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SPECIALIZATIONS OF COFINITE SUBALGEBRAS OF A POLYNOMIAL RING

Dedicated to Professor Hiroshi Nagao on his sixtieth birthday

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1. Introduction. Let K be a field of characteristic zero and let $R_K := K[x, y]$ be a polynomial ring in two variables over K . A normal K -subalgebra A of R_K is said to be *cofinite* if R_K is a finite A -module with the canonical A -module structure. In the case where K is an algebraically closed field, we know the following results:

(1) If A is regular, A is then a polynomial ring in two variables over K ; see [3] and [8].

(2) If A is singular, then there exist a polynomial subalgebra R'_K and a finite group G of linear K -automorphisms of R'_K such that $A = (R'_K)^G$ and G is a small subgroup of $GL(2, K)$; see [4] and [10].

In the present article, we shall show that the structures of normal cofinite subalgebras A of R_K are invariant under specializations, provided the quotient field extension $Q(R_K)/Q(A)$ is a quasi-Galois extension; see Definition 2.2. Our problem is formulated as follows: Let $\mathfrak{D} = k[[t]]$ be a formal power series ring in one variable over an algebraically closed field of characteristic zero and let $R := \mathfrak{D}[x, y]$ be a polynomial ring in two variables over \mathfrak{D} . Let A be an \mathfrak{D} -subalgebra of R . We say that A is *cofinite* if R is a finite A -module and that A is *geometrically \mathfrak{D} -normal* if $A_K := A \otimes_{\mathfrak{D}} K$ and $A_k := A/tA$ are normal domains, where K is the quotient field $Q(\mathfrak{D})$ of \mathfrak{D} . If A is a cofinite, geometrically \mathfrak{D} -normal subalgebra of R , then A_K and A_k are cofinite normal subalgebras in R_K and R_k , respectively. Let \bar{K} be an algebraic closure of K . We ask whether or not certain properties of a cofinite normal subalgebra $A_{\bar{K}}$ of $R_{\bar{K}}$ are inherited by the cofinite normal subalgebra A_k of R_k . We pose the following

Conjecture 1. *Let \mathfrak{D} and R be as above, and let A be a cofinite, geometrically \mathfrak{D} -normal subalgebra of R . Then there exist a cofinite \mathfrak{D} -subalgebra R' of R and a finite group G of \mathfrak{D} -automorphisms of R' such that:*

(i) *R' is a polynomial ring in two variables over \mathfrak{D} and contains A as an \mathfrak{D} -subalgebra;*

(ii) A is the G -invariant subalgebra $(R')^G$ of R' .

Our result, though partial, is the following:

Main Theorem. *Let \mathfrak{D} , R , K and \bar{K} be as above. Let A be a normal, cofinite \mathfrak{D} -subalgebra of R . Suppose that $Q(R)$ is a quasi-Galois extension of $Q(A)$ over K . Let G be the Galois group of the extension $Q(R) \otimes_{\bar{K}} Q(A) \otimes_{\bar{K}}$.*

Then the following assertions hold true:

- (1) G acts effectively on R , and $A=R^G$. Namely, R is a Galois extension of A with group G in the sense of [11].
- (2) A is geometrically \mathfrak{D} -normal.
- (3) R_k is a Galois extension of A_k with group G .
- (4) If $A_{\bar{K}}$ is a polynomial ring in two variables over \bar{K} , so is A_k over k .

We shall see later that Conjecture 1 is reduced to the following:

Conjecture 2. *Let \mathfrak{D} and R be as above. Let A be a normal, cofinite \mathfrak{D} -subalgebra of R such that A_K is a polynomial ring over K . Then A_k is a polynomial ring over k ; hence A is a polynomial ring over \mathfrak{D} by virtue of a result of Sathaye [14]; see also Kambayashi [6].*

Concerning the second conjecture, we can show that $\text{Spec } A_k$ has at most one singular point which has necessarily cyclic quotient singularity, provided A is geometrically \mathfrak{D} -normal; see Proposition 4.1 below.

2. Representability of a group functor

Let K be a field of characteristic zero, let L be a regular extension of K and let L' be a finite algebraic extension of L . Suppose that L' is a regular extension of K .

Let \mathcal{C} be the category of finite, reduced K -algebras. We define a group functor $\mathbf{Aut}_K(L'/L)$ on the dual category \mathcal{C}° by

$$\text{Spec}(S) \in \mathcal{C}^\circ \mapsto \mathbf{Aut}_K(L'/L)(S) := \text{Aut}(L' \otimes_K S / L \otimes_K S),$$

where $\text{Aut}(L' \otimes_K S / L \otimes_K S)$ denotes the group of all $L \otimes_K S$ -algebra automorphisms of $L' \otimes_K S$, which is a finite group. We then have the following:

Lemma 2.1. *The functor $\mathbf{Aut}_K(L'/L)$ is representable by a finite group scheme over K .*

Proof. Let X be a projective normal variety defined over K such that $L=K(X)$ and let X' be the normalization of X in L' . Let $\nu: X' \rightarrow X$ be the normalization morphism. We define a group functor $\mathbf{Aut}_K(X'/X)$ on the category of K -schemes by

$$T \in (\text{Sch}/K) \mapsto \mathbf{Aut}_K(X'/X)(T) := \text{Aut}(X' \times_K T / X \times_K T),$$

where $\text{Aut}(X' \times_K T / X \times_K T)$ denotes the group of all $X \times_K T$ -automorphisms of $X' \times_K T$. We claim that the restriction of $\mathbf{Aut}_K(X'/X)$ on the full subcategory \mathcal{C}° of (Sch/K) coincides with the group functor $\mathbf{Aut}_K(L'/L)^\circ$ which is the opposite of $\mathbf{Aut}_K(X'/X)$, i.e., the order of multiplication is reversed.

In fact, let S be a finite, reduced K -algebra. Then S is a direct product $S = \prod_{i=1}^n K_i$, where K_i is a finite algebraic extension of K . We have apparently

$$\mathbf{Aut}_K(X'/X)(S) = \prod_{i=1}^n \text{Aut}(X' \otimes_K K_i / X \otimes_K K_i), \text{ and}$$

$$\mathbf{Aut}_K(L'/L)(S) = \prod_{i=1}^n \text{Aut}(L' \otimes_K K_i / L \otimes_K K_i).$$

Hence we may (and shall) assume that S is a field. Note that $X \otimes_K S$ is a normal variety and $X' \otimes_K S$ is the normalization of $X \otimes_K S$ in the field $L' \otimes_K S$. Moreover, it is easy to show that the canonical homomorphism

$$\text{Aut}(X' \otimes_K S / X \otimes_K S)^\circ \rightarrow \text{Aut}(L' \otimes_K S / L \otimes_K S)$$

is an isomorphism.

Now, applying the representability criterion of Grothendieck [2; 221–19], $\mathbf{Aut}_K(X'/X)$ is representable by a K -group scheme, say $\text{Aut}_K(X'/X)$, which is locally of finite type over K^{cl} . However, since $|\mathbf{Aut}_K(X'/X)(K')| \leq [L' : L]$ for any finite algebraic extension K' of K , $\text{Aut}_K(X'/X)$ is a finite K -group scheme. Moreover, since $\text{char}(K)=0$, $\text{Aut}_K(X'/X)$ is reduced by a theorem of Cartier (cf. [12]). Therefore we know that $\mathbf{Aut}_K(L'/L)$ is representable by a finite K -group scheme $\text{Aut}_K(X'/X)^\circ$. Q.E.D.

We denote $\mathbf{Aut}_K(X'/X)^\circ$ by $\text{Aut}_K(L'/L)$ or simply by \mathcal{G} . Write $\mathcal{G} = \text{Spec}(\mathcal{A})$. Then the identity morphism $\text{id}_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{G}$ corresponds to an L -homomorphism

1) Define a functor $\mathbf{Hom}_K(X', X)$ on (Sch/K) by

$$T \in (\text{Sch}/K) \mapsto \mathbf{Hom}_K(X', X)(T) = \text{Hom}_T(X'_T, X_T)$$

(cf. [2], [16]). Then there exists the canonical morphism of functors

$$\phi: \mathbf{Aut}_K(X') \rightarrow \mathbf{Hom}_K(X', X)$$

such that, for $T \in (\text{Sch}/K)$ and $\alpha \in \text{Aut}_T(X'_T)$, $\phi_T(\alpha) = \nu_T \cdot \alpha$. Note that both $\mathbf{Aut}_K(X')$ and $\mathbf{Hom}_K(X', X)$ are representable by K -schemes locally of finite type, and hence ϕ is representable by a morphism of K -schemes,

$$f: \text{Aut}_K(X') \rightarrow \text{Hom}_K(X', X)$$

(cf. [15] and [16; Th. 3]). The K -scheme $\text{Hom}_K(X', X)$ has a K -rational point $\nu: X' \rightarrow X$. It is now apparent that $\mathbf{Aut}_K(X'/X)$ is representable by $f^{-1}(\nu)$, which is a K -group scheme locally of finite type.

$$\Delta: L' \rightarrow L' \otimes_K \mathcal{A},$$

and for any $S \in \mathcal{C}$ and any element $\alpha \in \text{Hom}_{K\text{-alg}}(\mathcal{A}, S) = \text{Aut}_K(L'/L)(S)$, the action of α on $L' \otimes_K S$ is given by $(id_{L'} \otimes \alpha)\Delta: L' \rightarrow L' \otimes_K S$. It is then easy to see that the homomorphism Δ defines an action of \mathcal{G} on $\text{Spec } L'$

$$\sigma: \mathcal{G} \times \text{Spec } L' \rightarrow \text{Spec } L'$$

which is a $\text{Spec } L$ -morphism. We denote by $(L')^{\mathcal{G}}$ the set

$$(L')^{\mathcal{G}} = \{z \in L' \mid \Delta(z) = z \otimes 1\},$$

which is a subfield of L' containing L .

DEFINITION 2.2. We say that L'/L is a quasi-Galois extension over K if $(L')^{\mathcal{G}} = L$.

Let K' be a finite algebraic field extension of K . Then it is straightforward to show that:

- (1) $\text{Aut}_{K'}(L' \otimes_K K' / L \otimes_K K') \simeq \text{Aut}_K(L'/L) \otimes K'$.
- (2) The action of $\text{Aut}_{K'}(L' \otimes_K K' / L \otimes_K K')$ on $\text{Spec } (L' \otimes_K K')$ is given by

$$\Delta \otimes K': L' \otimes_K K' \rightarrow (L' \otimes_K K') \otimes_{K'} (\mathcal{A} \otimes_K K'),$$

and we have $(L' \otimes_K K')^{\mathcal{G}'} = (L')^{\mathcal{G}} \otimes_K K'$, where $\mathcal{G}' = \mathcal{G} \otimes_K K'$.

Lemma 2.3. *The following conditions are equivalent:*

- (1) L'/L is a quasi-Galois extension over K .
- (2) For any finite algebraic field extension K' of K , $L' \otimes_K K' / L \otimes_K K'$ is a quasi-Galois extension over K' .
- (3) $L' \otimes_K \bar{K} / L \otimes_K \bar{K}$ is a Galois extension, where \bar{K} is an algebraic closure of K .

Proof. The equivalence of (1) and (2) is clear in view of the preceding observations. (2) \Rightarrow (3): There exists a finite algebraic extension K'/K such that $\mathcal{G}' := \mathcal{G} \otimes_K K'$ is a constant K' -group scheme with group $G := \mathcal{G}(K')$. Since $G = \text{Aut}(L' \otimes_K K' / L \otimes_K K')$ and $(L' \otimes_K K')^G = L \otimes_K K'$, $L' \otimes_K K' / L \otimes_K K'$ is a Galois extension with group G . Hence $L' \otimes_K K'' / L \otimes_K K''$ is a Galois extension with group G for any field extension K'' of K with $K'' \supseteq K'$. (3) \Rightarrow (1): The condition (3) implies that $L' \otimes_K K' / L \otimes_K K'$ is a Galois extension for some finite algebraic extension K'/K . Since $L \otimes_K K' = (L')^{\mathcal{G}} \otimes_K K'$ as noted above, we have $(L')^{\mathcal{G}} = L$. Namely, L'/L is a quasi-Galois extension over K . Q.E.D.

Corollary 2.4. L'/L is a quasi-Galois extension over K if and only if $|\mathcal{G}|$

($:=$ the rank of K -module \mathcal{A}) is equal to $[L': L]$.

A quasi-Galois extension is not necessarily a Galois extension as shown by the following trivial

EXAMPLE. Let K be the rational number field \mathbf{Q} , let $L=K(x)$ with indeterminate x and let $L'=K(y)$, where $y^n=x$ and $n>2$. Then $\mathcal{G}=\text{Aut}_K(L'/L) \simeq \text{Spec } \mathbf{Q}[\xi]/(\xi^n-1)$ and $\mathcal{G}(\mathbf{Q}) \cong \mathbf{Z}/n\mathbf{Z}$. Hence L'/L is a quasi-Galois extension, but not a Galois extension. In fact, let K' be the extension of \mathbf{Q} with all n -th roots of unity adjoined. Then $\mathcal{G}(K') \simeq \mathbf{Z}/n\mathbf{Z}$ and $L' \otimes_{K'} K'/L \otimes_{K'} K'$ is a Galois extension.

We don't know which conditions on K assure that a quasi-Galois extension L'/L over K is a Galois extension. In the next section, we shall, however, show that this is the case if K is the quotient field of a formal power series ring $k[[t]]$ in one variable over an algebraically closed field k of characteristic zero. We use only the property that $k[[t]]$ is strictly henselian.

3. Constancy of the K -group scheme $\text{Aut}_K(L'/L)$

Let $(\mathfrak{D}, t\mathfrak{D})$ be a discrete valuation ring of equicharacteristic zero, let $K = \mathbf{Q}(\mathfrak{D})$ be the quotient field and let k be the residue field. First of all, we shall prove:

Lemma 3.1. *Let A be a finitely generated, normal \mathfrak{D} -domain and let $L = \mathbf{Q}(A)$. Let L' be a finite Galois extension of L with group G and let A' be the integral closure of A in L' . Then the following assertions hold true:*

- (1) *G acts effectively on A'_k , and the canonical injection $A_k \hookrightarrow A'_k$ induces an isomorphism $A_k \simeq (A'_k)^G$.*
- (2) *Suppose A'_k is an integral domain. Then $\mathbf{Q}(A'_k)$ is a Galois extension of $\mathbf{Q}(A_k)$ with group G .*

Proof. Our proof consists of several steps.

(I) Note that A' is a finite A -module (cf. Matsumura [7]). Furthermore, A_k is a subring of A'_k . In fact, we have only to show that $A \cap tA' = tA$. Suppose $a = ta'$ with $a \in A$ and $a' \in A'$. Then $a' \in \mathbf{Q}(A)$ and a' is integral over A . Hence $a' \in A$ because A is normal. The Galois group G acts effectively on A' and $A = (A')^G$. Hence G acts on A'_k and $A_k \subseteq (A'_k)^G$.

(II) We shall show that G acts effectively on A'_k . Suppose, on the contrary, that an element $g \in G$ of order $n > 1$ acts trivially on A'_k . For any element $a' \in A'$, we have

$${}^g a' - a' = ta'_1 \quad \text{with } a'_1 \in A'.$$

Write ${}^g a'_1 = a'_1 + ta'_2$ with $a'_2 \in A'$. Inductively, we define $a'_i \in A'$ ($1 \leq i \leq n$) by ${}^g a'_{i-1} = a'_{i-1} + ta'_i$. Then it is easy to show

$$a' = {}^s a' = a' + n t a'_1 + \cdots + \binom{n}{i} t^i a'_i + \cdots + t^n a'_n .$$

Hence $a'_i \in tA'$. Namely, we can write ${}^s a' = a' + t^2 a'_1'$. This is true for every $a' \in A'$. By the same argument as above with t replaced by t^2 , we have $a'_1' \in t^2 A'$. Thus, we can show that ${}^s a' - a' \in \bigcap_{m \geq 0} t^m A'$. Since A' is a Noetherian integral domain, we have $\bigcap_{m \geq 0} t^m A' = (0)$ by Krull's intersection theorem (cf. [11]).

Namely, ${}^s a' = a'$ for every $a' \in A'$. This is a contradiction.

(III) We shall show that $A_k = (A'_k)^G$. In fact, suppose $\bar{a}' \in (A'_k)^G$, and write

$${}^s a' = a' + t b(g) \quad \text{with } b(g) \in A' ,$$

where $a' \in A'$ with $\bar{a}' = a' \pmod{tA'}$. Then we have

$$b(hg) = {}^h b(g) + b(h) \quad \text{for } g, h \in G .$$

Set $c = (\sum_{g \in G} b(g)) / |G|$. Then $b(g) = c - {}^s c$ for any $g \in G$, and $a' + t c \in (A')^G = A$.

Hence $\bar{a}' \in A_k$. Namely, we have $A_k = (A'_k)^G$. Now, the assertion (2) is readily ascertained. Q.E.D.

Hereafter, we assume that \mathfrak{D} is a formal power series ring $k[[t]]$ over an algebraically closed field k of characteristic zero. The constancy of the K -group scheme $Aut_K(L'/L)$ is assured by

Lemma 3.2. *Let $\mathfrak{D} = k[[t]]$ be as above and let $K = Q(\mathfrak{D})$. Let L be a regular extension of K and let L' be a quasi-Galois extension of L such that L' is a regular extension of K . Then L'/L is a Galois extension.*

Proof. We have only to prove that the K -group scheme $Aut_K(L'/L)$ is constant. Since the Puiseux field $\bigcup_{n > 0} k((t^{1/n}))$ is an algebraic closure of $k((t))$, where $k((t^{1/n}))$ is the quotient field of $k[[t^{1/n}]]$, there exists a cyclic extension $\mathfrak{D}' = k[[\tau]]$ of \mathfrak{D} ($\tau^n = t$) such that $Aut_K(L'/L) \otimes_K K' \simeq Aut_{K'}(L' \otimes_K K' / L \otimes_K K')$ is constant, where $K' = Q(\mathfrak{D}')$. Note that the morphism $\text{Spec } \mathfrak{D}' \rightarrow \text{Spec } \mathfrak{D}$ is a faithfully flat and finite morphism. Let $G = Aut_K(L'/L)(K')$. Then the constant K' -group scheme $G_{K'}$ has apparently a Néron model $G_{\mathfrak{D}'}$, a constant \mathfrak{D}' -group scheme with group G . Hence the K -group scheme $Aut_K(L'/L)$ has an \mathfrak{D} -Néron model \mathcal{G} ; see [13] for relevant results. By definition, the group scheme \mathcal{G} is smooth over \mathfrak{D} and satisfies $\mathcal{G} \otimes_{\mathfrak{D}} K \simeq Aut_K(L'/L)$. By virtue of [1; IV (18.10.16)], \mathcal{G} is finite and étale over \mathfrak{D} . Therefore \mathcal{G} must be a constant \mathfrak{D} -group scheme $H_{\mathfrak{D}}$, where $H \simeq \mathcal{G}(k) = \mathcal{G}(K)$. Since $G = \mathcal{G}(K') \simeq H_{\mathfrak{D}}(K') = H$, we know that $\mathcal{G} \simeq G_{\mathfrak{D}}$. Thus L'/L is a Galois extension with group G . Q.E.D.

Lemma 3.3. *Let the notations and the assumptions be the same as in Lemma 3.1. Assume that L is the quotient field of a finitely generated, normal \mathfrak{D} -domain A . Let A' be the normalization of A in L' , and let G be the Galois group of the extension $L'|L$. Then the following assertions hold true:*

- (1) A' is a Galois extension of A with group G .
- (2) Suppose that A' is geometrically \mathfrak{D} -normal. Then, so is A , and A'_k is a Galois extension of A_k with group G .

Proof. (1) is now clear. As for (2), A'_k is a normal domain by the hypothesis, and $A_k = (A'_k)^G$ by Lemma 3.1. Hence A_k is normal, and A is geometrically \mathfrak{O} -normal. The remaining assertion is clear by Lemma 3.1. Q.E.D.

Now, Main Theorem except the assertion (4) follows from Lemma 3.3. In fact, set $L := Q(A)$ and $L' := Q(R)$ with A and R as in Main Theorem. Then R is the normalization of A in L' , and R is geometrically \mathfrak{D} -normal. So, we can apply Lemma 3.3. We shall prove the assertion (4). Since $A_{\bar{K}}$ is a polynomial ring over \bar{K} , $A_{\bar{K}}$ is a polynomial ring over K by [5]. We can identify G as a finite subgroup of $GL(2, \bar{K})$, and it is well-known that G is then generated by pseudo-reflections. Recall that an element $g \in GL(2, \bar{K})$ is a pseudo-reflection if and only if the fixed-point locus $\Gamma(g)_{\bar{K}} := \text{Spec } \bar{K}[x, y]$ under the action of g has codimension ≤ 1 . Since g acts on $A_{\mathfrak{D}}^2 := \text{Spec } \mathfrak{D}[x, y]$, let $\Gamma(g)$ be the fixed-point locus in $A_{\mathfrak{D}}^2$ under the action of g . Namely, $\Gamma(g)$ is a closed subscheme of $A_{\mathfrak{D}}^2$ defined by an ideal I , where I is the smallest ideal of $\mathfrak{D}[x, y]$ generated by all elements of the form ${}^g a - a$ with $a \in \mathfrak{D}[x, y]$. Then we know that $\Gamma(g)_{\bar{K}} = \Gamma(g) \otimes_{\mathfrak{D}} \bar{K}$ and that $\Gamma(g) \otimes_{\mathfrak{D}} k$ is the fixed-point locus in $A_k^2 := \text{Spec } k[x, y]$ under the action of g . Hence $\Gamma(g) \otimes_{\mathfrak{D}} k$ has codimension ≤ 1 in A_k^2 . This implies that when one embeds G into $GL(2, k)$ upto conjugation in $\text{Aut}_k k[x, y]$, G is generated by pseudo-reflections. Hence the G -invariant subring A_k of $k[x, y]$ is a polynomial ring over k . This verifies the assertion (4) of Main Theorem.

4. Reduction from Conjecture 1 to Conjecture 2

Let \mathfrak{D} , R and A be as in Conjecture 1. Let $Y := A_{\mathfrak{D}}^2 = \text{Spec } R$, let $X := \text{Spec } A$ and let $\pi: Y \rightarrow X$ be the canonical finite morphism. For an algebraic closure \bar{K} of $K = Q(\mathfrak{D})$, $A_{\bar{K}}$ is a normal, cofinite \bar{K} -subalgebra of $\bar{K}[x, y]$. Note that $X_{\bar{K}} = \text{Spec } A_{\bar{K}}$ has at most one singular point. Let \bar{Z}' be the universal covering space of $X_{\bar{K}} - \text{Sing}(X_{\bar{K}})$. Then $\pi_{\bar{K}}: Y_{\bar{K}} - \pi^{-1}(\text{Sing } X_{\bar{K}}) \rightarrow X_{\bar{K}} - \text{Sing}(X_{\bar{K}})$ factors through \bar{Z}' because $Y_{\bar{K}} - \pi^{-1}(\text{Sing } X_{\bar{K}})$ is simply connected. Let \bar{Z} be the normalization of $X_{\bar{K}}$ in the function field $\bar{K}(\bar{Z}')$ of \bar{Z}' . Then $\bar{Z} \simeq A_{\bar{K}}^2$ and $\pi_{\bar{K}}: Y_{\bar{K}} \rightarrow X_{\bar{K}}$ factors through \bar{Z} ;

$$\pi_{\bar{K}}: Y_{\bar{K}} \xrightarrow{\alpha} \bar{Z} \xrightarrow{\beta} X_{\bar{K}}.$$

See [10] for the relevant results. Choose a K -rational point P of $Y_{\bar{K}} - \pi^{-1}(\text{Sing } X_{\bar{K}})$, and let $Q = \bar{\alpha}(P)$. We shall show that \bar{Z} descends down to a K -scheme. Namely, there exist a K -scheme Z and K -morphisms $\alpha: Y \rightarrow Z$ and $\beta: Z \rightarrow X$ such that $\bar{Z} = Z \otimes_{\bar{K}} \bar{K}$, $\bar{\alpha} = \alpha \otimes_{\bar{K}} \bar{K}$ and $\bar{\beta} = \beta \otimes_{\bar{K}} \bar{K}$. In fact, for $\sigma \in \text{Gal}(\bar{K}/K)$, let ${}^\sigma\bar{Z} = \text{Spec } \rho_\sigma(\mathcal{O}(\bar{Z}))$, where $\rho_\sigma: \bar{K}[x, y] \rightarrow \bar{K}[x, y]$ is $\sigma \otimes \text{id}_{K[x, y]}$ and $\mathcal{O}(\bar{Z})$ is the coordinate ring of \bar{Z} which is a \bar{K} -subalgebra of $\bar{K}[x, y]$. We denote by ${}^\sigma\bar{\alpha}: Y_{\bar{K}} \rightarrow {}^\sigma\bar{Z}$ and ${}^\sigma\bar{\beta}: {}^\sigma\bar{Z} \rightarrow X_{\bar{K}}$ the morphisms induced by $\rho_\sigma(\mathcal{O}(\bar{Z})) \hookrightarrow \bar{K}[x, y]$ and $A_{\bar{K}} \hookrightarrow \rho_\sigma(\mathcal{O}(\bar{Z}))$, respectively. Hence $\pi_{\bar{K}} = ({}^\sigma\bar{\beta}) \cdot ({}^\sigma\bar{\alpha})$. Let ${}^\sigma Q$ be the point of ${}^\sigma\bar{Z}$ which corresponds to Q under the canonical isomorphism $\text{Spec } \rho_\sigma(\mathcal{O}(\bar{Z})) \rightarrow \text{Spec } \mathcal{O}(\bar{Z})$. Then we have a unique \bar{K} -isomorphism $\phi_\sigma: {}^\sigma\bar{Z} \rightarrow \bar{Z}$ such that $\rho_\sigma({}^\sigma Q) = Q$, $\bar{\alpha} = \phi_\sigma \cdot {}^\sigma\bar{\alpha}$ and ${}^\sigma\bar{\beta} = \bar{\beta} \cdot \phi_\sigma$. Then it is easy to show that $\phi_{\tau\sigma} = \phi_\tau \cdot {}^\tau\phi_\sigma$ for $\sigma, \tau \in \text{Gal}(\bar{K}/K)$. In fact, this is the case for a finite Galois extension K'/K instead of \bar{K}/K . By the faithfully flat descent, we know that there exists a K -scheme Z such that $\bar{Z} = Z \otimes_{\bar{K}} \bar{K}$. Then $\bar{\alpha} = {}^\sigma\bar{\alpha}$ and ${}^\sigma\bar{\beta} = \bar{\beta}$ for any $\sigma \in \text{Gal}(\bar{K}/K)$.

Therefore $\bar{\alpha}$ and $\bar{\beta}$ descend down to K -morphisms $\alpha: X_K \rightarrow Z$ and $\beta: Z \rightarrow Y_K$ such that $\pi_K = \beta \cdot \alpha$. On the other hand, Z is K -isomorphic to A_K^2 by virtue of [5]. Identify the coordinate ring $\mathcal{O}(Z)$ with a K -subalgebra of $K[x, y]$ under α . Let B be the normalization of A in the function field $K(Z)$ of Z . Then B is a normal, cofinite \mathfrak{D} -subalgebra of R such that $B_K = \mathcal{O}(Z)$ is a polynomial ring over K . The Conjecture 2 then implies that B is a polynomial ring in two variables over \mathfrak{D} . Note that $Q(B)$ is a quasi-Galois extension of $Q(A)$ over K . Main Theorem then asserts that Conjecture 1 is affirmative.

As for the Conjecture 2, we know the following:

Proposition 4.1. *Let \mathfrak{D} , R and A be the same as in Conjecture 2, and let $X = \text{Spec } A$. Suppose that A is geometrically \mathfrak{D} -normal. Then X_k has at most one singular point which has necessarily cyclic quotient singularity.*

Proof. By the hypothesis, A_K is a polynomial ring $K[u, v]$. Let $\Delta = \frac{\partial}{\partial u}$, which is a locally nilpotent K -derivation of A_K . Since A is finitely generated over \mathfrak{D} , we find an integer $n \geq 0$ such that $t^n \Delta(A) \subseteq A$ and $t^n \Delta(A) \not\subseteq tA$. Define a k -derivation δ of A_k by

$$\delta(a) = t^n \Delta(a) \pmod{tA},$$

where $a = a \pmod{tA}$ with $a \in A$. Then δ is well-defined, and δ is a nontrivial, locally nilpotent k -derivation on A_k . Hence $X_k := \text{Spec } A_k$ is affine-ruled (cf. [9]) and X_k has at most one singular point which has necessarily cyclic quotient singularity (cf. [10]).

Q.E.D.

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