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ON THE EXISTENCE OF UNRAMIFIED p -EXTENSIONS

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Introduction

Let p be an odd prime. Let K be an algebraic number field of finite degree, and let L/K be a p -extension. Throughout this paper, a p -extension means a finite Galois extension whose Galois group is a p -group. In this paper, we study the existence of a p -extension $M/L/K$ such that M/L is unramified.

One of our results is the following.

Let k be the rational number field or an imaginary quadratic field with the class number prime to p (p is not equal to 3 when $k = \mathbf{Q}(\sqrt{-3})$). Let $L/K/k$ be a Galois tower satisfying the conditions (1), (2) and (3) in Theorem 1 below, and E be a non-split central extension of $\text{Gal}(L/k)$ by $\mathbf{Z}/p\mathbf{Z}$. Then there exists a Galois extension M/k such that M/K is unramified and $\text{Gal}(M/k)$ is isomorphic to E .

We try to proceed by means of the theory of central imbedding problems. In §1, we explain about the central imbedding problems. In §2, we study the existence of unramified p -extensions, and in §3 and §4, we have an application of results proved in §2. In §5, we study the central imbedding problem of exponent p .

1. Central imbedding problems

Let k be an algebraic number field of finite degree, \mathfrak{G} its absolute Galois group, and let L/k be a finite Galois extension with Galois group G . Let $(\varepsilon): 1 \rightarrow A \rightarrow E \xrightarrow{j} G \rightarrow 1$ be a central extension of finite groups. Then a central imbedding problem $(L/k, \varepsilon)$ is defined by the diagram

$$(*) \quad (\varepsilon): \begin{array}{ccccccc} & & & & \mathfrak{G} & & \\ & & & & \downarrow \varphi & & \\ & & & & G & & \\ & & & & \downarrow & & \\ & & & & 1 & & \end{array}$$

where φ is the canonical surjection. A solution of the imbedding problem $(L/k, \varepsilon)$ is, by definition, a continuous homomorphism ψ of \mathfrak{G} to E with $j \circ \psi = \varphi$. The

Galois extension over k corresponding to the kernel of any solution is called a solution field. A solution ψ is called a proper solution if it is surjective. The existence of a proper solution of $(L/k, \varepsilon)$ is equivalent to the existence of a Galois extension $M \supset L \supset k$ with Galois group $\text{Gal}(M/k)$ which is isomorphic to E and the canonical projection $\text{Gal}(M/k) \rightarrow \text{Gal}(L/k)$ coincides with the given homomorphism $j: E \rightarrow G$. We say the imbedding problem $(L/k, \varepsilon)$ is solvable (resp. properly solvable) if it has a solution (resp. proper solution). Now, we quote some results of central imbedding problems without proofs. For details, see Neukirch [1].

Let $(L/k, \varepsilon)$ be a central imbedding problem defined by the diagram (*).

Lemma 1. *If L/k is unramified or ε is split, then $(L/k, \varepsilon)$ is solvable.*

For any prime number p , denote by $G(p)$ one of the p -Sylow subgroups of G . Let $k^{(p)}$ be the fixed field of $G(p)$. Then the central imbedding problem $(L/k, \varepsilon)$ induces the p -Sylow problem $(L/k^{(p)}, \varepsilon(p))$, which is defined by the diagram

$$\varepsilon(p) : 1 \longrightarrow A \longrightarrow E(p) \xrightarrow{j|_{E(p)}} G(p) \longrightarrow 1$$

$$\begin{array}{c} \mathfrak{G}(p) \\ \downarrow \varphi|_{\mathfrak{G}(p)} \\ G(p) \end{array}$$

where $E(p)$ (resp. $\mathfrak{G}(p)$) is the inverse of $G(p)$ by j (resp. φ).

The following reduction holds.

Lemma 2. *If p -Sylow problems $(L/k^{(p)}, \varepsilon(p))$ are solvable for any prime number p , then $(L/k, \varepsilon)$ is solvable.*

REMARK. Lemma 2 holds for general imbedding problems. For example, let L/k be a Galois extension of a local field k and \mathfrak{G} the Galois group of \mathbf{L} over k , where \mathbf{L} is the maximal unramified extension of L . Then an imbedding problem $(L/k, \varepsilon)$ is defined, and Lemma 2 holds for this.

For any prime \mathfrak{p} of k , denote by $k_{\mathfrak{p}}$ (resp. $L_{\mathfrak{p}}$) the completion of k (resp. L) by \mathfrak{p} (resp. an extension of \mathfrak{p} to L). Then the local problem $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$ of $(L/k, \varepsilon)$ is defined by the diagram

$$(\varepsilon_{\mathfrak{p}}) : 1 \longrightarrow A \longrightarrow E_{\mathfrak{p}} \xrightarrow{j|_{E_{\mathfrak{p}}}} G_{\mathfrak{p}} \longrightarrow 1$$

$$\begin{array}{c} \mathfrak{G}_{\mathfrak{p}} \\ \downarrow \varphi|_{\mathfrak{G}_{\mathfrak{p}}} \\ G_{\mathfrak{p}} \end{array}$$

where $\mathfrak{G}_{\mathfrak{p}}$ is the absolute decomposition group of \mathfrak{p} which is isomorphic to the absolute Galois group of $k_{\mathfrak{p}}$, and $E_{\mathfrak{p}}$ is the inverse of $G_{\mathfrak{p}}$ by j . Similarly to the case of $(L/k, \varepsilon)$, we define a solution and a solution field of local problem $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$.

The following lemma is a generalization of Grunwald-Wang-Hasse's Theorem by Neukirch.

Lemma 3. (Neukirch [1; Example 1, Corollary 6.4]) *Assume that $(L/k, \varepsilon)$ is solvable. Let S be a finite set of primes of k . Let $M_{\mathfrak{p}}$ be a solution field of $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$ for \mathfrak{p} of S . Then there exists a solution field M of $(L/k, \varepsilon)$ such that the completion of M by \mathfrak{p} is equal to $M_{\mathfrak{p}}$ for any \mathfrak{p} of S .*

For a finite set S of primes of k , let $B_k(S) = \{\alpha \in k^* \mid (\alpha) = \mathfrak{a}^{\mathfrak{p}}$ for some ideal \mathfrak{a} of k , and $\alpha \in k_{\mathfrak{q}}^{\mathfrak{p}}$ for any \mathfrak{q} of $S\}$. Then the following lemma is well-known.

Lemma 4. (Safarevic [2; Theorem 1]) *Assume that $B_k(S) = k^{*p}$. Let \mathfrak{q} be a prime of k , not contained in S . If $N(\mathfrak{q})$, the absolute norm of \mathfrak{q} , is congruent to 1 (mod. p), then there exists a cyclic extension $k(\mathfrak{q})/k$ of degree p which is unramified outside $S \cup \{\mathfrak{q}\}$, and in which \mathfrak{q} is ramified.*

REMARK. Let k be either the rational number field or an imaginary quadratic field with the class number prime to p ($p \neq 3$, when $k = \mathbf{Q}(\sqrt{-3})$). In this case, $B_k(\mathfrak{p}) = k^{*p}$, and hence $B_k(S) = k^{*p}$ for any S .

2. On unramified extensions

In this section, let p be an odd prime and let k denote either the rational number field or an imaginary quadratic field with the class number prime to p ($p \neq 3$, when $k = \mathbf{Q}(\sqrt{-3})$).

The following theorem is our main result.

Theorem 1. *Let $L/K/k$ be a Galois tower satisfying the following conditions.*

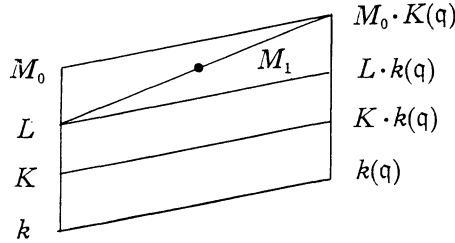
- (1) *The degree of K/k is prime to p .*
- (2) *L/K is an unramified p -extension.*
- (3) *For any prime \mathfrak{q} of k ramified in K/k , the inertia degree of \mathfrak{q} in K/k is equal to 1 or $L_{\mathfrak{q}}/k_{\mathfrak{q}}$ is cyclic.*

Put $G = \text{Gal}(L/k)$, and let $(\varepsilon): 1 \rightarrow \mathbf{Z}/p\mathbf{Z} \rightarrow E \rightarrow G \rightarrow 1$ be a non-split central extension. Then there exists a Galois extension M/k such that

- (i) *M/k gives a proper solution of the central imbedding problem $(L/k, \varepsilon)$,*
- (ii) *M/K is unramified.*

Proof. By Lemma 1 and Lemma 2, it is easy to see that $(L/k, \varepsilon)$ is solvable. Now, we consider the local problem $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$ for any prime \mathfrak{p} of k lying above p . By the remark of Lemma 2, as a solution field of $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$, we can take an

unramified extension M_p/K_p . By Lemma 3, there exists a solution field M_0/k of $(L/k, \varepsilon)$ such that its localization is equal to M_p for any prime \mathfrak{p} lying above p . Let $\psi: \mathfrak{G} \rightarrow E$ be a solution of $(L/k, \varepsilon)$ corresponding to the solution field M_0/k . Since ε is non-split, the central extension $\varepsilon(p): 1 \rightarrow \mathbf{Z}/p\mathbf{Z} \rightarrow E(p) \rightarrow G(p) \rightarrow 1$ of finite p -groups which is induced by ε , is also non-split. It is clear that the generator rank of $E(p)$ is equal to that of $G(p)$. Then the restriction $\psi|_{\mathfrak{G}(p)}: \mathfrak{G}(p) \rightarrow E(p)$ is surjective, and then $\psi: \mathfrak{G} \rightarrow E$ is surjective. Hence ψ is a proper solution of $(L/k, \varepsilon)$. By the choice of M_0/k , any prime of L lying above p is unramified in M_0/L . If M_0/L is unramified, then M_0/k is a required Galois extension. Suppose that M_0/L is not unramified, and take a prime \tilde{q} of M_0 ramified in M_0/L . Let q be a prime of k that is the restriction of \tilde{q} to k . We claim that $N(q) \equiv 1 \pmod{p}$. If q is ramified in K/k and the inertia degree in K/k is equal to 1, then $N(\tilde{q}) = N(q)^{p^r} \equiv 1 \pmod{p}$ for some integer r . Hence $N(\tilde{q}) \equiv 1 \pmod{p}$. Assume that q is not as above. We consider the extension $M_{0\tilde{q}}/k_q$ which is the localization of M_0/k with respect to \tilde{q} . Then $M_{0\tilde{q}}/k_q$ is abelian since $L_{\tilde{q}}/k_q$ is cyclic and $M_{0\tilde{q}}/k_q$ is a central extension of $L_{\tilde{q}}/k_q$. Thus \tilde{q} is ramified in a p -extension over k_q . Hence $N(q) \equiv 1 \pmod{p}$. This proves the claim. By Lemma 4, there exists a cyclic extension $k(q)/k$ of degree p which is unramified outside q and in which q is totally ramified. Then $k(q) \cap M_0 = k$ because q is unramified in L/K and the generator rank of $\text{Gal}(M_0/K)$ is equal to that of $\text{Gal}(L/K)$. Let q be an extension of q to $M_0 \cdot k(q)$, and let M_1 be the inertia field of q in $M_0 \cdot k(q)/L$.



Then M_1 is not equal to L , M_0 and $M_0 \cdot k(q)$ by the Hilbert theory of ramification. Since $\text{Gal}(M_0 \cdot k(q)/L)$ is contained in the center of $\text{Gal}(M_0 \cdot k(q)/k)$, M_1/k is a Galois extension. Moreover, $\text{Gal}(M_0/k)$ is isomorphic to $\text{Gal}(M_1/k)$ and M_1/k gives a proper solution of $(L/k, \varepsilon)$. By the choice of M_1 , any prime of L which is unramified in M_0/L is also unramified in M_1/L , and \tilde{q} is unramified in M_1/L . By continuing this procedure, we can take a required extension M/k . This proves the theorem.

3. An application to quadratic extensions

As in §2, let p be an odd prime, and let k be either the rational number field or an imaginary quadratic field with the class number prime to p ($p \neq 3$

when $k = \mathbf{Q}(\sqrt{-3})$). Let K be a quadratic extension over k . We first prove the following lemma.

Lemma 5. *Let K_1/K be an unramified cyclic extension of degree p , and \mathfrak{p} a prime of k lying above p . Then we have the following.*

- (1) K_1 is a Galois extension over k , and the Galois group is isomorphic to the group $\langle \sigma, \tau \mid \sigma^p = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$.
- (2) If \mathfrak{p} is ramified in K/k , the inertia degree of \mathfrak{p} in K_1/k is equal to 1.

Proof. (1) Suppose that K_1/k is non-Galois. Let K_2 be a conjugate field of K_1 over k , which is distinct from K_1 , and put $\text{Gal}(K_1 \cdot K_2/K_1) = \langle x \rangle$ and $\text{Gal}(K/k) = \langle y_0 \rangle$. Let y be an extension of y_0 to $K_1 \cdot K_2$. Then $\text{Gal}(K_1 \cdot K_2/K_2) = \langle xyx^{-1} \rangle$ and $\text{Gal}(K_1 \cdot K_2/k)$ is generated by x and y . The fixed field of $\langle yxy^{-1}x^{-1} \rangle$ is an abelian extension over k of degree $2p$. By considering the ramification index, we see that there exists an unramified cyclic extension over k of degree p . This is a contradiction. And it is easy to see that $\text{Gal}(K_1/k)$ is isomorphic to the group $\langle \sigma, \tau \mid \sigma^p = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (2) Let \mathfrak{p} be an extension of \mathfrak{p} to K , and put $G = \langle \sigma, \tau \mid \sigma^p = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. Suppose that \mathfrak{p} is also prime of K_1 . Then the inertia group of \mathfrak{p} in K_1/k is a normal subgroup of G , which is of order 2. But G has no normal subgroup of order 2. This is a contradiction.

Let H be a group of order p^3 defined by

$$\langle x, z \mid x^p = y^p = z^p = 1, z^{-1}xz = xy, y^{-1}xy = x, y^{-1}zy = z \rangle.$$

Let $Cl_K(p)$ be the p -Sylow subgroup of the ideal class group of K . Denote by $d_p Cl_K$, the p -rank of $Cl_K(p)$.

Then we have the following.

Theorem 2. *If $d_p Cl_K \geq 2$, then there exists an unramified Galois extension M/K with Galois group $\text{Gal}(M/K)$ isomorphic to H .*

Proof. Let $K_1/K, K_2/K$ be unramified cyclic extensions of degree p such that $K_1 \cap K_2 = K$. Then by Lemma 5, $K_1 \cdot K_2$ is a Galois extension over k , whose Galois group is isomorphic to the group

$$G := \langle u, v, w \mid u^p = v^p = w^p = 1, w^{-1}uw = u^{-1}, w^{-1}vw = v^{-1}, v^{-1}uv = u \rangle.$$

Now, we take a following group E of order $2p^3$,

$$E = \langle x, z, t \mid x^p = y^p = z^p = t^2 = 1, z^{-1}xz = xy, y^{-1}xy = x, y^{-1}zy = z, t^{-1}xt = x^{-1}, t^{-1}yt = y, t^{-1}zt = z^{-1} \rangle.$$

Then,

$$1 \rightarrow \langle y \rangle \rightarrow E \xrightarrow{j} G \rightarrow 1$$

is a non-split central extension, where j is defined by $x \rightarrow u, z \rightarrow v, t \rightarrow w$. By Theorem 1, there exists a Galois extension M/k such that $M/K_1 \cdot K_2$ is unramified

and that the Galois group $\text{Gal}(M/k)$ is isomorphic to E . Since the p -Sylow subgroup of E is isomorphic to H , $\text{Gal}(M/K)$ is isomorphic to H . This proves the theorem.

From the proof of Theorem 2, we have,

Corollary. *Assume that L/K is unramified extension with Galois group isomorphic to $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$. Then the class number of L is divisible by p .*

4. An application to cubic extensions

Let p be an odd prime which is congruent to $-1 \pmod{3}$, and k be the same as in §3. Let K/k be a cyclic extension of degree 3. In this section, we prove the following.

Theorem 3. *Assume that the class number of K is divisible by p . Then there exists an unramified Galois extension M/K with the Galois group isomorphic to H , where H is the same group as in §3.*

First we prove the following lemma.

Lemma 6. (1) *Let K_1/K be an unramified cyclic extension of degree p . Then K_1/k is non-Galois.* (2) *Let L/k be the Galois closure of K_1/k . Then $\text{Gal}(L/k)$ is isomorphic to the group*

$$G := \langle u, v, w \mid u^p = v^p = w^3 = 1, w^{-1}uw = v, w^{-1}vw = u^{-1}v^{-1}, u^{-1}vu = v \rangle.$$

Proof. (1) Assume that K_1/k is Galois. Since p is congruent to $-1 \pmod{3}$, the group of order $3p$ is abelian, so K_1/k is an abelian extension. Then it is easy to see that there exists an unramified cyclic extension of degree p . This is a contradiction. (2) The order of $\text{Gal}(L/k)$ is either $3p^2$ or $3p^3$. We notice that a p -Sylow subgroup of $\text{Gal}(L/k)$ is normal subgroup which is isomorphic to an elementary abelian p -group, and $\text{Gal}(L/k)$ does not have a normal subgroup of order 3. The group of order $3p^2$ or $3p^3$ with this property is isomorphic to G (see Western [3]).

Proof of Theorem 3. By Lemma 6, there exists a Galois extension $L/K/k$ such that L/K is unramified and $\text{Gal}(L/k)$ is isomorphic to G . Now, we take a following group E of order $3p^3$.

$$E = \left\langle x, z, t \mid \begin{array}{l} x^p = y^p = z^p = t^3 = 1, x^{-1}zx = yz, x^{-1}yx = y, y^{-1}zy = z \\ t^{-1}xt = z, t^{-1}zt = x^{-1}z^{-1}, y^{-1}ty = t \end{array} \right\rangle.$$

Then,

$$1 \rightarrow \langle y \rangle \rightarrow E \xrightarrow{j} G \rightarrow 1$$

is a non-split central extension, where j is defined by $x \rightarrow u, z \rightarrow v, t \rightarrow w$. In the

same manner of the proof of Theorem 2, we can take a required extension M/k .

5. On central imbedding problems of exponent p

Let p be an odd prime and L/k a Galois extension of an algebraic number field k with Galois group G . In this section, we assume that G is of exponent p (not necessarily abelian). Let S_0 be the set of primes of k which are ramified in L and prime to p .

We prove the following.

Theorem 4. *Assume that $B_k(S_0) = k^{*p}$, and that L_q/k_q is cyclic for any prime q of k . Let E be a p -group of exponent p and $(\varepsilon): 1 \rightarrow \mathbf{Z}/p\mathbf{Z} \rightarrow E \xrightarrow{j} G \rightarrow 1$ a non-split central extension. Then there exists a Galois extension M/k such that*

- (i) M/k gives a proper solution of the central imbedding problem $(L/k, \varepsilon)$,
- (ii) M/L is unramified.

Proof. For any prime q of k , ε_q is split by the assumption, so $(L_q/k_q, \varepsilon_q)$ is solvable. Then $(L/k, \varepsilon)$ is solvable. Now, we consider the local problem $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$ for any prime \mathfrak{p} of k lying above p . It is clear we can take $L_{\mathfrak{p}}/k_{\mathfrak{p}}$ as a solution field of $(L_{\mathfrak{p}}/k_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}})$. Then, by Lemma 3, there exists a Galois extension $M_1/L/k$ such that any prime \mathfrak{p} lying above p is unramified in M_1/L and that M_1/k gives a proper solution of $(L/k, \varepsilon)$. Let S_1 be the set of primes of k which are ramified in M_1 . Let q be a prime of S_1 not contained in $S_0 \cup \{p\}$. Then, in the same manner of the proof of Theorem 1, we can take a Galois extension M_2/k that is unramified outside $S_1 - \{q\}$, and can take a Galois extension M/k that is unramified outside $S_0 \cup \{p\}$.

Let q be a prime contained in S_0 and \tilde{q} a prime of L which is an extension of q . Since E is of exponent p , \tilde{q} is unramified in M by the Hilbert theory of ramification. Then M/L is unramified, therefore M/k is a required extension.

Let H be the same group as in §3. As a corollary of Theorem 4, we have

Corollary. *Let L/k be a Galois extension with the Galois group isomorphic to $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$. Assume that $B_k(S_0) = k^{*p}$. Then the following conditions (i) (ii) are equivalent.*

- (i) *There exists a Galois extension $M/L/k$ such that $\text{Gal}(M/k)$ is isomorphic to H and that M/L is unramified.*
- (ii) *Any prime of k which is ramified in L/k is decomposed in L/k .*

Proof. (ii) \rightarrow (i) is clear by Theorem 4. We prove (i) \rightarrow (ii). Assume that there exists a prime q of k which is ramified in L/k and is not decomposed in L/k . Let \tilde{q} be a prime of M which is an extension of q . Since H is of exponent p , the decomposition group of \tilde{q} in M/k is of order p^2 . Then the de-

composition group is normal subgroup of H , so the decomposition field is a cyclic extension over k of degree p . Therefore it is contained in L/k . This is a contradiction.

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