2} = e_{1,1} \cdot e_{2,1} \]
\[ e_{1,4} \cdot e_{1,4} = e_{1,2} \cdot e_{2,2} \]
\[ e_{1,8} \cdot e_{1,8} = e_{1,4} \cdot e_{2,4} + e_{2,1} \cdot e_{2,2} \cdot e_{3,1} \]
\[ [e_{1,1}, e_{1,64}] = e_{1,62} \cdot e_{2,1} + e_{1,58} \cdot e_{3,1} + e_{1,50} \cdot e_{4,1} + e_{1,34} \cdot e_{5,1} + e_{1,2} \cdot e_{6,1} \]
\[ e_{i,1} \cdot e_{i,1} = 0 \quad (i \geq 1), \text{ etc. (Cf. [4])} \]

It will be another interesting problem to give the explicit formulae expressing (2.3) and (2.4) by *admissible monomials* in the sense of §1, like the Adem relations [3].

3. Resolution

In this section we shall give a detailed proof of Theorem 1.2, since we had remained in showing only a sketchy proof in [8] for the case of the Landweber-Novikov algebra. Clearly the set of canonical monomials \( \langle e_J \rangle \) forms a \( \mathbb{Z}/2 \)-basis of \( P \). Then \( P = \sum_{s \geq 0} P_s \), where the submodule \( P_s \) is spanned by \( \langle e_J \rangle \) of length \( |J| = s \). We call \( |J| \) also the homological dimension of \( \langle e_J \rangle \).

We shall introduce in \( X = A \otimes P \) a boundary operator \( d = (d_s) \):

\[ d_s: X_s = A \otimes P_s \to X_{s-1} \]

and a contracting homotopy \( \sigma = (\sigma_s) \):

\[ \sigma_s: X_s \to X_{s+1}, \]

so that \( X \) becomes an acyclic differential \( A \)-module (a chain complex) with augmentation \( \varepsilon: X \to \mathbb{Z}/2 \).

First define an \( A \)-map \( d_1: X_1 = A \otimes sL \to X_0 = A \) by

\[ d_1(a \langle e_{i,2s} \rangle) = a \cdot e_{i,2s} \quad (a \langle e_{i,2s} \rangle \text{ means } a \otimes \langle e_{i,2s} \rangle), \]

and a \( \mathbb{Z}/2 \)-map \( \sigma_0: X_0 \to X_1 \) by

\[ \sigma_0(1) = 0 \]
\[ \sigma_0(e_{i_1,2^{k_1}} \cdots e_{i_n,2^{k_n}}) = e_{i_1,2^{k_1}} e_{i_{n-1},2^{k_{n-1}}} \langle e_{i_n,2^{k_n}} \rangle \]
for admissible monomials. Thus we have a direct sum decomposition

\[(3.3)\]

\[X_1 = \text{Im} \sigma_0 \oplus \text{Ker} \ d_1, \ \text{Ker} \ d_1 = \text{Im}(1 - \sigma_0 d_1),\]

\[\sigma_0 \eta = 0, \ \varepsilon d_1 = 0 \text{ and } d_1 \sigma_0 + \eta \varepsilon = 1,\]

where \(\eta: \mathbb{Z}/2^A \rightarrow A\) is the unit. Then \(d_2\) is easily defined by

\[(3.4)\]

\[d_2 \langle e_{j_1, 2l_1}, e_{j_2, 2l_2} \rangle = (1 - \sigma_0 d_1) (e_{j_2, 2l_2} \langle e_{j_1, 2l_1} \rangle) \quad ((j_1, l_1) \leq (j_2, l_2)).\]

On the other hand, it is laborious to find and formulate a proper candidate of possible contracting homotopy \(\sigma_1\). In order to overcome this difficulty, we begin with a careful observation of the construction \(X\).

Take the set of elements

\[(3.5)\]

\[e_I \langle e_J \rangle = e_{i_1, 2^k_1} \cdots e_{i_m, 2^k_m} \langle e_{j_1, 2^l_1} \cdots e_{j_s, 2^l_s} \rangle\]

with the index sequences \(I = (i_1, k_1) < \cdots < (i_m, k_m)\) and \(J: (j_1, l_1) \leq \cdots \leq (j_s, l_s)\) in the lexicographical order, and call it canonical basis of \(X = A \otimes P\).

Classify the canonical basis elements (c.b.e.'s) into the following types:

\[(3.6)\]

Type 1: \(\max I < \max J\) (i.e. \((i_m, k_m) < (j_s, l_s)\))

and

Type 2: \(\max I \geq \max J\).

Put

\[(3.7)\]

\[C_{1,s} = \mathbb{Z}/2 \{\text{c.b.e. of Type 1 in } X_s\}\]

and

\[C_{2,s} = \mathbb{Z}/2 \{\text{c.b.e. of Type 2 in } X_s\}\]

Then we have

\[(3.8)\]

\[X_s = C_{1,s} \oplus C_{2,s},\]

as a \(\mathbb{Z}/2\)-module, with obvious isomorphisms

\[\sigma_{s-1} \sim C_{1,s}, \ \ \ \sigma_{s}^{-1} = \tau_{s}^{-1},\]

defined by

\[(3.9)\]

\[\tau_{s}(e_I \langle e_J \rangle) = e_{I+(j_s, l_s)} \langle e_{J-(j_s, l_s)} \rangle \quad \text{for } e_I \langle e_J \rangle \in C_{1,s},\]
\[ \sigma'_{s-1}(e_J) = e_{J - (i_n, k_n)}(e_J + (i_n, k_n)) \] for \( e_J(e_J) \in C_{2,s-1} \).

We shall introduce here a partial order in the set of index sequences \( J \) of the same length \( |J| = s \) as follows:

\[ J' \leq J \text{ if } (j'_i, l'_i) \leq (j_i, l_i) \text{ for all } i, \quad \text{and} \]

\[ J' < J \text{ if, moreover, } (j'_i, l'_i) < (j_i, l_i) \text{ for at least one } i. \]

Now assume that \( (d_i, \sigma_{i-1}) \) are defined for \( 1 \leq i \leq s - 1 \) and satisfy the following conditions (for convenience, put \( d_0 = \varepsilon \) and \( \sigma_{-1} = \eta \)):

\[ (A_i) \quad \sigma_{i-1} = 0 \text{ and } \Im \sigma_{i-1} = C_{1,i}, \]

\[ (B_i) \quad X_i = \Im \sigma_{i-1} \oplus \Ker d_i, \]

\[ (C_i) \quad d_i \sigma_{i-1} + \sigma_{i-2} d_{i-1} = 1 \quad \text{and} \quad d_{i-1} d_i = 0, \]

\[ (D_i) \quad (i) \text{ There is a } \mathbb{Z}/2\text{-isomorphism } \varphi_i : C_{2,i} \to \Ker d_i, \text{ defined by } \]

\[ \varphi_i(e_J) = (1 - \sigma_{i-1} - d_{i-1})(e_{2i, i}(e_J)) \text{ for } e_J(e_J) \in C_{2,i} \]

\[ (ii) \text{ Further, we have } \varphi_i(e_J) = e_J + \sum a_J a_J(e_J), \]

where \( e_J(e_J) \) are suitable c.b.e.'s with conditions \( J_a > J \) and \( \max J_a \geq \max J \) (See (3.10)).

We temporarily assume (D), of which proof is reasonably postponed. Under this induction hypothesis (3.11) we shall define \( (d_s, \sigma_{s-1}) \) as follows.

First define \( d_s : X_s \to X_{s-1} \), as an \( A \)-map, by

\[ d_s(e_J) = \varphi_{s-1}(\tau_{s-1}(e_J)) = (1 - \sigma_{s-2} d_{s-1})(e_{2s, 2s}(e_J)) \]

where \( |J| = s \) and \( (j_s, l_s) = \max J \).

It follows immediately, from (C)\(_{s-1}\)

\[ d_{s-1} d_s = 0. \]

Next define

\[ \sigma_{s-1} = 0 \text{ on } \Im \sigma_{s-2} = C_{1,s-1}. \]

To define \( \sigma_{s-1} \) on \( \Ker d_{s-1} \), take the set \( \{ \varphi_{s-1}(e_J) ; e_J(e_J) \text{ c.b.e. of Type 2 in } X_{s-1} \} \) as a fixed basis of \( \Ker d_{s-1} \), by virtue of (D)\(_{s-1}\), and put

\[ \sigma_{s-1}(\varphi_{s-1}(e_J)) = \sigma'_{s-1}(e_J) = e_J(e_J) \]

where \( (i_n, k_n) = \max I \) (See (3.9)). Then \( \sigma_{s-1} \) is naturally extended to a \( \mathbb{Z}/2\)-map and gives an isomorphism.
Thus we have
\[
d_s = \begin{cases} 
\varphi_s - \tau_s & \text{on } C_{1,s} \\
0 & \text{on } \Ker d_s 
\end{cases}
\]

and verify \((A_s), (B_s), (C_s)\) for \((d_s, \sigma_{s-1})\). From (3.11), \((D_{s-1})\) and (3.14), it follows that

\[
(3.16) \quad \sigma_{s-1}(e_I \langle e_J \rangle) = e_I \langle e_{J + \max I} \rangle + \sum_{J_a > J, \max J_a \geq \max I_a} \sigma_{s-1}(e_{I_a} \langle e_{J_a} \rangle)
\]

for \(e_I \langle e_J \rangle \in C_{2,s-1}\), where the added conditions on the summand come from those of \(e_{I_a} \langle e_{J_a} \rangle \in C_{2,s-1}\), and as well

\[
(3.17) \quad d_s \langle e_J \rangle = e_{J_s,2s} \langle e_J \rangle + \sum_{J_y > J', \max J_y \geq (j_s, l_s) = \max J} e_{I_y} \langle e_{J_y} \rangle.
\]

**Lemma 3.18.**

\[
(3.18) \quad \sigma_{s-1}(e_I \langle e_J \rangle) = e_I \langle e_{J + \max I} \rangle + \sum_{J_a > J + \max I} e_{I_a} \langle e_{J_a} \rangle \quad \text{for } e_I \langle e_J \rangle \in C_{2,s-1}
\]

or, we write simply

\[
\sigma_{s-1}(e_I \langle e_J \rangle) = e_I \langle e_{J + \max I} \rangle + \Sigma \text{ higher terms.}
\]

**Proof.** In the right hand side of (3.16), using itself again, we have

\[
\sigma_{s-1}(e_I \langle e_{J_a} \rangle) = e_{I_a} \langle e_{J_a + \max I_a} \rangle + \sum_{J_{_B} > J_a, \max J_{_B} \geq \max I_{_B}} \sigma_{s-1}(e_{I_{_B}} \langle e_{J_{_B}} \rangle)
\]

here \(J_a > J\) and \(\max I_a \geq \max J_a \geq \max I\) so that \(J_a + \max I_a > J + \max I\). Repeating this process, we obtain Lemma 3.18.
Now $\varphi_s: C_{2,s} \to \text{Ker} \, d_s$ will be defined just as before:

\begin{equation}
\varphi_s(e_i(e_j)) = e_I(1 - \sigma_{s-1} d_s)(e_{i_n,2^n}(e_j)).
\end{equation}

To prove (D$_s$), (ii) it is sufficient to consider the special case $|I| = 1$:

\begin{equation}
\varphi_s(e_{i,2^k}(e_j)) = (1 - \sigma_{s-1} d_s)(e_{i,2^k}(e_j)) (i,k) \geq \max J.
\end{equation}

In view of (3.17), we have

\begin{equation}
\varphi_s(e_{i,2^k}(e_j)) = e_{i,2^k}(e_j) + \sigma_{s-1}(e_{i,2^k} d_s(e_j))
\end{equation}

\begin{equation}
= e_{i,2^k}(e_j) + \sigma_{s-1}(e_{i,2^k} e_{I,y}(e_j))
\end{equation}

\begin{equation}
+ \sum_{J_y > J', \max J_y \geq \max J} \sigma_{s-1}(e_{i,2^k} e_{I,y}(e_j)).
\end{equation}

Rewriting $e_{i,2^k} e_{I,y}$ and $e_{i,2^k} e_{I,y}$ in the admissible form:

\begin{equation}
e_{i,2^k} e_{I,y} = \sum_{\max I_y, I_z \geq (i,k)} e_{I_z}
\end{equation}

\begin{equation}
e_{i,2^k} e_{I,y} = \sum_{\max I_y, I_z \geq (i,k)} e_{I_y, I_z}
\end{equation}

we have, from Lemma 3.18,

\begin{equation}
\varphi_s(e_{i,2^k}(e_j))
\end{equation}

\begin{equation}
= e_{i,2^k}(e_j) + \sum_{\max I_z \geq (i,k)} \sigma_{s-1}(e_{I_z}(e_j)) + \sum_{J_y > J', \max J_y \geq \max J} \sigma_{s-1}(e_{I_y, I_z}(e_j))
\end{equation}

\begin{equation}
= e_{i,2^k}(e_j) + \sum_{\max I_z \geq (i,k)} \left( e_{I_z}(e_j + \max I_z) + \Sigma \text{higher terms} \right)
\end{equation}

\begin{equation}
+ \sum_{J_y > J', \max I_y, I_z \geq (i,k)} \left( e_{I_y, I_z}(e_{j_y + \max I_y, I_z}) + \Sigma \text{higher terms} \right).
\end{equation}

Then we have in general

\begin{equation}
\varphi_s(e_I(e_j)) = e_I \cdot \varphi_s(e_{i_n,2^n}(e_j))
\end{equation}

\begin{equation}
= e_I(e_j) + \sum_{J_z \geq J, \max J_z \geq \max I} e_{I_z}(e_{j_z}) \text{ for c.b.e. } e_I(e_j) \in C_{2,s}.
\end{equation}
Thus we have proved (3.11), (Dₜ), (ii).

To show (Dₜ), (i), first note that \(\psi_s(e_j\langle e_j\rangle)\in\ker d_s\) and the set \(\{\psi_s(e_j\langle e_j\rangle); \text{ c.b.e. } e_j\langle e_j\rangle\in C_{2,s}\}\) are linearly independent in virtue of (3.22). This means that \(\psi_s\) is injective. To show the surjectivity of \(\psi_s\), we replace each higher term \(e_i\langle e_j\rangle\) of Type 2 in (3.22) by \(\psi_s(e_i\langle e_j\rangle)\). Repeating this process, we should finally obtain

\[
\varphi_s(e_j\langle e_j\rangle) = e_j\langle e_j\rangle + \sum \psi_s(e_i\langle e_j\rangle) + u_{I,J},
\]

where \(u_{I,J}\in C_{1,s}\) and \(e_j\langle e_j\rangle + u_{I,J}\in\text{Im }\varphi_s\).

The difference \((1-\sigma_{s-1}d_s)(e_j\langle e_j\rangle) - (e_j\langle e_j\rangle + u_{I,J})\) belongs to \(\ker d_s\cap\text{Im }\sigma_{s-1} = 0\). Therefore we have

\[
(1-\sigma_{s-1}d_s)(e_j\langle e_j\rangle) = e_j\langle e_j\rangle + u_{I,J}\in\text{Im }\varphi_s.
\]

Since \((1-\sigma_{s-1}d_s)(C_{1,s}) = 0\) and \((1-\sigma_{s-1}d_s)(C_{2,s}) = (1-\sigma_{s-1}d_s)(X_s)\), we have \(\text{Im }\varphi_s = \text{Im }\sigma_{s-1}d_s\).

This proves (3.11), (Dₜ), (i).

Now, for the remaining case of \(n=1\), a proof of (D₁) can be performed in a literally parallel way as just described, so it will be omitted.

Thus we have completed the induction process and a proof of the theorem 1.2.

Here we shall show some simple examples of boundaries and contracting homotopies:

\[
\begin{align*}
\Delta(e_{1,1}e_{1,2}) &= e_{1,1}\langle e_{1,2}\rangle + e_{1,2}\langle e_{1,1}\rangle + e_{2,1}
\Delta(e_{1,1}e_{2,1}) &= e_{1,2}\langle e_{1,1}\rangle + e_{2,1}\langle e_{1,2}\rangle + \sigma_0(e_{1,2}\cdot e_{1,2})
\Delta(e_{1,2}e_{2,1}) &= e_{2,1}\langle e_{1,2}\rangle + e_{1,2}\langle e_{2,1}\rangle + \sigma_0[e_{1,2}, e_{1,2}]
\end{align*}
\]

for \((i,k) < (j,l)\), where \([,]\) means the commutator.

\[
\begin{align*}
\sigma(e_{1,2}\cdot e_{1,2}) &= (e_{1,2}\cdot e_{1,2})
\end{align*}
\]

for \((i,k)\geq\text{max }J\)

where the last example shows that \(\sigma_i \neq \sigma'_i\) in general.
4. Chain complex $P$ and its dual

The construction $P$ defined in §1 with the induced differential

\[ d = \mathbb{Z}/2 \otimes d : P \to P \]

becomes a chain complex.

Define natural $A$-linear chain maps $f: X \to B(A)$ and $g: B(A) \to X$ in the usual way ([2]), using contracting homotopy $\sigma$ of $X$ resp. $S$ of $B(A)$:

\[ f_0 = \text{id}, \quad f_s(e_j) = S f_{s-1} d(e_j) \quad \text{for} \quad s \geq 1, \]

and similar for $g$.

By induction on dimension, we see easily that

\[ g \circ f = \text{id} \quad \text{on} \quad X \quad \text{and} \quad f_s(e_j) \in \bar{B}(A). \]

This proves Prop. 1.3.

Similarly define a diagonal $\psi: X \to X \otimes X$ by

\[ \psi_0: X_0 = A \to A \otimes A = (X \otimes X)_0, \quad \text{the diagonal of} \quad A \]

\[ (i.e. \quad \psi_0(e_{i,k}) = \sum_j e_{i,k} \otimes e_{i,j}), \]

\[ \psi_s(e_j) = \tilde{\sigma} \psi_{s-1} d(e_j) \quad \text{for} \quad s \geq 1, \]

where $\tilde{\sigma} = \sigma \otimes 1 + \varepsilon \otimes \sigma$ is the induced contracting homotopy of $X \otimes X$.

This $\psi$ is a chain map, and there is a natural chain homotopy:

\[ (\psi \otimes 1) \psi - (1 \otimes \psi) \psi = d^{(3)} H + H d, \]

\[ \text{with} \quad d^{(3)} = d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d, \]

where $H: X \to X \otimes X \otimes X$ is a $\mathbb{Z}/2$-map of degree $(1,0)$.

The following example shows non-coassociativity of $\psi$.

\[ \psi(e_{1,4}) = e_{1,4} \otimes 1 + e_{1,1} e_{1,2} \otimes e_{1,1} + e_{1,2} \otimes e_{1,2} + e_{1,3} \otimes e_{1,3} + 1 \otimes e_{1,4} \]

\[ (((\psi \otimes 1) \psi - (1 \otimes \psi) \psi) \psi)(e_{1,4}) = e_{1,1} \otimes e_{1,2} \otimes e_{1,1}. \]
The diagonal $\psi$ induces a diagonal $\Delta: P \rightarrow P \otimes P$,

$$\Delta = (\rho \otimes \rho) \circ \psi, \quad \rho = \varepsilon_d \otimes 1_p: X \rightarrow P, \quad \text{with}$$

$$\Delta^2 \Delta = \Delta \delta.$$

From (4.5), it follows that $\Delta$ is also homotopy coassociative. We shall show a few examples of $\Delta \langle e_j \rangle$:

\begin{align*}
\Delta \langle e_{1,2k} \rangle &= \langle e_{1,2k} \rangle \otimes 1 + 1 \otimes \langle e_{1,2k} \rangle \\
\Delta \langle e_{1,2}, e_{1,2} \rangle &= \langle e_{1,2}, e_{1,2} \rangle \otimes 1 + \langle e_{1,2} \rangle \otimes \langle e_{1,2} \rangle + 1 \otimes \langle e_{1,2}, e_{1,2} \rangle \\
\Delta \langle e_{1,1}, e_{1,4} \rangle &= \langle e_{1,1}, e_{1,4} \rangle \otimes 1 + \langle e_{1,1} \rangle \otimes \langle e_{1,4} \rangle + \langle e_{1,4} \rangle \otimes \langle e_{1,1} \rangle \\
\text{(4.8)} &+ 1 \otimes \langle e_{1,1}, e_{1,4} \rangle + \langle e_{1,2} \rangle \otimes \langle e_{2,1} \rangle \\
\Delta \langle e_{1,4}, e_{2,2}, e_{3,1} \rangle &= \langle e_{1,4}, e_{2,2}, e_{3,1} \rangle \otimes 1 + \langle e_{1,4} \rangle \otimes \langle e_{2,2}, e_{3,1} \rangle \\
&+ \langle e_{2,2} \rangle \otimes \langle e_{1,4}, e_{3,1} \rangle \\
&+ \langle e_{3,1} \rangle \otimes \langle e_{1,4}, e_{2,2} \rangle + \langle e_{2,2}, e_{3,1} \rangle \otimes \langle e_{1,4} \rangle + \langle e_{1,4}, e_{3,1} \rangle \otimes \langle e_{2,2} \rangle \\
&+ \langle e_{1,4}, e_{2,2} \rangle \otimes \langle e_{3,1} \rangle + 1 \otimes \langle e_{1,4}, e_{2,2}, e_{3,1} \rangle + \langle e_{2,1}, e_{3,1} \rangle \otimes \langle e_{3,1} \rangle \\
\end{align*}

and, in general

$$\Delta \langle e_j \rangle = \text{shuffle} + \Sigma \text{ extra terms},$$

where an extra term $\langle e_{j_1} \rangle \otimes \langle e_{j_2} \rangle$, with $\langle e_{j_1} \rangle \cdot \langle e_{j_2} \rangle \neq \langle e_j \rangle$, is indicated by the underline.

Now the dual cochain complex $P^*$, with differential $\delta = \delta^*$, has a product $\Delta^*: P^* \otimes P^* \rightarrow P^*$, which is 'homotopy associative' and $\delta$ is a derivation there.

The product $\Delta^*$ of $P^*$ induces the usual associative product in the cohomology $H^*(P^*) = \text{Ext}^*(\mathbb{Z}/2, \mathbb{Z}/2)$ as stated in Corollary 1.5.

A few examples of boundaries are given by

\begin{align*}
\delta \langle e_{1,1}, e_{1,4}, e_{1,4} \rangle &= \langle e_{2,2}, e_{2,1} \rangle \\
\delta \langle e_{1,2}, e_{1,4}, e_{2,1} \rangle &= \langle e_{3,1}, e_{1,2} \rangle + \langle e_{2,2}, e_{2,1} \rangle \\
\delta \langle e_{1,1}, e_{1,2}, e_{2,2} \rangle &= \langle e_{3,1}, e_{1,2} \rangle + \langle e_{2,2}, e_{2,1} \rangle \\
\delta \langle e_{2,2}, e_{1,2} \rangle^* &= \langle e_{1,1}, e_{1,4}, e_{1,4} \rangle^* + \langle e_{1,2}, e_{1,4}, e_{2,1} \rangle^* + \langle e_{1,1}, e_{1,2}, e_{2,2} \rangle^* \\
\delta \langle e_{3,1}, e_{1,2} \rangle^* &= \langle e_{1,2}, e_{1,4}, e_{2,1} \rangle^* + \langle e_{1,1}, e_{1,2}, e_{2,2} \rangle^* \\
\delta \langle e_{2,2}, e_{1,2} \rangle^* + \langle e_{3,1}, e_{1,2} \rangle^* &= \langle e_{1,1}, e_{1,4}, e_{1,4} \rangle^* \\
\delta \langle e_{1,1}, e_{1,1}, e_{1,4} \rangle &= \langle e_{2,1}, e_{2,1} \rangle \\
\end{align*}
We shall define a filtration on $X$ which corresponds to May's filtration on $B(A)$ ([5]). This leads to a spectral sequence, essentially the same as the May spectral sequence.

Define a weight function $w$ on $X$ by

\[ w(\langle e_j \rangle) = \sum_{i_h} i_h + \sum_{j_m} j_m \text{ for a c.b.e. } e_j, \]

where $I = \{(i_1, k_1) < \cdots < (i_n, k_n)\}$ and $J = \{(j_1, l_1) \leq \cdots \leq (j_s, l_s)\}$, and put $w(x+y) = \max(w(x), w(y))$.

Define a filtration $F_u$ on $X$, for $u \leq 0$, by

\[ e_j \in F_u \text{ if } |J| - w(\langle e_j \rangle) \leq u. \]

Then we have

\[ X = F_0 \supset F_{-1} \supset F_{-2} \supset \cdots \]

and

\[ dF_u \subset F_u. \]

Putting $Z_u^r = \ker(F_u \to F_u/F_{u-r})$ for $r \geq 0$, we get a spectral sequence $\{ E_u^r \}$:

\[ E_u^r = Z_u^r + F_{u-1}/d(Z_u^r - 1) + F_{u-1}, \]

\[ d^r : E_u^r \to E_{u-r}^r, \text{ induced by } d. \]

It follows that

\[ E^0 X = \sum_{u \leq 0} F_u/F_{u-1} \cong E^0 A \otimes E^0 P, \]

\[ d^0 = 0. \]

Here $E^0 A$ is the primitively generated Hopf algebra, isomorphic to the enveloping algebra $V(E^0 L)$ of restricted Lie algebra $E^0 L$ (in [5] and [10],
$E^0 L$ is simply denoted by $L$.

From (5.5), we have

\[
(5.6) \quad E^1 X = E^0 X \quad \text{as} \quad E^0 A\text{-module},
\]

\[
d^1 \langle e_j \rangle = \sum_{(j,l)} e_{j,l} \langle e_{j-(j,l)} \rangle,
\]

where $(j,l)$ run over the index sequence $J$ without dublication.

Thus we have an isomorphism:

\[
(5.7) \quad (E^1 P_d, d^1 = E^1(d)) \cong (\Gamma(sE^0 L), d),
\]

the May complex (being divided polynomial algebra)

as a commutative DGA-coalgebra, in which $\langle e_{j,2l} \rangle^\gamma = \langle e_{j,2l}, \cdots, e_{j,2l} \rangle$ corresponds to $\gamma_\nu(P_j) \in \Gamma(sE^0 L)$. Thus we have $E^1 X \cong E^0 A \otimes \Gamma(sE^0 L)$, the May resolution.

Dualizing the above things, we shall have a filtration $\mathcal{F}_u$ on $X^* = A^* \otimes P^*$ such that

\[
(5.8) \quad \mathcal{F}_u = (X/F_{u-1})^*, \quad \text{for} \quad u \leq 0,
\]

\[
0 = \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}_{-1} \subset \cdots \subset \mathcal{F}_u \subset \mathcal{F}_{u-1} \subset \cdots \subset \mathcal{F}_{-\infty} = X^*,
\]

\[
\delta \mathcal{F}_u \subset \mathcal{F}_u,
\]

\[
Z_r^u = \operatorname{Ker}(\mathcal{F}_u \rightarrow \mathcal{F}_{u-1} / \mathcal{F}_{u+r}),
\]

\[
E_r^u = Z_r^u + \mathcal{F}_{u+1} / \delta Z_r^{u-r+1} + \mathcal{F}_{u+1},
\]

\[
\delta_r^u : E_r^u \rightarrow E_{r+r}^u.
\]

Thus we have

\[
E_0 X^* = E_0(A_*) \otimes E_0(P^*), \quad \delta_0 = 0,
\]

\[
E_1 X^* = E_0 X^* \quad \text{as a module},
\]

\[
(5.9) \quad E_1(P^*) \cong \Gamma(sE^0 L)^* = \mathcal{R} \quad \text{as a DGA-polynomial algebra ([5],[10])},
\]

\[
E_2 X^* \cong H^*(E^0 A),
\]

and $E_r X^*$ coincide with those of the May spectral sequence for $r \geq 2$. Here $\langle e_j \rangle^* \in E_1(P^*)$ corresponds to $R_{j_1}^{l_1} \cdots R_{j_r}^{l_r} \in \mathcal{R}$ ([5],[10]).

Returning to the complex $P^*$, we denote $\langle e_{j,2l} \rangle^*$ by $\varepsilon_{j,2l}$. Then we have
6. Appendix

Consider the case of the mod $p$ Steenrod algebra $A$ for an odd prime $p$. We shall sketch similar argument as in the preceding sections.

**Lemma 6.1.** (i) $A$ is multiplicatively generated by $\{e_{i,p^k}, f_j; i \geq 1, k \geq 0$ and $j \geq 0\}$, $e_{i,p^k} = (\xi f_i)^*$ (resp. $f_j = \tau_j$) the dual element $\xi f_i^*$ (resp. $\tau_j$) with respect to the Milnor monomial basis of the dual Hopf algebra $A^*$ of $A$. (ii) The set $\{1, e_1^l \cdot f_j = e_{i_1,p^k_1} \cdots e_{i_m,p^k_m} f_{j_1} \cdots f_{j_n};$ with index sequences $I: (i_1,k_1) < \cdots < (i_m,k_m)$, $L = (l_1, \cdots, l_m)$ with $1 \leq l_i < p$, and $J: j_1 < \cdots < j_n\}$ forms a basis of $A$.

Put $L^+ = \mathbb{Z}/p\langle e_{i,p^k}; (i,k) \geq (1,0)\rangle$, $L^- = \mathbb{Z}/p\langle f_j \rangle$. Let $sL^+ = \mathbb{Z}/p\langle e_{i,p^k} \rangle$, $sL^- = \mathbb{Z}/p\langle f_j \rangle$ be the suspensions with bideg $\langle e_{i,p^k} \rangle = (1,2p^k(p^k-1))$, bideg $\langle f_j \rangle = (1,2p^k-1)$ respectively. And let $s^2\pi L^+$ denote a vector space $\mathbb{Z}/p\langle y_{i,p^k}; (i,k) \geq (1,0)\rangle$ spanned by indeterminates $y_{i,p^k}$ of bidegree $(2,2p^k+1(p^k-1))$.

Define
\[
E(sL^+) = \text{the exterior algebra on } sL^+,
\]
\[
P(sL^-) = \text{the polynomial algebra on } sL^-,
\]
and
\[
P(s^2\pi L^+) = \text{the polynomial algebra on } s^2\pi L^+.
\]

**Theorem 6.2.** The $A$-module $X = A \otimes E(sL^+) \otimes P(s^2\pi L^+) \otimes P(sL^-)$ with an inductively defined differential $d$ gives an acyclic, $A$-free resolution of $\mathbb{Z}/p$: $X^d \to \mathbb{Z}/p$.

**Corollary 6.3.** A suitable filtration on $X$ induces a spectral sequence in which $E^1 X \cong E(sE^0 L^+) \otimes \Gamma(sE^0 L^-) \otimes \Gamma(s^2\pi E^0 L^+)$, the May's construction,
as a cocommutative DGA-coalgebra ([5]) and the $E^r$-terms are the same as those of May S.S. ($r \geq 2$).

We can prove this theorem quite similarly as in the mod 2 case, although we need here a more fine classification of the canonical basis elements $e^J_f f_j \langle e_G \rangle \langle y_M \rangle \langle f_K \rangle$ as follows.

Introduce first the following notation on the index sequences:

\begin{align*}
\alpha_1(I) &= \max I = (i_m, k_m) \quad \text{for} \quad I = (i_1, k_1) \prec \cdots \prec (i_m, k_m), \\
\text{and} \quad \alpha_1(\phi) &= (0, 0), \phi \text{ being the empty set.} \\
b(G) &= \max G \quad \text{for} \quad G = (g_1, h_1) \prec \cdots \prec (g_u, h_i), \\
\text{and} \quad b(\phi) &= (0, 0), \\
c(M) &= \max M \quad \text{for} \quad M = (m_1, q_1) \leq \cdots \leq (m_w, q_u), \\
\text{and} \quad c(\phi) &= (0, 0),
\end{align*}

and

\begin{align*}
a_2(J) &= \max J \quad \text{for} \quad J = (j_1 \prec \cdots \prec j_n), \\
\text{and} \quad a_2(\phi) &= -1, \\
d(K) &= \max K \quad \text{for} \quad K = (k_1 \leq \cdots \leq k_v), \\
\text{and} \quad d(\phi) &= -1.
\end{align*}

A c.b.e. $e^J_f f_j \langle e_G \rangle \langle y_M \rangle \langle f_K \rangle$ belongs to one of the following types:

Provided that $J = K = \phi$ the empty set,

\begin{align*}
\begin{cases}
I_1: a_1 \leq b \geq c \quad \text{and,} \\
\text{if} \quad a_1 = b, l_m < p - 1,
\end{cases} & \quad H_{I_1}: b < a_1 \geq c, \\
I_2: a_1 < c \geq b, & \quad H_{I_2}: a_1 = b \geq c, \text{ and } l_m = p - 1,
\end{align*}

Otherwise, if $J$ or $K \neq \phi$, put

\begin{align*}
I_3: a_2 < d, & \quad H_{I_3}: a_2 \geq d.
\end{align*}

Thus we have a direct sum decomposition

\begin{align*}
X &= C_I \oplus C_{II}, \quad C_I = C_{I_1} \oplus C_{I_2} \oplus C_{I_3}, \quad C_{II} = C_{II_1} \oplus C_{II_2} \oplus C_{II_3},
\end{align*}

where

\begin{align*}
C_{I_i} &= \mathbb{Z}/p\{\text{c.b.e. of type } I_i\} \quad \text{and} \quad C_{II_i} = \mathbb{Z}/p\{\text{c.b.e. of type } II_i\} \\
\text{for } i &= 1, 2, 3,
\end{align*}

with linear isomorphisms
\[ \tau_s \quad C_{I,s} \to C_{I,s-1} \]

defined by

\[
\tau_s(e^1_I \langle e_G \rangle y_M) = (-1)^{|G|-1} e^1_I \cdot e_{g_{s},p_{s}} \langle e_{G-(g_{s},h_{s})} \rangle y_M \\
\text{on c.b.e. of type } I_1 \quad ((g_s, h_s) = \max G) \\
\tau_s(e^1_I \langle e_G \rangle y_M) = e^1_I \cdot e_{p_{s},p_{s}^{\text{max}}} \langle e_{G+(m_{s},q_{s})} \rangle y_{M-(m_{s},q_{s})} \\
\text{on c.b.e. of type } I_2 \quad ((m_s, q_s) = \max M) \\
\tau_s(e^1_I f_{j} \langle e_G \rangle y_M \langle f_{k} \rangle) = (-1)^{|G|+|K|-1} e^1_I \cdot f_{j} \cdot f_{k} \cdot \langle e_G \rangle \cdot y_{M} \langle f_{K-(k)} \rangle \\
\text{on c.b.e. of type } I_3 \quad (k_s = \max K)
\]

where \(|G|\) denotes the length of the index sequence \(G\) and similarly for others, and \(s = |G| + 2|M| + |K|\) the homology dimension. The inverse \(\sigma'_{s-1}\) of \(\tau_s\) will be defined obviously.

Then, starting from

\[
d_1 \langle e_{j,p} \rangle = e_{j,p}, \quad d_1 \langle f_j \rangle = f_j,
\]

\[
\sigma_0(e^1_I) = (e^1_I)^{\prime} \cdot \langle e_{i_m,k_m} \rangle, \quad \text{with } (i_m,k_m) = \max I \text{ and}
\]

\[
(e^1_I)^{\prime} = \left\{ \begin{array}{ll}
e^1_{i_1,p^{k_1}} \cdot e^1_{i_2,p^{k_2}} \cdot \cdots \cdot e^1_{i_m,p^{k_m}} & \text{if } l_m > 1 \\
e^1_{i_1,p^{k_1}} \cdot e^1_{i_2,p^{k_2}} \cdot \cdots \cdot e^1_{i_m-1,p^{k_m-1}} & \text{if } l_m = 1
\end{array} \right.
\]

\[
\sigma_0(e^1_I \cdot f_j) = e^1_I \cdot f_j \cdot \langle f_{j_n} \rangle, \quad \text{with } j_n = \max J \text{ and } J^{\prime} = J - \{j_n\},
\]

we could define differential \(d\) and contracting homotopy \(\sigma\) inductively in \(X\) as before, and as well carry out all the parallel discussion.

References


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