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## SPECIALIZATIONS OF COFINITE SUBALGEBRAS OF A POLYNOMIAL RING

Dedicated to Professor Hiroshi Nagao on his sixtieth birthday

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**1. Introduction.** Let  $K$  be a field of characteristic zero and let  $R_K := K[x, y]$  be a polynomial ring in two variables over  $K$ . A normal  $K$ -subalgebra  $A$  of  $R_K$  is said to be *cofinite* if  $R_K$  is a finite  $A$ -module with the canonical  $A$ -module structure. In the case where  $K$  is an algebraically closed field, we know the following results:

(1) If  $A$  is regular,  $A$  is then a polynomial ring in two variables over  $K$ ; see [3] and [8].

(2) If  $A$  is singular, then there exist a polynomial subalgebra  $R'_K$  and a finite group  $G$  of linear  $K$ -automorphisms of  $R'_K$  such that  $A = (R'_K)^G$  and  $G$  is a small subgroup of  $GL(2, K)$ ; see [4] and [10].

In the present article, we shall show that the structures of normal cofinite subalgebras  $A$  of  $R_K$  are invariant under specializations, provided the quotient field extension  $Q(R_K)/Q(A)$  is a quasi-Galois extension; see Definition 2.2. Our problem is formulated as follows: Let  $\mathfrak{D} = k[[t]]$  be a formal power series ring in one variable over an algebraically closed field of characteristic zero and let  $R := \mathfrak{D}[x, y]$  be a polynomial ring in two variables over  $\mathfrak{D}$ . Let  $A$  be an  $\mathfrak{D}$ -subalgebra of  $R$ . We say that  $A$  is *cofinite* if  $R$  is a finite  $A$ -module and that  $A$  is *geometrically  $\mathfrak{D}$ -normal* if  $A_K := A \otimes_{\mathfrak{D}} K$  and  $A_k := A/tA$  are normal domains, where  $K$  is the quotient field  $Q(\mathfrak{D})$  of  $\mathfrak{D}$ . If  $A$  is a cofinite, geometrically  $\mathfrak{D}$ -normal subalgebra of  $R$ , then  $A_K$  and  $A_k$  are cofinite normal subalgebras in  $R_K$  and  $R_k$ , respectively. Let  $\bar{K}$  be an algebraic closure of  $K$ . We ask whether or not certain properties of a cofinite normal subalgebra  $A_{\bar{K}}$  of  $R_{\bar{K}}$  are inherited by the cofinite normal subalgebra  $A_k$  of  $R_k$ . We pose the following

**Conjecture 1.** *Let  $\mathfrak{D}$  and  $R$  be as above, and let  $A$  be a cofinite, geometrically  $\mathfrak{D}$ -normal subalgebra of  $R$ . Then there exist a cofinite  $\mathfrak{D}$ -subalgebra  $R'$  of  $R$  and a finite group  $G$  of  $\mathfrak{D}$ -automorphisms of  $R'$  such that:*

(i)  *$R'$  is a polynomial ring in two variables over  $\mathfrak{D}$  and contains  $A$  as an  $\mathfrak{D}$ -subalgebra;*

(ii)  $A$  is the  $G$ -invariant subalgebra  $(R')^G$  of  $R'$ .

Our result, though partial, is the following:

**Main Theorem.** *Let  $\mathfrak{D}$ ,  $R$ ,  $K$  and  $\bar{K}$  be as above. Let  $A$  be a normal, cofinite  $\mathfrak{D}$ -subalgebra of  $R$ . Suppose that  $Q(R)$  is a quasi-Galois extension of  $Q(A)$  over  $K$ . Let  $G$  be the Galois group of the extension  $Q(R) \otimes_{\bar{K}} \bar{K} / Q(A) \otimes_{\bar{K}} \bar{K}$ .*

*Then the following assertions hold true:*

- (1)  $G$  acts effectively on  $R$ , and  $A = R^G$ . Namely,  $R$  is a Galois extension of  $A$  with group  $G$  in the sense of [11].
- (2)  $A$  is geometrically  $\mathfrak{D}$ -normal.
- (3)  $R_k$  is a Galois extension of  $A_k$  with group  $G$ .
- (4) If  $A_{\bar{K}}$  is a polynomial ring in two variables over  $\bar{K}$ , so is  $A_k$  over  $k$ .

We shall see later that Conjecture 1 is reduced to the following:

**Conjecture 2.** *Let  $\mathfrak{D}$  and  $R$  be as above. Let  $A$  be a normal, cofinite  $\mathfrak{D}$ -subalgebra of  $R$  such that  $A_K$  is a polynomial ring over  $K$ . Then  $A_k$  is a polynomial ring over  $k$ ; hence  $A$  is a polynomial ring over  $\mathfrak{D}$  by virtue of a result of Sathaye [14]; see also Kambayashi [6].*

Concerning the second conjecture, we can show that  $\text{Spec } A_k$  has at most one singular point which has necessarily cyclic quotient singularity, provided  $A$  is geometrically  $\mathfrak{D}$ -normal; see Proposition 4.1 below.

## 2. Representability of a group functor

Let  $K$  be a field of characteristic zero, let  $L$  be a regular extension of  $K$  and let  $L'$  be a finite algebraic extension of  $L$ . Suppose that  $L'$  is a regular extension of  $K$ .

Let  $\mathcal{C}$  be the category of finite, reduced  $K$ -algebras. We define a group functor  $\mathbf{Aut}_K(L'/L)$  on the dual category  $\mathcal{C}^\circ$  by

$$\text{Spec}(S) \in \mathcal{C}^\circ \mapsto \mathbf{Aut}_K(L'/L)(S) := \text{Aut}(L' \otimes_K S / L \otimes_K S),$$

where  $\text{Aut}(L' \otimes_K S / L \otimes_K S)$  denotes the group of all  $L \otimes_K S$ -algebra automorphisms of  $L' \otimes_K S$ , which is a finite group. We then have the following:

**Lemma 2.1.** *The functor  $\mathbf{Aut}_K(L'/L)$  is representable by a finite group scheme over  $K$ .*

*Proof.* Let  $X$  be a projective normal variety defined over  $K$  such that  $L = K(X)$  and let  $X'$  be the normalization of  $X$  in  $L'$ . Let  $\nu: X' \rightarrow X$  be the normalization morphism. We define a group functor  $\mathbf{Aut}_K(X'/X)$  on the category of  $K$ -schemes by

$$T \in (\text{Sch}/K) \mapsto \mathbf{Aut}_K(X'/X)(T) := \text{Aut}(X' \times_K T / X \times_K T),$$

where  $\text{Aut}(X' \times_K T / X \times_K T)$  denotes the group of all  $X \times_K T$ -automorphisms of  $X' \times_K T$ . We claim that the restriction of  $\mathbf{Aut}_K(X'/X)$  on the full subcategory  $\mathcal{C}^\circ$  of  $(\text{Sch}/K)$  coincides with the group functor  $\mathbf{Aut}_K(L'/L)^\circ$  which is the opposite of  $\mathbf{Aut}_K(X'/X)$ , i.e., the order of multiplication is reversed.

In fact, let  $S$  be a finite, reduced  $K$ -algebra. Then  $S$  is a direct product  $S = \prod_{i=1}^n K_i$ , where  $K_i$  is a finite algebraic extension of  $K$ . We have apparently

$$\mathbf{Aut}_K(X'/X)(S) = \prod_{i=1}^n \text{Aut}(X' \otimes_K K_i / X \otimes_K K_i), \text{ and}$$

$$\mathbf{Aut}_K(L'/L)(S) = \prod_{i=1}^n \text{Aut}(L' \otimes_K K_i / L \otimes_K K_i).$$

Hence we may (and shall) assume that  $S$  is a field. Note that  $X \otimes_K S$  is a normal variety and  $X' \otimes_K S$  is the normalization of  $X \otimes_K S$  in the field  $L' \otimes_K S$ . Moreover, it is easy to show that the canonical homomorphism

$$\text{Aut}(X' \otimes_K S / X \otimes_K S)^\circ \rightarrow \text{Aut}(L' \otimes_K S / L \otimes_K S)$$

is an isomorphism.

Now, applying the representability criterion of Grothendieck [2; 221–19],  $\mathbf{Aut}_K(X'/X)$  is representable by a  $K$ -group scheme, say  $\text{Aut}_K(X'/X)$ , which is locally of finite type over  $K^1$ . However, since  $|\mathbf{Aut}_K(X'/X)(K')| \leq [L': L]$  for any finite algebraic extension  $K'$  of  $K$ ,  $\text{Aut}_K(X'/X)$  is a finite  $K$ -group scheme. Moreover, since  $\text{char}(K)=0$ ,  $\text{Aut}_K(X'/X)$  is reduced by a theorem of Cartier (cf. [12]). Therefore we know that  $\mathbf{Aut}_K(L'/L)$  is representable by a finite  $K$ -group scheme  $\text{Aut}_K(X'/X)^\circ$ . Q.E.D.

We denote  $\text{Aut}_K(X'/X)^\circ$  by  $\text{Aut}_K(L'/L)$  or simply by  $\mathcal{G}$ . Write  $\mathcal{G} = \text{Spec}(\mathcal{A})$ . Then the identity morphism  $\text{id}_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{G}$  corresponds to an  $L$ -homomorphism

1) Define a functor  $\mathbf{Hom}_K(X', X)$  on  $(\text{Sch}/K)$  by

$$T \in (\text{Sch}/K) \mapsto \mathbf{Hom}_K(X', X)(T) = \text{Hom}_T(X'_T, X_T)$$

(cf. [2], [16]). Then there exists the canonical morphism of functors

$$\phi: \mathbf{Aut}_K(X') \rightarrow \mathbf{Hom}_K(X', X)$$

such that, for  $T \in (\text{Sch}/K)$  and  $\alpha \in \text{Aut}_T(X'_T)$ ,  $\phi_T(\alpha) = \nu_T \cdot \alpha$ . Note that both  $\mathbf{Aut}_K(X')$  and  $\mathbf{Hom}_K(X', X)$  are representable by  $K$ -schemes locally of finite type, and hence  $\phi$  is representable by a morphism of  $K$ -schemes,

$$f: \text{Aut}_K(X') \rightarrow \text{Hom}_K(X', X)$$

(cf. [15] and [16; Th. 3]). The  $K$ -scheme  $\text{Hom}_K(X', X)$  has a  $K$ -rational point  $\nu: X' \rightarrow X$ . It is now apparent that  $\mathbf{Aut}_K(X'/X)$  is representable by  $f^{-1}(\nu)$ , which is a  $K$ -group scheme locally of finite type.

$$\Delta: L' \rightarrow L' \otimes_K \mathcal{A},$$

and for any  $S \in \mathcal{C}$  and any element  $\alpha \in \text{Hom}_{K\text{-alg}}(\mathcal{A}, S) = \text{Aut}_K(L'/L)(S)$ , the action of  $\alpha$  on  $L' \otimes_K S$  is given by  $(id_{L'} \otimes \alpha)\Delta: L' \rightarrow L' \otimes_K S$ . It is then easy to see that the homomorphism  $\Delta$  defines an action of  $\mathcal{Q}$  on  $\text{Spec } L'$

$$\sigma: \mathcal{Q} \times \text{Spec } L' \rightarrow \text{Spec } L'$$

which is a  $\text{Spec } L$ -morphism. We denote by  $(L')^{\mathcal{Q}}$  the set

$$(L')^{\mathcal{Q}} = \{z \in L' \mid \Delta(z) = z \otimes 1\},$$

which is a subfield of  $L'$  containing  $L$ .

**DEFINITION 2.2.** We say that  $L'/L$  is a quasi-Galois extension over  $K$  if  $(L')^{\mathcal{Q}} = L$ .

Let  $K'$  be a finite algebraic field extension of  $K$ . Then it is straightforward to show that:

- (1)  $\text{Aut}_{K'}(L' \otimes_K K' / L \otimes_K K') \simeq \text{Aut}_K(L'/L) \otimes_K K'$ .
- (2) The action of  $\text{Aut}_{K'}(L' \otimes_K K' / L \otimes_K K')$  on  $\text{Spec } (L' \otimes_K K')$  is given by

$$\Delta \otimes K': L' \otimes_K K' \rightarrow (L' \otimes_K K') \otimes_{K'} (\mathcal{A} \otimes_K K'),$$

and we have  $(L' \otimes_K K')^{\mathcal{Q}'} = (L')^{\mathcal{Q}} \otimes_K K'$ , where  $\mathcal{Q}' = \mathcal{Q} \otimes_K K'$ .

**Lemma 2.3.** *The following conditions are equivalent:*

- (1)  $L'/L$  is a quasi-Galois extension over  $K$ .
- (2) For any finite algebraic field extension  $K'$  of  $K$ ,  $L' \otimes_K K' / L \otimes_K K'$  is a quasi-Galois extension over  $K'$ .
- (3)  $L' \otimes_K \bar{K} / L \otimes_K \bar{K}$  is a Galois extension, where  $\bar{K}$  is an algebraic closure of  $K$ .

*Proof.* The equivalence of (1) and (2) is clear in view of the preceding observations. (2)  $\Rightarrow$  (3): There exists a finite algebraic extension  $K'/K$  such that  $\mathcal{Q}' := \mathcal{Q} \otimes_K K'$  is a constant  $K'$ -group scheme with group  $G := \mathcal{Q}(K')$ . Since  $G = \text{Aut}(L' \otimes_K K' / L \otimes_K K')$  and  $(L' \otimes_K K')^G = L \otimes_K K'$ ,  $L' \otimes_K K' / L \otimes_K K'$  is a Galois extension with group  $G$ . Hence  $L' \otimes_K K'' / L \otimes_K K''$  is a Galois extension with group  $G$  for any field extension  $K''$  of  $K$  with  $K'' \supseteq K'$ . (3)  $\Rightarrow$  (1): The condition (3) implies that  $L' \otimes_K \bar{K} / L \otimes_K \bar{K}$  is a Galois extension for some finite algebraic extension  $K'/K$ . Since  $L \otimes_K K' = (L')^{\mathcal{Q}} \otimes_K K'$  as noted above, we have  $(L')^{\mathcal{Q}} = L$ . Namely,  $L'/L$  is a quasi-Galois extension over  $K$ . Q.E.D.

**Corollary 2.4.**  $L'/L$  is a quasi-Galois extension over  $K$  if and only if  $|\mathcal{Q}|$

( $\cdot$  := the rank of  $K$ -module  $\mathcal{A}$ ) is equal to  $[L' : L]$ .

A quasi-Galois extension is not necessarily a Galois extension as shown by the following trivial

EXAMPLE. Let  $K$  be the rational number field  $\mathbf{Q}$ , let  $L = K(x)$  with indeterminate  $x$  and let  $L' = K(y)$ , where  $y^n = x$  and  $n > 2$ . Then  $\mathcal{G} = \text{Aut}_K(L'/L) \simeq \text{Spec } \mathbf{Q}[\xi]/(\xi^n - 1)$  and  $\mathcal{G}(\mathbf{Q}) \neq \mathbf{Z}/n\mathbf{Z}$ . Hence  $L'/L$  is a quasi-Galois extension, but not a Galois extension. In fact, let  $K'$  be the extension of  $\mathbf{Q}$  with all  $n$ -th roots of unity adjoined. Then  $\mathcal{G}(K') \simeq \mathbf{Z}/n\mathbf{Z}$  and  $L' \otimes_{K'} K'/L \otimes_{K'} K'$  is a Galois extension.

We don't know which conditions on  $K$  assure that a quasi-Galois extension  $L'/L$  over  $K$  is a Galois extension. In the next section, we shall, however, show that this is the case if  $K$  is the quotient field of a formal power series ring  $k[[t]]$  in one variable over an algebraically closed field  $k$  of characteristic zero. We use only the property that  $k[[t]]$  is strictly henselian.

### 3. Constancy of the $K$ -group scheme $\text{Aut}_K(L'/L)$

Let  $(\mathfrak{D}, t\mathfrak{D})$  be a discrete valuation ring of equicharacteristic zero, let  $K = Q(\mathfrak{D})$  be the quotient field and let  $k$  be the residue field. First of all, we shall prove:

**Lemma 3.1.** *Let  $A$  be a finitely generated, normal  $\mathfrak{D}$ -domain and let  $L = Q(A)$ . Let  $L'$  be a finite Galois extension of  $L$  with group  $G$  and let  $A'$  be the integral closure of  $A$  in  $L'$ . Then the following assertions hold true:*

(1)  *$G$  acts effectively on  $A'_k$ , and the canonical injection  $A_k \hookrightarrow A'_k$  induces an isomorphism  $A_k \simeq (A'_k)^G$ .*

(2) *Suppose  $A'_k$  is an integral domain. Then  $Q(A'_k)$  is a Galois extension of  $Q(A_k)$  with group  $G$ .*

Proof. Our proof consists of several steps.

(I) Note that  $A'$  is a finite  $A$ -module (cf. Matsumura [7]). Furthermore,  $A_k$  is a subring of  $A'_k$ . In fact, we have only to show that  $A \cap tA' = tA$ . Suppose  $a = ta'$  with  $a \in A$  and  $a' \in A'$ . Then  $a' \in Q(A)$  and  $a'$  is integral over  $A$ . Hence  $a' \in A$  because  $A$  is normal. The Galois group  $G$  acts effectively on  $A'$  and  $A = (A')^G$ . Hence  $G$  acts on  $A'_k$  and  $A_k \subseteq (A'_k)^G$ .

(II) We shall show that  $G$  acts effectively on  $A'_k$ . Suppose, on the contrary, that an element  $g \in G$  of order  $n > 1$  acts trivially on  $A'_k$ . For any element  $a' \in A'$ , we have

$${}^g a' - a' = ta'_1 \quad \text{with } a'_1 \in A'.$$

Write  ${}^g a'_1 = a'_1 + ta'_2$  with  $a'_2 \in A'$ . Inductively, we define  $a'_i \in A'$  ( $1 \leq i \leq n$ ) by  ${}^g a'_{i-1} = a'_{i-1} + ta'_i$ . Then it is easy to show

$$a' = {}^s a' = a' + nta'_1 + \cdots + \binom{n}{i} t^i a'_i + \cdots + t^n a'_n.$$

Hence  $a'_1 \in tA'$ . Namely, we can write  ${}^s a' = a' + t^2 a'_1'$ . This is true for every  $a' \in A'$ . By the same argument as above with  $t$  replaced by  $t^2$ , we have  $a'_1' \in t^2 A'$ . Thus, we can show that  ${}^s a' - a' \in \bigcap_{m \geq 0} t^m A'$ . Since  $A'$  is a Noetherian integral domain, we have  $\bigcap_{m \geq 0} t^m A' = (0)$  by Krull's intersection theorem (cf. [11]).

Namely,  ${}^s a' = a'$  for every  $a' \in A'$ . This is a contradiction.

(III) We shall show that  $A_k = (A'_k)^G$ . In fact, suppose  $\bar{a}' \in (A'_k)^G$ , and write

$${}^s a' = a' + tb(g) \quad \text{with } b(g) \in A',$$

where  $a' \in A'$  with  $\bar{a}' = a' \pmod{tA'}$ . Then we have

$$b(hg) = {}^h b(g) + b(h) \quad \text{for } g, h \in G.$$

Set  $c = (\sum_{g \in G} b(g)) / |G|$ . Then  $b(g) = c - {}^g c$  for any  $g \in G$ , and  $a' + tc \in (A')^G = A$ .

Hence  $\bar{a}' \in A_k$ . Namely, we have  $A_k = (A'_k)^G$ . Now, the assertion (2) is readily ascertained. Q.E.D.

Hereafter, we assume that  $\mathfrak{D}$  is a formal power series ring  $k[[t]]$  over an algebraically closed field  $k$  of characteristic zero. The constancy of the  $K$ -group scheme  $\text{Aut}_K(L'/L)$  is assured by

**Lemma 3.2.** *Let  $\mathfrak{D} = k[[t]]$  be as above and let  $K = Q(\mathfrak{D})$ . Let  $L$  be a regular extension of  $K$  and let  $L'$  be a quasi-Galois extension of  $L$  such that  $L'$  is a regular extension of  $K$ . Then  $L'/L$  is a Galois extension.*

*Proof.* We have only to prove that the  $K$ -group scheme  $\text{Aut}_K(L'/L)$  is constant. Since the Puiseux field  $\bigcup_{n>0} k((t^{1/n}))$  is an algebraic closure of  $k((t))$ , where  $k((t^{1/n}))$  is the quotient field of  $k[[t^{1/n}]]$ , there exists a cyclic extension  $\mathfrak{D}' = k[[\tau]]$  of  $\mathfrak{D}$  ( $\tau^n = t$ ) such that  $\text{Aut}_K(L'/L) \otimes_{\bar{K}} K' \simeq \text{Aut}_{K'}(L' \otimes_{\bar{K}} K' / L \otimes_{\bar{K}} K')$  is constant, where  $K' = Q(\mathfrak{D}')$ . Note that the morphism  $\text{Spec } \mathfrak{D}' \rightarrow \text{Spec } \mathfrak{D}$  is a faithfully flat and finite morphism. Let  $G = \text{Aut}_K(L'/L)(K')$ . Then the constant  $K'$ -group scheme  $G_{K'}$  has apparently a Néron model  $G_{\mathfrak{D}'}$ , a constant  $\mathfrak{D}'$ -group scheme with group  $G$ . Hence the  $K$ -group scheme  $\text{Aut}_K(L'/L)$  has an  $\mathfrak{D}$ -Néron model  $\mathcal{G}$ ; see [13] for relevant results. By definition, the group scheme  $\mathcal{G}$  is smooth over  $\mathfrak{D}$  and satisfies  $\mathcal{G} \otimes_{\mathfrak{D}} K \simeq \text{Aut}_K(L'/L)$ . By virtue of [1; IV (18.10.16)],  $\mathcal{G}$  is finite and étale over  $\mathfrak{D}$ . Therefore  $\mathcal{G}$  must be a constant  $\mathfrak{D}$ -group scheme  $H_{\mathfrak{D}}$ , where  $H \simeq \mathcal{G}(k) = \mathcal{G}(K)$ . Since  $G = \mathcal{G}(K') \simeq H_{\mathfrak{D}}(K') = H$ , we know that  $\mathcal{G} \simeq G_{\mathfrak{D}}$ . Thus  $L'/L$  is a Galois extension with group  $G$ . Q.E.D.

**Lemma 3.3.** *Let the notations and the assumptions be the same as in Lemma 3.1. Assume that  $L$  is the quotient field of a finitely generated, normal  $\mathfrak{D}$ -domain  $A$ . Let  $A'$  be the normalization of  $A$  in  $L'$ , and let  $G$  be the Galois group of the extension  $L'/L$ . Then the following assertions hold true:*

- (1)  *$A'$  is a Galois extension of  $A$  with group  $G$ .*
- (2) *Suppose that  $A'$  is geometrically  $\mathfrak{D}$ -normal. Then, so is  $A$ , and  $A'_k$  is a Galois extension of  $A_k$  with group  $G$ .*

*Proof.* (1) is now clear. As for (2),  $A'_k$  is a normal domain by the hypothesis, and  $A_k = (A'_k)^G$  by Lemma 3.1. Hence  $A_k$  is normal, and  $A$  is geometrically  $\mathfrak{O}$ -normal. The remaining assertion is clear by Lemma 3.1. Q.E.D.

Now, Main Theorem except the assertion (4) follows from Lemma 3.3. In fact, set  $L := Q(A)$  and  $L' := Q(R)$  with  $A$  and  $R$  as in Main Theorem. Then  $R$  is the normalization of  $A$  in  $L'$ , and  $R$  is geometrically  $\mathfrak{D}$ -normal. So, we can apply Lemma 3.3. We shall prove the assertion (4). Since  $A_{\bar{K}}$  is a polynomial ring over  $\bar{K}$ ,  $A_K$  is a polynomial ring over  $K$  by [5]. We can identify  $G$  as a finite subgroup of  $GL(2, \bar{K})$ , and it is well-known that  $G$  is then generated by pseudo-reflections. Recall that an element  $g \in GL(2, \bar{K})$  is a pseudo-reflection if and only if the fixed-point locus  $\Gamma(g)_{\bar{K}} := \text{Spec } \bar{K}[x, y]$  under the action of  $g$  has codimension  $\leq 1$ . Since  $g$  acts on  $A_{\mathfrak{D}}^2 := \text{Spec } \mathfrak{D}[x, y]$ , let  $\Gamma(g)$  be the fixed-point locus in  $A_{\mathfrak{D}}^2$  under the action of  $g$ . Namely,  $\Gamma(g)$  is a closed subscheme of  $A_{\mathfrak{D}}^2$  defined by an ideal  $I$ , where  $I$  is the smallest ideal of  $\mathfrak{D}[x, y]$  generated by all elements of the form  ${}^g a - a$  with  $a \in \mathfrak{D}[x, y]$ . Then we know that  $\Gamma(g)_{\bar{K}} = \Gamma(g) \otimes_{\mathfrak{D}} \bar{K}$  and that  $\Gamma(g) \otimes_{\mathfrak{D}} k$  is the fixed-point locus in  $A_k^2 := \text{Spec } k[x, y]$  under the action of  $g$ . Hence  $\Gamma(g) \otimes k$  has codimension  $\leq 1$  in  $A_k^2$ . This implies that when one embeds  $G$  into  $GL(2, k)$  upto conjugation in  $\text{Aut}_k k[x, y]$ ,  $G$  is generated by pseudo-reflections. Hence the  $G$ -invariant subring  $A_k$  of  $k[x, y]$  is a polynomial ring over  $k$ . This verifies the assertion (4) of Main Theorem.

#### 4. Reduction from Conjecture 1 to Conjecture 2

Let  $\mathfrak{D}$ ,  $R$  and  $A$  be as in Conjecture 1. Let  $Y := A_{\mathfrak{D}}^2 = \text{Spec } R$ , let  $X := \text{Spec } A$  and let  $\pi: Y \rightarrow X$  be the canonical finite morphism. For an algebraic closure  $\bar{K}$  of  $K = Q(\mathfrak{D})$ ,  $A_{\bar{K}}$  is a normal, cofinite  $\bar{K}$ -subalgebra of  $\bar{K}[x, y]$ . Note that  $X_{\bar{K}} = \text{Spec } A_{\bar{K}}$  has at most one singular point. Let  $\bar{Z}'$  be the universal covering space of  $X_{\bar{K}} - \text{Sing}(X_{\bar{K}})$ . Then  $\pi_{\bar{K}}: Y_{\bar{K}} \rightarrow X_{\bar{K}}$  factors through  $\bar{Z}'$  because  $Y_{\bar{K}} \rightarrow X_{\bar{K}}$  is simply connected. Let  $\bar{Z}$  be the normalization of  $X_{\bar{K}}$  in the function field  $\bar{K}(\bar{Z}')$  of  $\bar{Z}'$ . Then  $\bar{Z} \simeq A_{\bar{K}}^2$  and  $\pi_{\bar{K}}: Y_{\bar{K}} \rightarrow X_{\bar{K}}$  factors through  $\bar{Z}$ ;

$$\pi_{\bar{K}}: Y_{\bar{K}} \xrightarrow{\alpha} \bar{Z} \xrightarrow{\beta} X_{\bar{K}}.$$



See [10] for the relevant results. Choose a  $K$ -rational point  $P$  of  $Y_{\bar{K}} - \pi^{-1}(\text{Sing } X_{\bar{K}})$ , and let  $Q = \bar{\alpha}(P)$ . We shall show that  $\bar{Z}$  descends down to a  $K$ -scheme. Namely, there exist a  $K$ -scheme  $Z$  and  $K$ -morphisms  $\alpha: Y \rightarrow Z$  and  $\beta: Z \rightarrow X$  such that  $\bar{Z} = Z \otimes_{\bar{K}} \bar{K}$ ,  $\bar{\alpha} = \alpha \otimes_{\bar{K}} \bar{K}$  and  $\bar{\beta} = \beta \otimes_{\bar{K}} \bar{K}$ . In fact, for  $\sigma \in \text{Gal}(\bar{K}/K)$ , let  ${}^{\sigma}\bar{Z} = \text{Spec } \rho_{\sigma}(\mathcal{O}(\bar{Z}))$ , where  $\rho_{\sigma}: \bar{K}[x, y] \rightarrow \bar{K}[x, y]$  is  $\sigma \otimes \text{id}_{K[x, y]}$  and  $\mathcal{O}(\bar{Z})$  is the coordinate ring of  $\bar{Z}$  which is a  $\bar{K}$ -subalgebra of  $\bar{K}[x, y]$ . We denote by  ${}^{\sigma}\bar{\alpha}: Y_{\bar{K}} \rightarrow {}^{\sigma}\bar{Z}$  and  ${}^{\sigma}\bar{\beta}: {}^{\sigma}\bar{Z} \rightarrow X_{\bar{K}}$  the morphisms induced by  $\rho_{\sigma}(\mathcal{O}(\bar{Z})) \hookrightarrow \bar{K}[x, y]$  and  $A_{\bar{K}} \hookrightarrow \rho_{\sigma}(\mathcal{O}(\bar{Z}))$ , respectively. Hence  $\pi_{\bar{K}} = ({}^{\sigma}\bar{\beta}) \cdot ({}^{\sigma}\bar{\alpha})$ . Let  ${}^{\sigma}Q$  be the point of  ${}^{\sigma}\bar{Z}$  which corresponds to  $Q$  under the canonical isomorphism  $\text{Spec } \rho_{\sigma}(\mathcal{O}(\bar{Z})) \rightarrow \text{Spec } \mathcal{O}(\bar{Z})$ . Then we have a unique  $\bar{K}$ -isomorphism  $\phi_{\sigma}: {}^{\sigma}\bar{Z} \rightarrow \bar{Z}$  such that  $\rho_{\sigma}({}^{\sigma}Q) = Q$ ,  $\bar{\alpha} = \phi_{\sigma} \cdot {}^{\sigma}\bar{\alpha}$  and  ${}^{\sigma}\bar{\beta} = \bar{\beta} \cdot \phi_{\sigma}$ . Then it is easy to show that  $\phi_{\tau\sigma} = \phi_{\tau} \cdot {}^{\tau}\phi_{\sigma}$  for  $\sigma, \tau \in \text{Gal}(\bar{K}/K)$ . In fact, this is the case for a finite Galois extension  $K'/K$  instead of  $\bar{K}/K$ . By the faithfully flat descent, we know that there exists a  $K$ -scheme  $Z$  such that  $\bar{Z} = Z \otimes_{\bar{K}} \bar{K}$ . Then  $\bar{\alpha} = {}^{\sigma}\bar{\alpha}$  and  ${}^{\sigma}\bar{\beta} = \bar{\beta}$  for any  $\sigma \in \text{Gal}(\bar{K}/K)$ .

Therefore  $\bar{\alpha}$  and  $\bar{\beta}$  descend down to  $K$ -morphisms  $\alpha: X_K \rightarrow Z$  and  $\beta: Z \rightarrow Y_K$  such that  $\pi_K = \beta \cdot \alpha$ . On the other hand,  $Z$  is  $K$ -isomorphic to  $A_K^2$  by virtue of [5]. Identify the coordinate ring  $\mathcal{O}(Z)$  with a  $K$ -subalgebra of  $K[x, y]$  under  $\alpha$ . Let  $B$  be the normalization of  $A$  in the function field  $K(Z)$  of  $Z$ . Then  $B$  is a normal, cofinite  $\mathfrak{D}$ -subalgebra of  $R$  such that  $B_K = \mathcal{O}(Z)$  is a polynomial ring over  $K$ . The Conjecture 2 then implies that  $B$  is a polynomial ring in two variables over  $\mathfrak{D}$ . Note that  $Q(B)$  is a quasi-Galois extension of  $Q(A)$  over  $K$ . Main Theorem then asserts that Conjecture 1 is affirmative.

As for the Conjecture 2, we know the following:

**Proposition 4.1.** *Let  $\mathfrak{D}$ ,  $R$  and  $A$  be the same as in Conjecture 2, and let  $X = \text{Spec } A$ . Suppose that  $A$  is geometrically  $\mathfrak{D}$ -normal. Then  $X_k$  has at most one singular point which has necessarily cyclic quotient singularity.*

*Proof.* By the hypothesis,  $A_K$  is a polynomial ring  $K[u, v]$ . Let  $\Delta = \frac{\partial}{\partial u}$ , which is a locally nilpotent  $K$ -derivation of  $A_K$ . Since  $A$  is finitely generated over  $\mathfrak{D}$ , we find an integer  $n \geq 0$  such that  $t^n \Delta(A) \subseteq A$  and  $t^n \Delta(A) \not\subseteq tA$ . Define a  $k$ -derivation  $\delta$  of  $A_k$  by

$$\delta(a) = t^n \Delta(a) \pmod{tA},$$

where  $a = a \pmod{tA}$  with  $a \in A$ . Then  $\delta$  is well-defined, and  $\delta$  is a nontrivial, locally nilpotent  $k$ -derivation on  $A_k$ . Hence  $X_k := \text{Spec } A_k$  is affine-ruled (cf. [9]) and  $X_k$  has at most one singular point which has necessarily cyclic quotient singularity (cf. [10]).

Q.E.D.

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