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THE *P*-IDEAL LINKING CONCEPT IN CRITICAL POINT THEORY. NON EQUIVARIANT CASE

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0. Introduction

Our main objective in this work is to develop the linking concept via the P-Ideal Index Theory developed in [6] and to show that that concept is useful in critical point theory. This linking concept is based on the Fadell-Husseini linking concept that was developed in [9]. They have employed the numerical-valued cohomological index theory. More precisely, in this work we first shall announce the P-Ideal Valued Cohomological Index Theory. Second, we shall develop the P-Ideal linking concept, and some computational examples of P-Ideal linking between two sets A and B will be provided. Finally, the P-Ideal Linking Concept will be employed in critical point theory in order to obtain a general version of the Li's three critical point theorem.

1. P-Ideal Valued Cohomological Index Theory

The objective of this section is to announce the P-Ideal Valued Cohomological Index Theory that was developed by Dos Santos in [6]. Such a theory gives us the flexibility to choose $H^*(E)$-submodule P of $H^*(A)$, where A is a closed subset of our ambient space E, permitting the development of some useful algebraic topological concepts such as P-Ideal linking between two sets A and B which will be developed in the next section.

Let E be a paracompact space and $(X, A)\in\mathcal{E}_E$ where $\mathcal{E}_E$ is the category of paracompact pair (X, A) in E for a fixed closed subset A of E. Let $H^*(\ )$ be the Alexander-Spanier cohomology theory with a field coefficient K.

Recall that the cup product defines a multiplication on $H^*(X, A)$ as follows:

$$\begin{array}{c} H^*(X, A)\otimes H^*(E) \\ \downarrow 1\otimes i^* \\ H^*(X, A)\otimes H^*(X)\rightarrow H^*(X, A)\end{array}$$

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where 1 is the identity on $H^*(X, A)$ and i is the inclusion map $X \xrightarrow{i} E$. Therefore, $H^*(X, A)$ is an $H^*(E)$-module. In particular, $H^*(A)$ is also an $H^*(E)$-module.

DEFINITION 1.1. Let E be a paracompact space $(X, A) \in \mathcal{E}_E$. For an $H^*(E)$-submodule P of $H^*(A)$ the P-Ideal Value Cohomological Index of (X, A) over K is an ideal denoted by

$$\begin{aligned} P\text{-Index}_E(X, A) &= \operatorname{Ann} M^*(X, A) \quad \text{in} \quad \Lambda = H^*(E) \\ &= \{\lambda \in \Lambda \mid u \cdot \lambda = 0, \ \forall u \in M^*(X, A)\} \end{aligned}$$

where $M^q(X, A) = \delta^q(P)$ for $q \geq -1$, $M^\circ(X, A) = \mathcal{E}(K)$, δ^* is the coboundary operator for the pair (X, A) and $\mathcal{E}$ is the augmentation.

1.2. The corresponding numerical value index is

$$|P\text{-Index}_E(X, A)| = \dim_K \frac{\Lambda}{P\text{-Index}_E(X, A)} .$$

The notion of P-Ideal Index Theory is a generalization of the Facell-Husseini δ-index theory (see [9] for the numerical value notion of the δ-Index theory and [10] for the ideal value notion). In fact, if $P = H^*(A)$ then

$$M^*(X, A) = \operatorname{Im}\{\delta : H^*(A) \to H^*(X, A)\} \quad \text{and} \quad M^\circ(X, A) = \mathcal{E}(K) .$$

Recall that Fadell-Husseini in [10] have defined the δ-Index of the pair $(X, A) \in \mathcal{E}_E$ as being

$$\delta\text{-Index}\,(X, A) = \operatorname{Ann} \operatorname{Im} \delta \quad \text{in} \quad H^*(E) ,$$

and the corresponding numerical value as being

$$|\delta\text{-Index}\,(X, A)| = \dim_K \frac{H^*(E)}{\delta\text{-Index}\,(X, A)} .$$

Therefore,

$$\delta\text{-Index}\,(X, A) = P\text{-Index}_E(X, A) \quad \text{when} \quad P = H^*(A) .$$

Numerical valued cohomological index theories have been applied in critical point theory (more precisely in minimax theory) successfully by Fadell-Husseini-Rabinowitz in [11], [12], [9], [23]. In addition, Fadell-Husseini have observed that for the Lasry-Magil example (which is a problem of the Borsuk-Ulam type), the numerical value index theory does not provide a solution to it. On the other hand, the Ideal-Valued theory provides a nice solution (see Fadell-Husseini in [10]).

REMARKS. The preceeding definition could be considered in a more general setting. In fact, one can consider any category $\mathcal{E}$ of topological pairs (X, A)

in E (E is a topological space) and of maps, and let $h^*(X, A)$ denote a multiplicative cohomology theory on $\mathcal{E}$ i.e. $h^*(\ ,\ )$ is a contravariant functor into graded algebra over a field K, and $h^*(\ ,\ )$ is equipped with long exact sequences, excision, the homotopy property, and the unit in $h^*(X)$.

1.3. In case that $A=\phi$. We can consider the following index theory [10]. Let $(X, \phi)\in\mathcal{E}$ and $X\xrightarrow{i}E$ be the inclusion map. Therefore

$$\begin{array}{c} H^*(X) \otimes H^*(E) \\ \downarrow 1\otimes i_X^* \\ H^*(X) \otimes H^*(X) \rightarrow H^*(X) \end{array}$$

$H^*(X)$ is a $H^*(E)$-module.

Definition 1.4. The Ideal-Valued Cohomological Index of X over K is the ideal

$$\mathrm{Index}_E\, X = \mathrm{Ann}\, H^*(X) \quad \text{in} \quad H^*(E) = \{\lambda\in H^*(E)\,|\, u\lambda = 0,\ \forall u\in H^*(X)\}\ .$$

1.5. We observe that $\mathrm{Index}_E\, X = \ker\, i_X^*$. In fact if $\lambda\in\mathrm{Ker}\, i_X^*$, then $i_X^*(\lambda)=0$. Therefore

$$u\cdot i_X^*(\lambda) = 0,\ \forall u\in H^*(X)$$

showing that $\mathrm{Ker}\, i_X^*\subset\mathrm{Index}_E\, X$. On the other hand, given $\lambda\in\mathrm{Index}_E\, X$ then $u\cdot i_X^*(\lambda)=0$, $\forall u\in H^*(X)$. Since $1\in H^*(X)$ implies

$$1\cdot i_X^*(\lambda) = 0\,,$$

therefore,

$$\mathrm{Index}_E\, X\subset\mathrm{Ker}\, i_X^* \quad \text{then}$$
$$\mathrm{Index}_E\, X = \ker\, i_X^*\,.$$

It is important to observe that the above index theory satisfies those important properties of the Ljusternik-Schnirelmann theory.

Perhaps the first question that comes to mind is the following:

1.6. When is the P-$\mathrm{Index}_E(X, A)$ a finitely generated ideal over K?

In order to answer this question it is important to observe that $H^*(E)$ is a connected skew commutative grades K-algebra (see Dos Santos in [5]). Thus one needs to have the concept of Noetherian ring R for a non-commutative ring R. Following Passman in [22] (pages 423–424) one has:

1.7. Let R denote a non-commutative right ring.

Definition. R is a right Noetherian ring if and only if all right ideals are finitely generated over R.

Theorem 1.8. *Let S be a ring, $R\subset S$ a subring and $u\in S$. Assume*
(1) *R is right Noetherian.*
(2) *$R+uR=R+Ru$.*
(3) *$S=(R, u)\equiv$ ring generated by R and u.*
Then S is right Noetherian.

Invoking the above theorem (1.8) one can prove the following property:

Property 1.9. *Let $H=\bigoplus_{i\geq 0} H_{i0}H_{o}=K$ denotes a graded skew commutative algebra over K with a finite set $b_1, b_2, \cdots, b_k$ of generators as an algebra. Then H is Noetherian.*

Proof. See Dos Santos in [6].

An immediate consequence of (1.9) is the following Corollary.

1.10. *Let M be a finitely generated right R-module and R be a right Noetherian ring, then M is Noetherian (as in R-module).*

Therefore, the answer to the question (1.6) is if E is a connected paracompact Banach manifold such that $H^*(E)$ is finitely generated as an algebra over K then by (1.9), $H^*(E)$ is Noetherian. Consequently

(1.11) $\qquad P\text{-Index}_E(X, A)$ is finitely generated over K.

The P-Ideal Valued Cohomological Index Theory satisfies those important properties of the Ljusternik-Schnirelmann theory. In fact,

Monotonicity Property 1.12. *Given a commutative diagram in $\mathcal{E}_E$*

$$\begin{tikzcd}
(X_1, A) \arrow[rr, "f"] \arrow[dr, "i_{X_1}"'] & & (X_2, A) \arrow[dl, "i_{X_2}"] \\
& E &
\end{tikzcd}$$

Inducing a commutative diagram in cohomology level

$$\begin{tikzcd}
H^*(X_1, A) & & H^*(X_2, A) \arrow[ll, "f^*"'] \\
& H^*(E) \arrow[ul, "i^*_{X_1}"] \arrow[ur, "i^*_{X_2}"'] &
\end{tikzcd}$$

If

$$\begin{gathered}\mathrm{Id} = (f_{|A})^*\colon H^*(A) \to H^*(A) \quad \textit{then} \\ P \mapsto P \\ P\text{-Index}_E(X_1, A) \supset P\text{-Index}_E(X_2, A)\,.\end{gathered}$$

Proof. See Dos Santos in [6].

Subadditivity Property 1.13. *Given a commutative diagram in* $\mathcal{E}_E(X = X_1 \cup X_2)$

$$\begin{tikzcd} (X_1, A) \arrow[r] \arrow[dr, "i_{X_1}"'] & (X_1 \cup X_2, A) \arrow[d] & X_2 \arrow[l] \arrow[dl, "i_{X_2}"] \\ & E & \end{tikzcd}$$

then

$$P\text{-Index}_E(X_1 \cup X_2, A) \supset P\text{-Index}_E(X_1, A).\ \mathrm{Index}_E\, X_2\,.$$

Proof. See Dos Santos in [6].

Invariance Property 1.14. *Given a morphism* φ *in* $\mathcal{E}_E$

$$\varphi\colon (X_1, A) \to (X_2, A)$$

such that

$$\varphi^*\colon H^*(X_2, A) \to H^*(X_1, A)$$

is an isomorphism and

$$\mathrm{Id} = (\varphi_{|}A)^*\colon H^*(A) \to H^*(A)\,,$$

then

$$P\text{-Index}_E(\varphi(X_1), A) = P\text{-Index}_E(X_2, A)\,.$$

The proof (1.14) is an application of the (1.12) twice.

1.15. Recall that the Alexander-Spanier Cohomology Theory [26] satisfies the Continuity Property.

C.1 Suppose that $(X, A) \in \mathcal{E}_E$, If $\mathcal{N} = \{(V_\alpha, A)\}$ is a family of neighborhoods (V_α, A) of (X, A), $(V_\alpha, A) \in \mathcal{E}_E$ is directed downward to (X, A) by inclusion. Then

$$\varinjlim_{\alpha} H^*(V_\alpha, A) = H^*(X, A)$$

since E is a paracompact space.

C.2 Given any open set U such that $A \subset X \subset U \subset E$ there is an open set V such that

$$A \subset V \subset \bar{V} \subset U \quad \text{and} \quad (\bar{V}, A) \in \mathcal{N} .$$

Continuity Property 1.16. *Let E be a paracompact space such that $H^*(E)$ is finitely generated over K as an algebra. Given any open set U such that $A \subset X \subset U \subset E$, there is an open set V such that $A \subset X \subset V \subset \bar{V} \subset U$ and*

$$P\text{-Index}_E(X, A) = P\text{-Index}_E(\bar{V}, A) .$$

Proof. See Dos Santos in [6].

(1.17) *Let $X \in \mathcal{E}_E$. If* $\text{Index}_E X \subsetneq \check{H}^*(E)$ *then X has a positive cohomology, in particular $X \neq \phi$*

Proof. Recall that

$$\text{Index}_E X = \ker\{i_X^*: H^*(E) \to H^*(X)\} \subsetneq \check{H}^*(E) .$$

Therefore there is $q>0$: $H^q(X) \neq 0$ i.e. X has positive cohomology. Then $X \neq \phi$.

2. The P-Ideal linking concept between two sets A and B

Various concepts of linking have been employed successfully by many mathematicians in critical point theory, e.g. see [1], [2], [3], [8], [9], [18], [19], [25] and [16]. Our main objective in this section is to develop linking concepts via the P-Ideal Valued Cohomological Index Theory anounced in 1. These linking concepts are based on the Fadell-Husseini linking concept that was developed in [9]. They have employed the numerical-valued cohomological index theory. More precisely, in this section we shall develop the linking concept via the P-Ideal Valued Index Theory in the non-equivariant case called the P-Ideal Linking concept. This linking concept is related to two conditions: the geometric condition (HO). and the cohomological condition (CO). And some important computational examples of P-Ideal Linking between two set A and B will be provided. These examples of the P-Ideal linking between two sets A and B will be useful in critical point theory.

Definition 2.1. Let E be a paracompact space and $(X, A) \in \mathcal{E}_E$. Let A and B be two disjoint closed sets in E. We say that A is P-Ideal linking to B if and only if

$$P\text{-Index}_E(E \backslash B, A) \supsetneq P\text{-Index}_E(E, A) .$$

Proposition 2.2. *Let E be a Banach space, and A and B be disjoint closed sets such that*

(1) $\check{H}^*(A) \neq 0$, *A has positive cohomology*

(2) $H^*(E \backslash B) \to H^*(A)$ *is an epimorphism*

then A is P-Ideal linking to B where $P=H^*(A)$.

Proof. $P\text{-Index}_E(E\backslash B, A)=\text{Ann Im}\ \{\delta: H^*(A)\to H^*(E\backslash B, A)\}=K$. Therefore

$$P\text{-Index}_E(E\backslash B, A) = K \supsetneqq P\text{-Index}_E(E, A) = (0)\,.$$

This shows that A is P-Ideal Linking (over K) to B.

Corollary 2.3. *Let E be a Banach space, and A and B two disjoint closed sets such that $i_*: H_*(A)\to H_*(E\backslash B)$ is a monomorphism. If $P=H^*(A)$ then A is P-Ideal linking to B.*

Proof. $i^*: H^*(E\backslash B)\to H^*(A)$ is an epimorphism since i_* is a monomorphism. Therefore, the proof of (2.3) is completed by (2.2).

We shall generalize the above result. First, we need to introduce two geometric conditions called Geometric Condition (HO). and Cohomological Condition (CO). It is important to observe that in the Geometric Condition (HO). we shall employ the Singular Homology Theory while in the Cohomological Condition (CO). we can employ the Alexander-Spanier Cohomology Theory or Singular Cohomology Theory since $A\subset E$ will be a space locally contractible.

Geometric Condition (HO) 2.4. Let E be a paracompact Banach manifold, A and B be two disjoint closed sets. Let ξ denote a singular cycle in A and $\eta=[\eta]$ its homology class (over a field K). Assume that $H_*(\ )$ is the singular homology theory.

We say that A and B satisfy the Geometric Condition (HO) if there exists an $\eta\in H^*(A)$ such that $i_2(\eta)\neq 0$ and $i_1(\eta)=0$ where

$$\begin{tikzcd}
 & H_*(E\backslash B) \arrow[dd, "i_*"] \\
H_*(A) \arrow[ur, "i_2"] \arrow[dr, "i_1"'] & \\
 & H_*(E)\,.
\end{tikzcd}$$

Cohomological Condition (CO) 2.5. Let E, A and B be as in (2.4). Let $H^*(\ )$ be the Alexander-Spanier Cohomology Theory or the Singular Cohomology Theory.

We say that A and B satisfy the Cohomology Condition (CO) if there is an $H^*(E)$-module $P\subset H^*(A)$ such that $\delta_1 P\neq 0$ and $\delta_2 P=0$ where:

$$\begin{tikzcd}[column sep=large] & H^*(E,A) \arrow[dd, "j^*"] \\ H^*(A) \arrow[ur, "\delta_1"] \arrow[dr, "\delta_2"'] & \\ & H^*(E\backslash B, A)\,. \end{tikzcd}$$

Proposition 2.6. *Let E, A and B be as in* (2.5). *If A and B satisfy the Geometric Condition* (HO)., *A is locally contractible and*

$$P = \text{Image}\,\{H^*(E\backslash B) \xrightarrow{i^*} H^*(A)\}\ .$$

Then

A and B satisfy the Cohomological Condition (CO).

Proof. By the Geometric Condition (HO) we have the following diagram

$$\begin{tikzcd}[column sep=large] & H_*(E\backslash B) \arrow[dd, "j^*"] \\ H_*(A) \arrow[ur, "i_2"] \arrow[dr, "i_1"'] & \\ & H_*(E) \end{tikzcd}$$

and there exist $\eta \in H_*(A)$ such that $i_2(\eta) \neq 0$ and $i_1(\eta) = 0$. Denote

$$i_2(\eta) = \gamma$$

$$P \equiv H^*(E)\text{-module generated by} \quad \eta^*$$

where η^* is the dual class of η (over a field K). Note that

$$\begin{tikzcd}[column sep=large] \mathcal{H}om(H_*(E\backslash B; K)) \arrow[dr, "i_2^*"] & \\ & \mathcal{H}om(H_*(A); K)\,. \\ \mathcal{H}om(H_*(E); K) \arrow[ur, "i_1^*"'] & \end{tikzcd}$$

Moreover

$$i_2^*(\gamma^*)\,\eta = \langle i_2^*\,\gamma^*, \eta\rangle = \langle \gamma^*, i_2\,\eta\rangle = 1 \tag{2.7}$$

and for any $\xi \in (\gamma^{\perp})$ we have

$$
\begin{aligned}
i_2^*(\gamma)\,\eta &= \langle i_2^*\,\gamma^*, \xi\rangle = \langle \gamma^*, i_2\,\xi\rangle = 0 \\
i_1^*(\rho^*)\,\eta &= \langle i_1^*(\rho^*), \eta\rangle = \langle \rho^*, i_1\,\eta\rangle = 0\,.
\end{aligned} \tag{2.8}
$$

Furthermore, the following diagram holds

$$
\begin{tikzcd}
 & H^*(E, A) & \\
H^*(E\backslash B) \arrow[r, "i_2^*"] & H^*(A) \arrow[u, "\delta_1"] \arrow[r, "\delta_2"] & H^*(E\backslash B, A) \\
 & H^*(E) \arrow[u, "i_1^*"] &
\end{tikzcd} \tag{2.9}
$$

hence (2.7) and (2.3) guarantee that $i_2^*(\gamma^*)$ is not in the image of $H^*(E)$ by i_1^*. Therefore

$$\delta_1\,P \neq 0 \quad \text{and} \quad \delta_2\,P = 0\,.$$

2.10. Let E, A and B be as in (2.7) assuming that A and B satisfy the geometrical condition (HO). Is A P-Ideal linking to B for some $H^*(E)$-module $P \subset H^*(A)$?

In order to answer the above question, one needs to compute P-Index_E $(E\backslash B, A)$ and P-$\text{Index}_E(E, A)$. Let us take

$$P = \text{Im}\{H^*(E\backslash B) \xrightarrow{i^*} H^*(A)\}\,.$$

By (2.6) A and B satisfy the Condition (CO)., since A and B satisfy the Geometrical Condition (HO). Therefore $\delta_1\,P \neq 0$ and $\delta_2\,P=0$. Then

$$P\text{-Index}_E(E, A) = \text{Ann}\,\delta_1\,P = \{\lambda \in H^*(E) \colon u\lambda = 0,\ \forall u \in \delta_1\,P\} = \alpha$$

Furthermore

$$
\begin{aligned}
P\text{-Index}_E(E\backslash B, A) &= \text{Ann}\,\delta_2\,P \quad \text{in} \quad H^*(E) \\
&= \{\lambda \in H^*(E) \colon u\lambda = 0,\ \forall u \in \delta_2\,P = 0\} \\
&= H^*(E)
\end{aligned}
$$

then A might be P-Ideal linking to B. In order to get A and B P-Ideal linking one needs to have

$$H^*(E) \supset \tilde{H}^*(E) \supsetneqq \alpha$$

2.11. For example, let E be a Banach space. Hence

$$\cdots \to H^*(E) \to H^*(A) \to H^{*+1}(E, A) \to H^{*+1}(E) \to \cdots\,.$$

Therefore

$$P\text{-Index}_E(E, A) = \{\lambda \in H^*(E) : u\lambda = 0,\ \forall u \in \delta_1 P\} = 0$$

since $\delta_1 P \neq 0$ then

$$P\text{-Index}_E(E \backslash B, A) = H^*(E) \supset \check{H}^*(E) \supseteq P\text{-Index}_E(E, A) = (0)$$

i.e. A is P-Ideal linking to B.

Proposition 2.12. *Assume that E is a Banach space, A and B are two disjoint sets in E. If A is P-Ideal Linking to B, then*

(1) *$\check{H}^*(A) \neq 0$, A has positive cohomology.*

(2) *A can not be contractible to a point in $E \backslash B$.*

Proof. We observe that $P\text{-Index}_E(E \backslash B, A) = K$ and $P\text{-Index}_E(E, A) = (0)$ since A is P-Ideal Linking to B in a Banach space E.

By the exact sequence of the (E, A):

$$\cdots \to H^*(E) \to H(A) \xrightarrow{\delta_1} H^*(E, A) \to H^*(E) \to \cdots$$

we have $\check{H}^*(A) \simeq H^*(E, A) \neq 0$ then A has positive cohomology. On the other hand, if A is contractible to point in $E \backslash B$, we have

$$\cdots \to H^*(E - B) \xrightarrow{i^*} H^*(A) \xrightarrow{\delta_2} H^*(E \backslash B, A) \to \cdots$$

then δ_2 is a monomorphism. Therefore

$$P\text{-Index}_E(E \backslash B, A) = 0 \quad \text{since} \quad \delta_2 P \neq 0\,.$$

This is a contradiction. Then the above proposition is proved.

3. A Critical Point Theory

3.1. A General Version of Li's Theorem

The aim of this section is to prove an abstract critical point theorem which is a general version of Li's theorem in [16]. The P-Ideal Linking concept is applied successfully in order to obtain a critical value of a functional J by the minimax procedure. Furthermore, if J is bounded from below we shall use the P-Ideal Linking concept to obtain a third critical point of J.

Theorem 3.2. *Let E be a connected paracompact Banach manifold and J be a C^1-functional. Assume that there are $c_0, a, b, c_\infty \in \bar{R}$:*

$$-\infty \leq c_0 < a < b < c_\infty \leq \infty$$

such that

(J_1) *J satisfies the $(P.S)$ condition in $J^{-1}(]c_0, c_\infty[)$.*

(J_2) *There are two disjoint sets A and B such that A is P-Ideal Linking to B, $P \subset H^*(A)$*

(J_3) $J(u) \geq b$ *on* B

(J_4) $J(u) \leq a$ *on* A

(J_5) *There exists a closed set $\tilde{X} \supset A$ in E such that $\tilde{X} \backslash A$ is precompact and* $P\text{-Index}_E(\tilde{X}, A) = P\text{-Index}_E(E, A)$.

Then J possesses at least one critical value $c \geq b$.

Proof. Denote

$$\alpha = P\text{-Index}_E(E, A) \quad \text{and} \quad \beta = P\text{-Index}_E(E \backslash B, A) .$$

Since A is P-Ideal Linking to B, hence $\beta \supsetneqq \alpha$. Define

$$c = \inf_{X \in \Sigma_\alpha} \sup_{u \in X} J(u)$$

where

$$\Sigma_\alpha = \{(X, A) \in \mathcal{E}_E : P\text{-Index}_E(X, A) = \alpha\}$$

and $\mathcal{E}_E$ is the class of all paracompact pairs (X, A) in E. Note that $\Sigma_\alpha \neq \phi$ since $(\tilde{X}, A) \in \Sigma_\alpha$.

STEP 1. c is well defined as a real number. First, note that

$$c < \infty .$$

In fact

$$\sup_{u \in \tilde{X}} J(u) < \infty \quad \text{and} \quad P\text{-Index}_E(\tilde{X}, A) = \alpha \quad \text{by} \quad (J_5)$$

therefore

$$c = \inf_{X \in \Sigma_\alpha} \sup_{u \in X} J(u) \leq \sup_{u \in \tilde{X}} J(u) < \infty . \tag{3.3}$$

The following intersection property is verified:

$$\forall X \in \Sigma_\alpha : X \cap B \neq \phi, \text{ since } A \text{ is } P\text{-Ideal Linking to } B.$$

Otherwise, there is a $X_1 \in \Sigma_\alpha : X_1 \cap B = \phi$. Thus

$$(X_1, A) \subset (E \backslash B, A) .$$

Hence

$$P\text{-Index}_E(X_1, A) \supset P\text{-Index}_E(E \backslash B, A) \supsetneqq \alpha$$

therefore

$$(X_1, A) \notin \Sigma_\alpha$$

which is a contradiction. This shows that

$$\forall X \in \Sigma_\alpha \Rightarrow X \cap B \neq \phi$$

and hence

$$b \leq \sup_{u \in X \cap B} J(u) \leq \sup_{x \in X} J(u), \quad \forall X \in \Sigma_\alpha$$

implying

$$c = \inf_{X \in \Sigma_\alpha} \sup_{u \in X} J(u) \geq b \tag{3.4}$$

by (3.3) and (3.4) c is indeed a real number. Moreover

$$a < b \leq c < \infty .$$

Step 2. c is indeed a critical value of J. Suppose

$$c = \inf_{X \in \Sigma_\alpha} \sup_{u \in X} J(u)$$

is a regular value of J. Therefore, there is a set $X_1 \in \Sigma_\alpha$:

$$\sup_{u \in X_1} J(u) \leq c + \varepsilon$$

by taking ε as in the deformation theorem. Suppose otherwise:

$$\sup J(u) > c + \varepsilon , \quad \forall X \in \Sigma_\alpha$$

therefore

$$c = \inf_{X \in \Sigma_\alpha} \sup_{u \in X} J(u) \geq c + \varepsilon$$

that is a contradiction. Indeed, there is a set $X_1 \in \Sigma_\alpha$:

$$X_1 \subset \dot{J}^{c+\varepsilon}$$

by invoking the deformation theorem (see [25]). One can get a homeomorphism

$$\eta_t : E \to E, \ \forall^{t \in [0,1]}$$

$$\eta_t(u) = u , \quad \forall u \colon |J(u) - c| \geq \varepsilon \tag{3.5}$$

$$\eta_1(X_1) \subset J^{c-\varepsilon} . \tag{3.6}$$

Let us choose ε such that: $a < c - \varepsilon$. Hence

$$J(u) \leq a < c - \varepsilon , \quad \forall u \in A \tag{3.7}$$

and

$$\eta_t(u) = u , \quad \forall u \in A , \quad \forall t \in [0, 1] .$$

From (3.7) and the invariance property of P-Ideal valued index theory:

$$P\text{-Index}_E(\eta_1(X_1), A) = P\text{-Index}_E(X_1, A) = \alpha .$$

Therefore

$$Y = \eta_1(X_1) \in \Sigma_\alpha \quad \text{and} \quad Y \subset J^{c-\varepsilon}\,.$$

Furthermore

$$\sup_{u\in Y} J(u) \leq \sup_{u\in J^{c-\varepsilon}} J(u) \leq c-\varepsilon$$

therefore

$$c = \inf_{X\in\Sigma_\alpha} \sup_{u\in X} J(u) \leq c-\varepsilon$$

which is a contradiction. Consequently c is indeed a critical value of J.

Corollary 3.8. *Let E be a Banach space and J be a C^1-functional. Assume all hypotheses of the preceding theorem. Assume either (J_5) or A is compact. If J is bounded from below then J possesses at least three critical points.*

Proof. Let m be the minimum value of J

$$m = \min_{u\in E} J(u)\,.$$

m is indeed a real number, since J is bounded from below. Furthermore, there exists a $u_0 \in E$:

$$J(u_0) = m = \min_{u\in E} J(u)$$

and

$$J'(u_0) = 0\,,$$

since J is bounded from below and satisfies the (P.S) condition. In fact, according to Zeidler in [27] (p. 158). $\forall \varepsilon > 0$, $\exists u \in E$:

$$J(u) \leq \inf_{v\in E} J(v) + \varepsilon$$

and

$$\|J'(u)\| \leq \varepsilon$$

since J is lower semicontinuous, G-differentiable, and bounded from below. Define now

$$m = \inf_{v\in E} J(v)$$

m is well defined, since J is bounded from below. Taking $\varepsilon = \dfrac{1}{n}$, there exists a sequence $\{v_n\}$

$$m \leq J(v_n) \leq m + \frac{1}{n}$$

and

$$0 \leq \|J'(v_n)\| \leq \frac{1}{n}$$

therefore

$$J(v_n) \to m \quad \text{and} \quad \|J'(v_n)\| \to 0 \text{ as } n \to \infty .$$

By (P.S) condition there is a subsequence $\{v_{n_k}\}$ of the sequence $\{v_n\}$

$$v_{n_k} \to v_0 , \quad \text{as} \quad k \to \infty .$$

By continuity of J and J'

$$J(v_{n_k}) \to J(v_0) \quad \text{and} \quad J'(v_{n_k}) \to J'(v_0)$$

in norm operator. Then there is a $v_0 \in E$:

$$J(v_0) = m = \min_{v \in E} J(v)$$

and

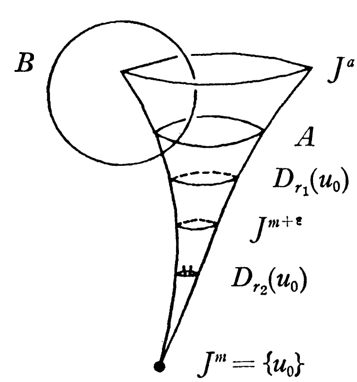
$$J'(v_0) = 0 .$$

This assures us that m is indeed a critical value of J. Observe now that $-\infty < m \leq a$, since $J(u) \leq a$ for all $u \in A$ and m is the minimum value of J.

3.9. Suppose that the minimum m is attached at more than one point u_0 then by Theorem (3.2), J possesses at least one critical value c_2, $m \leq a < b \leq c_2$, and the result is immeditae.

3.10. Now, let us suppose that there exists a unique point u_0 such that the minimum value of J is attained. Thus it is enough to show that J has a critical value $c_1 \in (m, a]$. In fact, $m < a$. Otherwise $m = a$. Note $m \leq J(u) \leq m$, $\forall u \in A$. Hence $J(u) = m$, $\forall u \in A$. By (J_2) and (2.12) A has positive cohomology. Hence A must have more than a point. This contradicts the fact that there exists a unique minimum point u_0.

3.11. In order to show that there is a critical value of J $c_1 \in (m, a]$, let us assume that J has no critical value in $(m, a]$.



By invoking the deformation theorem there is $\varepsilon\in(0, \bar{\varepsilon})$ and a homeomorphism $\eta\in C([0, 1]\times E, E)$ such that

$$\eta_t: J^a \to J^a \quad \text{and} \quad \eta_t(J^a)\subset J^{m+\varepsilon}$$

for all $t\in[0, 1]$. Let $D_{r_1}(u_0)$ and $D_{r_2}(u_0)$ be two disk neighborhoods of u_0 for $0<r_1<r_2$ such that $D_{r_2}(u_0)\subset J^a$.

Claim 3.12. For ε small enough $J^{m+\varepsilon}\subset D_{r_2}(u_0)$. In order to prove the claim, first note that J has no critical point in $J^a\backslash D_{r_1}(u_0)$, since otherwise the corollary is proved.

By (P.S) condition there is an $r>0$ such that

$$\|J'(u)\|\geq r,\ \forall u\in J^a\backslash D_{r_1}(u_0)\,. \tag{3.13}$$

Otherwise there exists a sequence of $r_n\to 0$ and $u_n\in J^a\backslash D_{r_1}(u_0)$ such that

$$\|J'(u_n)\|<r_n\,.$$

By (P.S) condition there is a subsequence u_{n_k} of $\{u_n\}$ such that $u_{n_k}\to u\in J^a\backslash D_{r_1}(u_0)$ but u is a critical point of J in $J^a\backslash D_{r_1}(u_0)$ which is a contradiction to the hypothesis (3.11).

Based on an analogous argument in the proof of Theorem A.4 in [25] pp. 82–86, one can be assured that there is a pseudogradient vector field v for J at u. *i.e*:

$$\|v\|\leq 2\,\|J'(u)\| \tag{3.14}$$

3.15. $J'(u).\,v\geq\|J'(u)\|^2$ and v is Lipschitz for any $u\in J^a\backslash D_{r_1}(u_0)$. Define the flow φ by

$$\begin{cases} \dfrac{d}{dt}\varphi = -\dfrac{v(\varphi(t))}{\|J'(\varphi(t))\|^2} \\ \varphi(0) = u \end{cases}$$

therefore

$$\begin{aligned} \left\|\int_0^t \frac{d\varphi(s)}{ds}\,ds\right\| &= \left\|-\int_0^t \frac{v(\varphi(s))}{\|J'(\varphi(s))\|^2}\,ds\right\| \\ \|\varphi(t)-\varphi(0)\| &\leq \int_0^t \frac{\|v(\varphi)\,(s)\|}{\|J'(\varphi(s))\|^2}\,ds \\ &\leq \int_0^t 2\,\frac{\|J(\varphi)\,(s)\|}{\|J'(\varphi(s))\|^2}\,ds \\ &\leq 2\int_0^t \frac{1}{\|J'(\varphi(s))\|}\,ds \end{aligned}$$

$$\leq 2\int_0^t \frac{1}{r}\,ds = \frac{2t}{r}$$

taking $T \leq \frac{r}{2}(r_2 - r_1)$. Hence

$$\|\varphi(T) - u\| \leq r_2 - r_1$$

then for $t \in [0, T]$ the flow φ_t is not in $D_{r_1}(u_0)$. Furthermore

$$\begin{aligned}\frac{d}{dt}J(\varphi(t)) &= \langle J'(\varphi(t)), \frac{d}{dt}\varphi(t)\rangle \\ &= \langle J'(\varphi(t)), -\frac{v(\varphi(t))}{\|J'(\varphi(t))\|^2}\rangle \\ &= -\frac{1}{\|J'(\varphi(t))\|^2}\langle J'(\varphi(t)), v(\varphi(t))\rangle\end{aligned}$$

and

$$\int_0^t \frac{d}{ds}J(\varphi(s))\,ds = \int_0^t -\frac{1}{\|J'(\varphi(s))\|^2}\langle J'(\varphi(s)), v(\varphi(s))\rangle\,ds$$

(by (3.15))
$$\leq -\int_0^t \frac{\|J'(\varphi(s))\|^2}{\|J'(\varphi(s))\|^2}\,ds$$
$$\leq -t\,.$$

Therefore

$$J(\varphi(t) - J(\varphi(0)) \leq -t \tag{3.16}$$
$$J(\varphi(T) \leq J(u) - T$$

suppose that $J^{m+\varepsilon} \not\subset D_{r_2}(u_0)$ for all $\varepsilon > 0$. Then there is a $u \in J^{m+\varepsilon} \backslash D_{r_2}(u_0)$ such that

$$J(u) < m + T \quad \text{for some small} \quad \varepsilon > 0$$

if not

$$J(u) \geq m + \varepsilon + T \quad \text{for every} \quad \varepsilon \geq 0$$

and $u \in J^{m+\varepsilon} \backslash D_{r_2}(u_0)$. By taking the infimum in ε

$$J(u) \geq m + T,\ \forall u \in J^{m+\varepsilon} \backslash D_{r_2}(u_0)$$

which is a contradiction. Therefore, there exists an ε and $u \in J^{m+\varepsilon} \backslash D_{r_2}(u_0)$ such that

$$J(u) < m + T \tag{3.17}$$

by (3.16) and (3.17)

$$J(\varphi(T)) \leq J(u) - T < m + T - T$$

$$J(\varphi(T))\leq m$$

which is a contradiction since m is the minimum value of J. Therefore, for some small $\varepsilon>0$:

$$\eta(A)\subset J^{m+\varepsilon}\subset D_{r_2}(u_0)\,.$$

That is, A is deformable into a disk neighborhood $D_{r_2}(u_0)$ in the complement of B. Therefore A contains a q-cycle S such that there exists a $(q+1)$-chain ξ in E which is also a $(q+1)$-chain in $E\backslash B$ with boundary S or A can be deformable to a point in $E\backslash B$. This assures us of a contradiation with (J_2) therefore (2.12) are verified. Then one must have a critical value $c_1\in(m, a]$ of J. Consequently, J possesses at least three distinct critical points.

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