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Osaka University
ON AUSLANDER-REITEN COMPONENTS FOR CERTAIN GROUP MODULES

SHIGETO KAWATA

(Received January 7, 1992)

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$. Let $\Gamma_i(kG)$ be the stable Auslander-Reiten quiver of the group algebra $kG$. By Webb's theorem, the tree class of a connected component $\Delta$ of $\Gamma_i(kG)$ is a Euclidean diagram, a Dynkin diagram or one of the infinite trees $A_\infty, B_\infty, C_\infty, D_\infty,$ or $A_\infty$. Moreover if $\Delta$ contains the trivial $kG$-module $k$, then the graph structure of $\Delta$ has been investigated (see [21], [16] and [17]). In this paper we study a connected component of $\Gamma_i(kG)$ containing an indecomposable $kG$-module whose $k$-dimension is not divisible by $p$. Suppose that $M$ is an indecomposable $kG$-module and $p \nmid \dim_k M$. In Section 2, we will show that $M$ lies in a connected component isomorphic to $ZA_\infty$ if $k$ is algebraically closed and a Sylow $p$-subgroup of $G$ is not cyclic, dihedral, semidihedral or generalized quaternion. In Section 3 we make some remarks on tensoring the component containing the trivial $kG$-module $k$ with $M$. In Sections 4 and 5 we consider the situation where $p = 2$ and a Sylow 2-subgroup of $G$ is dihedral of order at least 8 or semidihedral.

The notation is almost standard. All modules considered here are finite dimensional over $k$. We write $W \approx W' (\text{mod projectives})$ for $kG$-modules $W$ and $W'$ if the projective-free part of $W$ is isomorphic to that of $W'$. For an indecomposable non-projective $kG$-module $W$, we write $A(W)$ to denote the Auslander-Reiten sequence ($AR$-sequence) $0 \to \Omega^2 W \to m(W) \to W \to 0$ terminating at $W$, where $\Omega$ is the Heller operator, and we write $m(W)$ to denote the middle term of $A(W)$. If an exact sequence of $kG$-modules $S$ is of the form $0 \to \Omega^2 W \oplus U' \to m(W) \oplus U \oplus U' \to W \oplus U \to 0$, where $W$ is an indecomposable non-projective $kG$-module, and $U, U'$ are projective or 0, we say that $S$ is the $AR$-sequence $A(W)$ modulo projectives. The symbol $\otimes$ denotes the tensor product over the coefficient field $k$. For an exact sequence of $kG$-modules $S: 0 \to A \to B \to C \to 0$ and a $kG$-module $W$, we write $S \otimes W$ to denote the tensor sequence $0 \to A \otimes W \to B \otimes W \to C \otimes W \to 0$. Concerning some basic facts and terminologies used here, we refer to [2], [10] and [11].
1. Preliminaries

We start by summarizing results on the graph structure of connected components of $\Gamma_{s}(kG)$.

**Theorem 1.1** ([21], [17], [5], [9]). Let $\Delta$ be a connected component of $\Gamma_{s}(kG)$. Then the tree class of $\Delta$ is $A_{n}$, $A_{l,2}$, $B_{3}$, $A_{o}$, $B_{o}$, $C_{o}$, $D_{o}$ or $A_{\sim}$. If $k$ is algebraically closed, then the tree class is not $B_{3}$, $B_{o}$ or $C_{o}$. Moreover if the tree class or the reduced graph of $\Delta$ is Euclidean, then the modules in $\Delta$ lie in a block whose defect group is a Klein four group.

**Theorem 1.2** ([21], [16], [17], [7]). Let $\Delta_{0}$ be the connected component containing the trivial $kG$-module $k$, and let $P$ be a Sylow $p$-subgroup of $G$. Then;

1. If $P$ is not cyclic, dihedral, semidihedral or generalized quaternion, then $\Delta_{0} \cong ZA_{\sim}$ and $k$ lies at the end of $\Delta_{0}$.
2. If $P$ is a dihedral 2-group of order at least 8, then $\Delta_{0} \cong ZA_{\sim}$.
3. If $P$ is a semidihedral 2-group, then $\Delta_{0} \cong ZD_{o}$ and $k$ lies at the end of $\Delta_{0}$.
4. If $P$ is a generalized quaternion 2-group, then $\Delta_{0}$ is a 2-tube.

We will need the following result on tensoring the AR-sequence by Auslander and Carlson [1].

**Theorem 1.3** ([1], see also [3]). Assume that $k$ is algebraically closed. Let $\mathcal{A}(k) : 0 \rightarrow \Omega k \rightarrow m(k) \rightarrow k \rightarrow 0$ be the AR-sequence terminating at the trivial $kG$-module $k$. Let $M$ be an indecomposable $kG$-module. Then the tensor sequence $\mathcal{A}(k) \otimes M : 0 \rightarrow \Omega k \otimes M \rightarrow m(k) \otimes M \rightarrow M \rightarrow 0$ has the following properties.

1. If $p \nmid \dim_{k} M$, the tensor sequence $\mathcal{A}(k) \otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives.
2. If $p | \dim_{k} M$, then the tensor sequence $\mathcal{A}(k) \otimes M$ is split.

Concerning tensor products, we will also need the following result by Benson and Carlson [3].

**Theorem 1.4** ([3], see also [1]). Assume that $k$ is algebraically closed. Let $M$ and $N$ be indecomposable $kG$-modules. Then;

1. The following are equivalent.
   a. $k | M \otimes N$.
   b. $p \nmid \dim_{k} M$ and $N \cong M^{*}$. Here $M^{*} = \text{Hom}_{k}(M, k)$ is the dual of $M$.
      Moreover if $p \nmid \dim_{k} M$, then the multiplicity of $k$ in $M \otimes M^{*}$ is one.
2. Suppose that $p | \dim_{k} M$. Then for any indecomposable direct summand $U$ of $M \otimes N$, we have $p \mid \dim_{k} U$.

As an immediate consequence of Theorem 1.3, we have;

**Lemma 1.5.** Assume that $k$ is algebraically closed. Let $M$ be an indecom-
posable $kG$-module with $p \neq \dim_k M$ and $\mathcal{A}(M) : 0 \to \Omega^2 M \to m(M) \to M \to 0$ be the AR-sequence terminating at $M$. Let $W$ be a $kG$-module, and let $M \otimes W = (\oplus_i M_i) \oplus (\oplus_j N_j) \oplus U$, where $M_i$ and $N_j$ are non-projective indecomposable $kG$-modules (possibly 0) such that $p \neq \dim_k M_i$ and $p \mid \dim_k N_j$, and $U$ is projective or 0. Then the tensor sequence $\mathcal{A}(M) \otimes W : 0 \to \Omega^2 M \otimes W \to m(M) \otimes W \to M \otimes W \to 0$ is a direct sum $\bigoplus_i \mathcal{A}(M_i)$ of the AR-sequences $\mathcal{A}(M_i)$ plus a split sequence $0 \to \left( \oplus_j \Omega^2 N_j \right) \oplus U' \to \left( \oplus_j \Omega^2 N_j \right) \oplus U \oplus U' \to \left( \oplus_j N_j \right) \oplus U \to 0$, where $U$ and $U'$ are projective or 0.

Let $( , )$ denote the inner product of the Green ring $a(kG)$ induced from $\dim \text{Hom}( , )$ (see [4]). For an exact sequence of $kG$-modules $\mathcal{S} : 0 \to A \to B \to C \to 0$, let $[\mathcal{S}] \in a(kG)$ be the element $[\mathcal{S}] = B - A - C$. Using the results of Benson and Parker [4, Section 3], we have the following two lemmas.

**Lemma 1.6.** Assume that $k$ is an algebraically closed field. Let $M$ be a non-projective indecomposable $kG$-module and $H$ a subgroup of $G$. Suppose that exactly $n$ non-isomorphic indecomposable $kH$-modules $L_i (i=1, 2, \ldots, n)$ satisfy $M \mid L_i \uparrow G$. Let $t_i$ be the multiplicity of $M$ in $L_i \uparrow G$. Then $[\mathcal{A}(M) \downarrow_H] = \Sigma_{i=1}^n t_i [\mathcal{A}(L_i)]$ as elements of the Green ring $a(kH)$. ($n$ may be zero, and in this case, the right hand side of the above is understood to be zero.) In particular we have:

1. Let $Q$ be a vertex of $M$ and $S$ a $Q$-source of $M$. Let $N = N_G(Q)$ and $T = \{ g \in G \mid S^g \simeq S \}$. Let $t$ be the multiplicity of $M$ in $S \uparrow G$. Then $[\mathcal{A}(M) \downarrow Q] = t (\Sigma_{g \in G/T} [\mathcal{A}(S^g)])$.

2. Suppose that $H$ is a normal subgroup of $G$ and $M$ is $H$-projective. Let $S$ be an $H$-source of $M$. Let $T = \{ g \in G \mid S^g \simeq S \}$ and $t$ the multiplicity of $M$ in $S \uparrow G$. Then $[\mathcal{A}(M) \downarrow H] = t (\Sigma_{g \in G/T} [\mathcal{A}(S^g)])$.

3. ([2, Proposition 2.17.10]) The AR-sequence $\mathcal{A}(M)$ splits on restriction to $H$ if and only if $M$ is not $H$-projective.

Proof. By [4, Theorem 3.4], it suffices to show that $(V, [\mathcal{A}(M) \downarrow_H] - \Sigma_{i=1}^n t_i [\mathcal{A}(L_i)]) = 0$ for any indecomposable $kH$-module $V$. Using the Frobenius reciprocity, we have $(V, [\mathcal{A}(M) \downarrow H] - \Sigma_{i=1}^n t_i [\mathcal{A}(L_i)]) = (V, [\mathcal{A}(M) \downarrow H]) - (V, \Sigma_{i=1}^n t_i [\mathcal{A}(L_i)]) = (V \uparrow G, [\mathcal{A}(M)]) - \Sigma_{i=1}^n t_i (V, [\mathcal{A}(L_i)])$. Now $M \mid V \uparrow G$ if and only if $V$ is isomorphic to some $L_i$. Since $k$ is algebraically closed, we have $(V \uparrow G, [\mathcal{A}(M)]) = t_i$ in this case, and hence $(V, [\mathcal{A}(M) \downarrow H] - \Sigma_{i=1}^n t_i [\mathcal{A}(L_i)]) = 0$ as desired.

**Lemma 1.7.** Let $M$ be a non-projective indecomposable $kG$-module. Let $\mathcal{E} : 0 \to \Omega^2 M \to X \to M \to 0$ be an exact sequence. Then:

1. $\mathcal{E}$ is the AR-sequence $\mathcal{A}(M)$ if and only if $(M, [\mathcal{E}]) = d_M$. Here $d_M = \dim_k (\text{End}_{kG}(M) / \text{Rad}(\text{End}_{kG}(M)))$.

2. $\mathcal{E}$ is the AR-sequence $\mathcal{A}(M)$ if and only if $\mathcal{E}$ does not split and $(m(M),$
Proof. (1) Suppose that \( \mathcal{E} \) is the AR-sequence. Then by [2, 2.18.4 Theorem] we have \( (M, [\mathcal{E}]) = d_M \). To show the converse assume by way of contradiction that \( (M, [\mathcal{E}]) = d_M \) but \( \mathcal{E} \) is not the AR-sequence \( \mathcal{A}(M) \). Now the exact sequence \( \mathcal{E} \) does not split since \( (M, [\mathcal{E}]) > 0 \). Letting \( \mathcal{A}(M): 0 \to \Omega^2 M \to m(M) \to M \to 0 \) be the AR-sequence terminating at \( M \), we have the following commutative diagram.

\[
0 \to \Omega^2 M \to X \to M \to 0
\]

Since the left-hand square is a pushout diagram, we get an exact sequence \( \mathcal{E}': 0 \to \Omega^2 M \to X \oplus \Omega^2 M \to m(M) \to 0 \). Since \( \mathcal{E}' \) is not the AR-sequence \( \mathcal{A}(M) \), \( \mathcal{E}' \) does not split: if \( \mathcal{E}' \) is a split sequence, then \( X \) is isomorphic to \( m(M) \) but this implies that \( \mathcal{E} \) is the AR-sequence \( \mathcal{A}(M) \), a contradiction. Thus we also have the following commutative diagram.

\[
0 \to \Omega^2 M \to m(M) \to M \to 0
\]

Since the right-hand square is a pullback diagram, we get an exact sequence \( \mathcal{E}'' : 0 \to m(M) \to X \oplus \Omega^2 M \to m(M) \to 0 \). Thus we get \( [\mathcal{E}] = \mathcal{A}(M) + [\mathcal{E}'] \mathcal{A}(M) + [\mathcal{E}'']. \) Hence we have \( (M, [\mathcal{E}]) = (M, \mathcal{A}(M)) + (M, [\mathcal{E}]) = 2d_M + (M, [\mathcal{E}'']) > d_M, \) a contradiction.

(2) Suppose that \( \mathcal{E} \) is the AR-sequence. Then by [2, 2.18.4 Theorem] we have \( (m(M), [\mathcal{E}]) = 0 \) since \( M \not\cong m(M) \). Conversely suppose that \( \mathcal{E} \) does not split and \( (m(M), [\mathcal{E}]) = 0 \). Let \( [\mathcal{E}'] \) be as in the proof of (1). Since \( [\mathcal{E}] = \mathcal{A}(M) + [\mathcal{E}] \mathcal{A}(M) + [\mathcal{E}'] \mathcal{A}(M) + [\mathcal{E}'']. \) Hence we have \( (M, [\mathcal{E}']) = (M, \mathcal{A}(M)) + (M, [\mathcal{E}']) = 0, \) which implies that \( \mathcal{E}' \) splits. Thus \( X \) is isomorphic to \( m(M) \), and hence \( \mathcal{E} \) is the AR-sequence \( \mathcal{A}(M) \).

Remark. If \( k \) is algebraically closed, then \( d_M = 1 \) for any indecomposable \( kG \)-module \( M \).

The following two lemmas are useful for our investigation.

**Lemma 1.8.** Let \( \Delta \) be a connected component of \( \Gamma_s(kG) \). Suppose that the tree class of \( \Delta \) is \( A_\infty \). Let \( T: M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_s \leftarrow \cdots \) be a tree in \( \Delta \) such that \( \Delta \cong \mathbb{Z}T/\Pi \) for some admissible group of automorphisms \( \Pi \subseteq \text{Aut} \mathbb{Z}T \). Then \( \dim_k M_s \equiv n(\dim_k M_1) \mod p \) for all \( n \geq 1 \).

Proof. We proceed by induction on \( n \). Clearly \( \dim_k M_1 = 1 \times \dim_k M_1 \) and
dim_k\Omega^2 M_i \equiv \dim_k M_i \pmod{p}$. Since the AR-sequence $\mathcal{A}(M_i)$ is of the form $0 \rightarrow \Omega^2 M_i \rightarrow M_2 \oplus U \rightarrow M_1 \rightarrow 0$, where $U$ is projective or 0, we have $\dim_k M_2 \equiv 2(\dim_k M_i) \pmod{p}$.

Suppose then that $\dim_k M_i \equiv \dim_k \Omega^2 M_i \equiv i(\dim_k M_i) \pmod{p}$ for all $i$ with $1 \leq i \leq n-1$. Now we have the AR-sequence $\mathcal{A}(M_{n-1}): 0 \rightarrow \Omega^2 M_{n-1} \rightarrow \Omega^2 M_{n-2} \oplus M_2 \oplus U \rightarrow M_{n-1} \rightarrow 0$, where $U$ is projective or 0. Therefore $\dim_k M_{n-1} = \dim_k \Omega^2 M_{n-2} - \dim_k \Omega^2 M_{n-2} \equiv n(\dim_k M_i) \pmod{p}$.

**Lemma 1.9.** Let $\Theta$ be a connected component of $\Gamma(kG)$.

(1) If the tree class of $\Theta$ is $A^\infty$, then $\dim_k M \equiv \dim_k M' \pmod{p}$ for all indecomposable $kG$-modules $M$ and $M'$ in $\Theta$.

(2) Suppose that the tree class of $\Theta$ is $Z\lambda_0$. Let $M^\downarrow \rightarrow \Omega \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \rightarrow M' \downarrow \rightarrow$ be a tree in $\Theta$ with $\Theta \cong ZT$. Then $\dim_k M \equiv \dim_k M' \pmod{p}$ and $\dim_k M_n \equiv 2(\dim_k M) \pmod{p}$ for all $n \geq 2$.

**Proof.** Let $x$ be an element of $G$ of order $p$ and let $H = \langle x \rangle$. Then the group algebra $kH$ has only $p$ non-isomorphic indecomposable modules, say $V_1, V_2, \ldots, V_{p-1}$ and $V_p$, where $\dim_k V_t = t$ ($1 \leq t \leq p$) and $V_p$ is projective. For a $kG$-module $M$, let $a(t, M)$ be the multiplicity of $V_t$ in $M^\downarrow$.

(1) We show that $a(t, M) = a(t, M')$ for any indecomposable $kG$-modules $M$ and $M'$ in $\Theta$ and $1 \leq t \leq p-1$. Let $a_i$ be the smallest integer in $\{a(t, M) \mid M \in \Theta\}$ and let $M_i$ be a $kG$-module in $\Theta$ such that $a(t, M_i) = a_i$. Let $T: \cdots \rightarrow W_n \rightarrow \cdots \rightarrow W_2 \rightarrow W_1 \rightarrow W_0$ be a tree in $\Theta$ such that $\Theta \cong ZT/\Pi$ for some admissible group of automorphisms $\Pi \subseteq \text{Aut} ZT$. Then we have the AR-sequence $\mathcal{A}(M_i): 0 \rightarrow \Omega^2 M_i \rightarrow \Omega^2 M_2 \oplus M_1 \oplus U \rightarrow M_0 \rightarrow 0$, where $U$ is projective or 0. Since the connected component containing $M_i$ is not a tube, $M_i$ is not periodic and in particular $M_i$ is not $H$-projective. Thus $\mathcal{A}(M_i)$ splits on restriction to $H$ by Lemma 1.6(3) and it follows that $\Omega^2 M_i \oplus M_2 \oplus U \cong M_1 \oplus \Omega^2 M_2$. This implies that $a(t, W_2) = a(t, M_2) = a_i$. Since $a(t, W_2) \geq a_i$, we proceed inductively, we obtain $a(t, M_n) = a(t, W_n) = a_i$ for all $n \geq 2$ and all $t$ with $1 \leq t \leq p-1$. Thus the result follows.

(2) Since the tree class of $\Theta$ is $D_\infty$, all indecomposable modules in $\Theta$ are not $H$-projective. Hence for any indecomposable $kG$-module $M$ in $\Theta$, the AR-sequence $\mathcal{A}(M)$ splits on restriction to $H$ by Lemma 1.6(3). We have the AR-sequences $\mathcal{A}(M): 0 \rightarrow \Omega^2 M \rightarrow M_2 \oplus U \rightarrow M_1 \rightarrow 0$ and $\mathcal{A}(M'): 0 \rightarrow \Omega^2 M' \rightarrow M_2 \oplus U' \rightarrow M' \rightarrow 0$, where $U$ and $U'$ are projective or 0. Since both $\mathcal{A}(M)$ and $\mathcal{A}(M')$ split on restriction to $H$, we have $\Omega^2 M \oplus M_2 \oplus U \cong M_1 \oplus \Omega^2 M_2$ and $\Omega^2 M' \oplus M' \oplus U' \cong M_2 \oplus U' \oplus \Omega^2 M_2$. Thus we get $a(t, M_2) = 2a(t, M) = 2a(t, M')$ for $1 \leq t \leq p-1$. 


Next we show that \( a(t, M_n) = 2a(t, M) \) for \( 1 \leq t \leq p - 1 \) and all \( n \geq 2 \) by induction on \( n \). We have the AR-sequence \( \mathcal{A}(M_2): 0 \to \Omega^2 M_2 \to M_3 \oplus \Omega^2 M' \oplus U_2 \to M_4 \to 0 \), where \( U_2 \) is projective or 0. Since \( \mathcal{A}(M_2) \) splits on restriction to \( S \), we get \( a(t, M_3) = a(t, M_2) + a(t, \Omega^2 M_2) - a(t, \Omega^2 M') = a(t, M_2) \) for \( 1 \leq t \leq p - 1 \). Suppose then that \( a(t, M_i) = a(t, M_2) \) for all \( i \) with \( 2 \leq i \leq n - 1 \). We have the AR-sequence \( \mathcal{A}(M_{n-1}): 0 \to \Omega^2 M_{n-1} \to \Omega^2 M_{n-2} \oplus M_n \oplus U'' \to M_{n-1} \to 0 \), where \( U'' \) is projective or 0. As \( \mathcal{A}(M_{n-1}) \) splits on restriction to \( S \), we get \( a(t, M_n) = a(t, M_{n-1}) \) for \( 1 \leq t \leq p - 1 \). Hence the result follows.

In the rest of this section, we consider the following situation.

(*) Assume that \( k \) is an algebraically closed field of characteristic \( p > 0 \) and a Sylow \( p \)-subgroup \( P \) of \( G \) is normal. Let \( \Xi \) be a connected component of \( \Gamma(kP) \). Assume that every module in \( \Xi \) is \( G \)-invariant. Assume furthermore that \( \Xi \) is not a tube and every arrow in \( \Xi \) is multiplicity free. Let \( S \) be an indecomposable \( kP \)-module in \( \Xi \) and \( M \) an indecomposable \( kG \)-module having \( S \) as a \( P \)-source. Let \( \Theta \) be the connected component of \( \Gamma(kG) \) containing \( M \).

**Remark.** The assumption (*) implies that \( P \) is not a Klein four group and \( \Xi \) is isomorphic to \( ZA_1, ZD_5 \) or \( ZA_6^* \).

**Lemma 1.10.** Assume (*). Then all the \( P \)-sources of the indecomposable modules in \( \Theta \) lie in \( \Xi \).

**Proof.** Let \( W \) be an indecomposable \( kG \)-module in \( \Theta \). Then there is a sequence of indecomposable \( kG \)-modules \( M = M_1, M_2, \ldots, M_n = W \) such that \( M_i \) and \( M_{i+1} \) are connected by an irreducible map \( (1 \leq i \leq n - 1) \). We proceed by induction on \( n \).

By the assumption, a \( P \)-source \( S \) of \( M = M_1 \) lies in \( \Xi \). Suppose then that a \( P \)-source \( S_{n-1} \) of \( M_{n-1} \) lies in \( \Xi \). Now \( M_n \) is \( \Omega^2 M_{n-1} \) or \( \Omega^{-2} M_{n-1} \), where \( m(M_{n-1}) \) \( (\text{resp. } m(\Omega^{-2} M_{n-1}) \) is the middle term of the AR-sequence \( \mathcal{A}(M_{n-1}) \) \( (\text{resp. } \mathcal{A}(\Omega^{-2} M_{n-1}) \). By Lemma 1.6 (2), we have \( [\mathcal{A}(M_{n-1})] = t[\mathcal{A}(S_{n-1})] \) and \( [\mathcal{A}(\Omega^{-2} M_{n-1})] = t[\mathcal{A}(\Omega^{-2} S_{n-1})] \), where \( t \) is the multiplicity of \( M_{n-1} \) in \( S_{n-1} \). This implies that a \( P \)-source of \( M_n = W \) lies in \( \Xi \).

For an indecomposable \( kG \)-module \( W \) in \( \Theta \), let \( \phi W \) be a (unique) \( P \)-source of \( W \). The following fact is an immediate consequence of the result of Uno [20, Section 3].

**Lemma 1.11.** Assume (*). Then \( \phi \) induces a graph isomorphism from \( \Theta \) onto \( \Xi \).

**Proof.** By [20, Theorem 3.5], the multiplicity of \( S \) in \( M \downarrow P \) is equal to
that of $M$ in $S^\dagger$. From Lemma 1.10 and [20, Theorem 3.7], we get the result.

2. $ZA_w$-Components

In this section we consider a connected component of $\Gamma_s(kG)$ containing an indecomposable $kG$-module whose $k$-dimension is not divisible by $p$ under the following hypothesis:

(4) $k$ is an algebraically closed field of characteristic $p>0$ and a Sylow $p$-subgroup $P$ of $G$ is not cyclic, dihedral, semidihedral or generalized quaternion.

Theorem 2.1. Assume (4). Suppose that $\Theta$ is a connected component of $\Gamma_s(kG)$ and $\Theta$ contains an indecomposable $kG$-module whose $k$-dimension is not divisible by $p$. Then $\Theta$ is isomorphic to $ZA_w$.

Proof. The tree class of $\Theta$ is $A_m$, $D_m$ or $A_\infty$ by Theorem 1.1.

Step 1. The tree class of $\Theta$ is not $A_\infty$.

Proof. We shall derive a contradiction assuming that the tree class of $\Theta$ is $A_\infty$. Let $T : \cdots \rightarrow W_n \rightarrow \cdots \rightarrow W_2 \rightarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$ be a tree in $\Theta$ with $\Theta \cong ZT$. Note that $p \nmid \dim_k M$, $p \nmid \dim_k M_n$ and $p \nmid \dim_k W_n$ for all $n \geq 2$ from Lemma 1.9(1). On the other hand the connected component $\Delta_0$ containing $k$ is isomorphic to $ZA_w$ by Theorem 1.2. Let $T_0 : k = L_1 \leftarrow L_2 \leftarrow \cdots \leftarrow L_n \leftarrow \cdots$ be a tree in $\Delta_0$ with $\Delta_0 \cong ZT_0$. Let $\mathcal{A}(k) : 0 \rightarrow \Omega^2 k \rightarrow L_2 \oplus U \rightarrow k \rightarrow 0$ be the AR-sequence terminating at $k$, where $U$ is projective or 0. Then the tensor sequence $\mathcal{A}(k) \otimes M : 0 \rightarrow \Omega^2 k \otimes M \rightarrow (L_2 \oplus U) \otimes M \rightarrow M \rightarrow 0$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives by Theorem 1.3. Hence it follows that $L_2 \otimes M \cong M_2 \oplus W_2$ (mod projectives).

In case $p=2$, this is a contradiction, since $2|\dim_k L_2$ by Lemma 1.8 and thus $L_2 \otimes M$ does not have any odd dimensional indecomposable direct summand from Theorem 1.4(2).

In case $p>2$, applying Lemma 1.5, we have the tensor sequence $\mathcal{A}(L_2) \otimes M : 0 \rightarrow \Omega^2 L_2 \otimes M \rightarrow (\Omega^2 k \oplus L_3) \otimes M \rightarrow L_2 \otimes M \rightarrow 0$, which is a direct sum $\mathcal{A}(M_2) \oplus \mathcal{A}(W_2)$ modulo projectives, as $p \nmid \dim_k L_2$, $p \nmid \dim_k M_2$ and $p \nmid \dim_k W_2$. Hence we have $L_3 \otimes M \cong M_3 \oplus W_3 \oplus \Omega^2 M$ (mod projectives). Repeating this argument until $n=p$, we have $\mathcal{A}(L_{p-1}) \otimes M$ is a direct sum of the AR-sequences modulo projectives and $M_{n} \oplus W_{n} \mid L_{n} \otimes M$ for $n \leq p$. In particular we obtain $M_{p} \oplus W_{p} \mid L_{p} \otimes M$. But this is also a contradiction, since $p|\dim_k L_p$ from Lemma 1.8 and thus $L_p \otimes M$ has no indecomposable direct summand whose $k$-dimension is not divisible by $p$ from Theorem 1.4(2).

Step 2. The tree class of $\Theta$ is not $D_m$.

Proof. Assume contrary that the tree class of $\Theta$ is $D_m$. Let
Let \( T: M \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots \) be a tree in \( \Theta \) with \( \Theta \in \text{ZT} \).

Note that \( p \not| \dim_k M \) and \( p \not| \dim_k W \) from Lemma 1.9(2). Let \( \mathcal{A}(k): 0 \rightarrow \Omega^2 k \rightarrow m(k) \rightarrow k \rightarrow 0 \) be the AR-sequence terminating at \( k \). By Theorem 1.3 the tensor sequences \( \mathcal{A}(k) \otimes M \) and \( \mathcal{A}(k) \otimes W \) are the AR-sequences \( \mathcal{A}(M) \) modulo projectives and \( \mathcal{A}(W) \) modulo projectives respectively. Hence we have \( M_2 \cong m(k) \otimes M \cong m(k) \otimes W \) (mod projectives). Thus \( m(k) \otimes M \otimes M^* \cong m(k) \otimes W \otimes M^* \) (mod projectives). Note that \( m(k) \otimes M \otimes M^* \) and \( m(k) \otimes W \otimes M^* \) are the middle terms of the tensor sequences \( \mathcal{A}(k) \otimes M \otimes M^* \) and \( \mathcal{A}(k) \otimes W \otimes M^* \) respectively.

Let \( M \otimes M^* = k \oplus (\oplus_i L_i) \oplus (\oplus_j L'_j) \oplus N \), where \( L_i \) is an indecomposable \( kG \)-module lying in \( \Delta_0 \) such that \( p \not| \dim_k L_i \) and \( L'_i \) is an indecomposable \( kG \)-module lying in \( \Delta_0 \) such that \( p | \dim_k L'_i \) and \( N \) has no indecomposable direct summand lying in \( \Delta_0 \). Since the multiplicity of \( k \) in \( M \otimes M^* \) is one, \( L_i \) is not isomorphic to \( k \). By Lemma 1.5, we have \( m(k) \otimes M \otimes M^* \cong m(k) \oplus (\oplus_i m(L_i)) \oplus (\oplus_j (\Omega L'_j \oplus L'_j)) \oplus N' \) for some \( kG \)-module \( N' \). Note that \( N' \) does not have any indecomposable direct summand lying in \( \Delta_0 \). Therefore the number of indecomposable direct summands of \( m(k) \otimes M \otimes M^* \) lying in \( \Delta_0 \) is odd. On the other hand \( k \) is not a direct summand of \( W \otimes M^* \). Therefore the number of indecomposable direct summands of \( m(k) \otimes W \otimes M^* \) lying in \( \Delta_0 \) is even, a contradiction.

By Steps 1 and 2, the tree class of \( \Theta \) is \( A_\infty \). Since a Sylow \( p \)-subgroup \( P \) of \( G \) is not generalized quaternion, indecomposable \( kG \)-modules whose \( k \)-dimension is not divisible by \( p \) are not periodic. Hence \( \Theta \) is isomorphic to \( ZA_\infty \).

**Lemma 2.2.** Assume \((\#)\). Suppose that \( \Theta \) is a connected component of \( \Gamma(kG) \) and \( \Theta \) contains an indecomposable \( kG \)-module whose \( k \)-dimension is not divisible by \( p \). Then all modules in \( \Theta \) have the same vertex \( P \).

**Proof.** By Theorem 2.1, \( \Theta \) is isomorphic to \( ZA_\infty \). Let \( M_1 \) be an indecomposable \( kG \)-module lying at the end of \( \Theta \). Then Lemma 1.8 implies that \( p \not| \dim_k M_1 \). Hence a Sylow \( p \)-subgroup \( P \) of \( G \) is a vertex of \( M_1 \) and the result follows from [20, Theorem 4.3].

Let \( M \) be an indecomposable \( kG \)-module having a Sylow \( p \)-subgroup \( P \) of \( G \) as vertex, and let \( S \) be a \( P \)-source of \( M \). Then \( p \not| \dim_k M \) if and only if \( p \not| \dim_k S \) from [3, Proposition 2.4].

**Proposition 2.3.** Assume \((\#)\). Suppose that \( \Theta \) is a connected component of \( \Gamma(kG) \) containing an indecomposable \( kG \)-module whose \( k \)-dimension is not divisible by \( p \), and let \( T: M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots \) be a tree in \( \Theta \) with \( \Theta \in \text{ZT} \). Let \( S_1 \) be a \( P \)-source of \( M_1 \) and \( \Xi \) the connected component of \( \Gamma(kP) \) containing \( S_1 \). Then we
have P-source $S_n$ of $M_n$ ($n \geq 1$) and a tree $T'$: $S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_n \leftarrow \cdots$ with $\Xi \cong \mathbb{Z}T$.

Proof. Lemma 1.8 implies that $p \not| \dim M_n$, and thus by the remark preceding Proposition 2.3 we have $p \not| \dim S_1$. Hence both $\Theta$ and $\Xi$ are isomorphic to $ZA_\infty$ by Theorem 2.1.

Step 1. We may assume that $P$ is a normal subgroup of $G$.

Proof. Let $N = N^G(P)$ and $f$ the Green correspondence with respect to $(G, P, N)$. Let $\Theta'$ be the connected component of $\Gamma_\infty(kN)$ containing $fM$. Since $p \not| \dim fM$, $\Theta'$ is isomorphic to $ZA_\infty$ and all modules in $\Theta'$ have the same vertex $P$ by Theorem 2.1 and Lemma 2.2. Therefore $f$ induces a graph isomorphism between $\Theta$ and $\Theta'$ by [13, Theorem].

Step 2. We may assume that every module in $\Xi$ is $G$-invariant.

Proof. Let $H = \{g \in G \mid W^g \in \Xi \text{ for all } W \in \Xi\}$ be the inertia group of $\Xi$. Since $\Xi \cong ZA_\infty$, $H$ acts on $\Xi$ trivially. Hence $H$ is the inertia group of $S_1$ and all modules in $\Xi$ are $H$-invariant.

Suppose that $S_1^H - R_1 \oplus R_2 \oplus \cdots \oplus R_n$ is an indecomposable direct sum decomposition such that $R_1^G = M_1$ (Note that each $R_i^G$ is indecomposable by [12, VII. 9.6 Theorem]). Let $\Theta''$ be the connected component of $\Gamma_\infty(kH)$ containing $R_1$. Then the inducing from $H$ to $G$ gives a graph isomorphism from $\Theta''$ onto $\Theta$ by [14, Theorem].

Now we may assume that $P$ is normal and every module in $\Xi$ is $G$-invariant. Hence we can apply Lemma 1.11 and the conclusion holds.

As an immediate consequence of Proposition 2.3, we have;

**Corollary 2.4.** Assume (¶). Let $M$ be an indecomposable $kG$-module whose $k$-dimension is not divisible by $p$, and let $S$ be a $P$-source of $M$. Then $M$ lies at the end of a $ZA_\infty$-component if and only if $S$ lies at the end of a $ZA_\infty$-component.

In the rest of this section, we give examples of indecomposable $kG$-modules lying at the end of a $ZA_\infty$-component.

**Lemma 2.5.** Suppose that $\Theta$ is a connected component isomorphic to $ZA_\infty$. Let $T$: $M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$ be a tree in $\Theta$ with $\Theta \cong \mathbb{Z}T$. Suppose that all modules in $\Theta$ have the same vertex $P$. Let $Q$ be a proper subgroup of $P$, and let $N$ be the projective-free part of $M_1 \downarrow Q$. Then $M_n \downarrow Q = \bigoplus_{i=0}^{n-1} \Omega^i N \mod projectives$ for all $n \geq 1$.

Proof. We proceed by induction on $n$. Clearly $M_1 \downarrow Q = N$ (mod projectives) and $\Omega^2 M_1 \downarrow Q = \Omega^2 N$ (mod projectives). Now the AR-sequence $\mathcal{A}(M_1)$ is of the form $0 \rightarrow \Omega M_1 \rightarrow M_2 \oplus U \rightarrow M_1 \rightarrow 0$, where $U$ is projective or 0. Since $\mathcal{A}(M_1)$ splits on restriction to $Q$ by Lemma 1.6(3), we have $M_2 \downarrow Q = \bigoplus_{i=0}^{1} \Omega^i N$ (mod projectives).
Suppose then that $M_i M_n = \Omega_i \Omega N (\text{mod } \text{projectives})$ for all $i$ with $1 \leq i \leq n - 1$. We have the $AR$-sequence $\mathcal{A}(M_{n-1})$: $0 \rightarrow \Omega^2 M_{n-1} \rightarrow M_n \oplus \Omega^2 M_{n-2} \oplus U \rightarrow M_{n-1} \rightarrow 0$, where $U$ is projective or $0$. Since $\mathcal{A}(M_{n-1})$ splits on restriction to $Q$ by Lemma 1.6(3), we have $(M_n \oplus \Omega^2 M_{n-2} \oplus U) \downarrow Q \cong M_{n-1} \ominus Q \oplus \Omega^2 M_{n-1} \downarrow Q$. This implies that $M_i M_n = \Omega_i \Omega N (\text{mod } \text{projectives})$.

From Theorem 2.1 and Lemmas 2.2 and 2.5, we have:

Lemma 2.6. Assume $(\#)$. Let $Q$ be a proper subgroup of $P$. Let $M$ be an indecomposable $kG$-module whose $k$-dimension is not divisible by $p$. Suppose that $M \ominus \Omega S \not\simeq M \ominus Q$ and $N \oplus \Omega^{-2} N \not\simeq M \ominus Q$ for some non-projective indecomposable direct summand $N$ of $M \ominus Q$. Then $M$ lies at the end of a $ZA_{\infty}$-component.

Corollary 2.7. Assume $(\#)$. Let $M$ be an indecomposable $kG$-module with vertex $P$ and $S$ a $P$-source of $M$.

1. Suppose that $p$ is odd and $\dim_S S = 2$. Then $M$ lies at the end of a $ZA_{\infty}$-component.

2. Suppose that $p \neq 3$ and $\dim_S S = 3$. Then $M$ lies at the end of a $ZA_{\infty}$-component.

3. Suppose that $p \neq 5$ and $\dim_S S = 5$. Then $M$ lies at the end of a $ZA_{\infty}$-component.

Proof. There exists an element $x$ of $P$ such that $x$ does not act on $S$ trivially. Let $Q = \langle x \rangle$. Then $S \downarrow Q$ satisfies the assumption in Lemma 2.6. Therefore $S$ lies at the end of a $ZA_{\infty}$-component, and $M$ lies at the end of a $ZA_{\infty}$-component by Corollary 2.4.

Remark. In [8], Erdmann proved that there are infinitely many $kP$-modules of dimension 2 or 3 lying at the ends of $ZA_{\infty}$-components under the hypothesis $(\#)$ ([8, Propositions 4.2 and 4.4]). Consequently she showed that for a block $B$ over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many components isomorphic to $ZA_{\infty}$ if a defect group of $B$ is not cyclic, dihedral, semidihedral or generalized quaternion ([8, Theorem 5.1]).

3. Remarks on Tensoring with a Certain Module

Suppose that $M$ is an indecomposable $kG$-module such that $p \not\mid \dim_S M$, and let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Let $\Delta_0$ be the connected component of $\Gamma_s(kG)$ containing the trivial $kG$-module $k$. In this section we consider tensoring modules in $\Delta_0$ with $M$ under the same hypothesis as in Section 2:

$(\#) k$ is an algebraically closed field of characteristic $p > 0$ and a Sylow
\( p \)-subgroup \( P \) of \( G \) is not cyclic, dihedral, semidihedral or generalized quaternion.

Thus both \( \Theta \) and \( \Delta_0 \) are isomorphic to \(ZA_m\) by Theorem 2.1. We fix some notation: \( T_0: k=L_1 \leftarrow L_2 \leftarrow L_3 \leftarrow \cdots \leftarrow L_n \leftarrow \) is a tree in \( \Delta_0 \) with \( \Delta_0 \cong ZT_0 \).

**Proposition 3.1.** Assume \((\#)\). Suppose that \( M \) is an indecomposable \( kG \)-module such that \( p \not| \dim_k M \) and \( M \) lies at the end of its component \( \Theta \). Let \( S \) be a \( P \)-source of \( M \). Let \( B \) and \( \Lambda_0 \) be the connected components of \( T_S(kP) \) containing \( S \) and the trivial \( kP \)-module \( k \) respectively. Then tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \) if and only if tensoring with \( S \) induces a graph isomorphism from \( \Lambda_0 \) onto \( \Xi \).

**Remark.** The assumption in Proposition 3.1 implies that both \( \Lambda_0 \) and \( \Xi \) are isomorphic to \(ZA^*\) and \( S \) lies at the end of \( H \) by Theorem 2.1 and Corollary 2.4.

**Proof of Proposition 3.1.** Let \( T: M=M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \) be a tree in \( \Theta \) with \( \Theta \cong ZT \). Then we have \( P \)-sources \( S_n \) of \( M_n \) (\( n \geq 1 \)) and a tree \( T': S=S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_n \) with \( \Xi \cong ZT' \) by Proposition 2.3. Let \( T'': k=H_1 \leftarrow H_2 \leftarrow \cdots \leftarrow H_n \) be a tree in \( \Delta_0 \) with \( \Delta_0 \cong ZT'' \).

Suppose that the tensoring with \( S \) induces a graph isomorphism from \( \Lambda_0 \) onto \( H \). This means that \( H_n \otimes S \cong S_n \) (mod projectives) and \( J_1(H_n) \otimes S \) is the AR-sequence \( J_1(M_n) \) modulo projectives for \( n > 1 \). We show that \( L_n \otimes M \cong M_n \) (mod projectives) for all \( n \) by induction on \( n \). Clearly \( L_1 \otimes M_1 \cong k \otimes M_1 \cong M_1 \).

By Theorem 1.3, \( J_1(k) \otimes M \) is the AR-sequence \( J_1(M^*) \) modulo projectives. Hence \( L_2 \otimes M_1 \cong M_2 \) (mod projectives). Suppose then that \( L_i \otimes M_i \cong M_i \) (mod projectives) for all \( i \) with \( 1 \leq i \leq n-1 \). We claim that \( J_1(L_{n-1}) \otimes M_i \) is the AR-sequence \( J_1(M_{n-1}) \otimes M_i \) modulo projectives: Since \( L_{n-1} | M_{n-1} \otimes M_i \otimes M_i^* \) by Theorem 1.4, we have \( 0= \mbox{Tor}(L_{n-1} \otimes M_i \otimes M_i^*, J_1(L_{n-1} \otimes M_i))= \mbox{Tor}(L_{n-1} \otimes M_i, J_1(L_{n-1} \otimes M_i)) \). This implies that \( J_1(L_{n-1}) \otimes M_i \) does not split. Thus in order to show that \( J_1(L_{n-1}) \otimes M_i \) is the AR-sequence \( J_1(M_{n-1}) \otimes M_i \) modulo projectives, it is enough to show that \( \mbox{Tor}(L_{n-1} \otimes M_i, J_1(M_{n-1}))=0 \) by Lemma 1.7(2). From Proposition 2.3, we have \( m(M_{n-1}) \mid m(S_{n-1} \otimes M_i) \) and \( M_i \mid S_{n-1} \). Thus it follows that \( m(S_{n-1} \otimes M_i) \cong (m(S_{n-1} \otimes M_i) \otimes S_{n-1} \otimes M_i) \cong 0 \). Now we have \( m(S_{n-1} \otimes M_i) \cong (m(S_{n-1} \otimes S_{n-1} \otimes M_i)) \cong m(S_{n-1} \otimes S_{n-1}) \) from the Frobenius reciprocity. By the Mackey decomposition theorem, we have \( (S_{n-1} \otimes M_i) \otimes \bigotimes_{\tilde{g} \in N_G(P)/P} (S_{n-1} \otimes M_i) \otimes \bigotimes_{\tilde{g} \in N_G(P)/P} (S_{n-1} \otimes M_i) \) as elements of the Green ring \( a(kP) \) by Lemma 1.6(1), we get \( \mbox{Tor}(L_{n-1} \otimes M_i \otimes S_{n-1} \otimes M_i) = \mbox{Tor}(L_{n-1} \otimes M_i \otimes S_{n-1} \otimes M_i) = 0 \) by our assumption. Since \( S_{n-1} \not\subset M \) and \( m(S_{n-1}) \) for any \( g \) in \( N_G(P) \), we get \( m(S_{n-1}) \), \( \mbox{Tor}(L_{n-1} \otimes M_i \otimes S_{n-1} \otimes M_i) = 0 \). Thus we obtain \( m(M_{n-1}) \cong (m(S_{n-1}) \otimes M_i) \cong 0 \) as desired. Therefore \( J_1(L_{n-1}) \otimes M_i: \Omega^2 L_{n-1} \otimes M_i \rightarrow (\Omega^2 L_n \otimes L_n) \otimes M_i \rightarrow L_{n-1} \otimes M_i \rightarrow 0 \) is the AR-sequence \( J_1(M_{n-1}) \) modulo projectives. This implies
that $L_n \otimes M_1 \cong M_n$ (mod projectives).

Conversely suppose that the tensoring with $M_1$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$. We show that $H_n \otimes S_1 \cong S_n$ (mod projectives) for all $n \geq 1$ by induction on $n$. Clearly $H_1 \otimes S_1 = k \otimes S_1 \cong S_1$. By Theorem 1.3, $\mathcal{A}(k) \otimes S_1$ is the AR-sequence $\mathcal{A}(S_1)$ modulo projectives. Hence $H_2 \otimes S_1 \cong S_2$ (mod projectives). Suppose then that $H_i \otimes S_1 \cong S_i$ (mod projectives) for all $i$ with $1 \leq i \leq n-1$. We claim that $\mathcal{A}(H_{n-1}) \otimes S_1$ is the AR-sequence $\mathcal{A}(S_{n-1})$ modulo projectives. Since $H_{n-1} \otimes S_1 \cong S_{n-1}$ (mod projectives) and $\Omega^2 H_{n-1} \otimes S_1 \cong \Omega^2 S_{n-1}$ (mod projectives), it is enough to show that $(m(S_{n-1}), [\mathcal{A}(H_{n-1} \otimes S_1)]) = 0$ by Lemma 1.7(2). From Lemma 1.6(1), we have $m(S_{n-1}) \mid m(M_{n-1}) \otimes (M_1 \otimes P)$ and $[\mathcal{A}(H_{n-1})] = [\mathcal{A}(L_{n-1}) \otimes P]$. Hence it follows that $(m(S_{n-1}), [\mathcal{A}(L_{n-1}) \otimes P] \otimes (M_1 \otimes P)) \geq (m(S_{n-1}), [\mathcal{A}(H_{n-1}) \otimes S_1]) \geq 0$. Using the Frobenius reciprocity, we have $(m(S_{n-1}), [\mathcal{A}(L_{n-1}) \otimes P] \otimes (M_1 \otimes P)) = (m(S_{n-1}) \otimes \mathcal{A}(M_1)) = (m(S_{n-1}) \otimes \mathcal{A}(M_1))$, which is zero since $m(S_{n-1}) = S_n \otimes \Omega^2 S_n = 0$. This implies that $m(S_{n-1}) \mid \mathcal{A}(H_{n-1} \otimes S_1) = 0$ as desired. Therefore $\mathcal{A}(H_{n-1}) \otimes S_1: 0 \rightarrow \Omega^2 H_{n-1} \otimes S_1 \rightarrow (\Omega^2 H_{n-2} \otimes H_n) \otimes S_1 \rightarrow H_{n-1} \otimes S_1 \rightarrow 0$ is the AR-sequence $\mathcal{A}(S_{n-1})$ modulo projectives. This implies that $H_n \otimes S_1 \cong S_n$ (mod projectives).

**Corollary 3.2.** Let $M$ be a trivial source module with vertex $P$. Let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Then $\Theta$ is isomorphic to $ZA_n$ and $M$ lies at the end of $\Theta$. Moreover tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

Proof. Proposition 2.3 and Corollary 2.4 imply that $\Theta$ is isomorphic to $ZA_n$ and $M$ lies at the end of $\Theta$. The second statement follows by Proposition 3.1.

In the following, we give some conditions each of which implies that tensoring an indecomposable $kG$-module $M$ induces a graph isomorphism from $\Delta_0$ onto a component isomorphic to $ZA_n$.

**Proposition 3.3.** Assume $(\#)$. Let $M$ be an indecomposable $kG$-module such that $p \not\mid \dim_k M$, and let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Let $Q$ be a proper subgroup of $P$. Suppose that $M$ satisfies the following conditions (with respect to $Q$).

1. The trivial $kQ$-module $k$ is a direct summand of $(M \otimes M^*) \downarrow_{Q}$ with multiplicity one;
2. If $Q$ is generalized quaternion, then $\Omega^2 k \not\mid (M \otimes M^*) \downarrow_{Q}$.

Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

Remark. (i) From Theorem 1.4, the above condition (1) is equivalent to the following condition:

1'. We have an indecomposable direct sum decomposition $N \oplus (\oplus_i W_i)$ of
\( M \downarrow_{\Theta}, \) where \( p \not\mid \dim N \) and \( p \not\mid \dim W_t \) for all \( t \).

(ii) \( \Theta \) is isomorphic to \( \mathbb{Z}A_m \) by Theorem 2.1. Moreover \( M \) lies at the end of \( \Theta \) by Lemma 2.6.

In order to prove Proposition 3.3, we need the following.

**Lemma 3.4.** Under the same assumption as in Proposition 3.3, \( L_n \) is a direct summand of \( L_n \otimes M \otimes M^* \) with multiplicity one for all \( n \geq 1 \).

**Proof.** Note that \( L_n \) is a direct summand of \( L_n \otimes M \otimes M^* \) since \( k \not\mid M \otimes M^* \).

From Lemma 2.5, we have \( L_n \downarrow_\Theta = \oplus \otimes^{\Pi}_{i=1} \Omega^2 k \) (mod projectives). Since the multiplicity of \( k \) in \( (M \otimes M^*) \downarrow_\Theta \) is one (and \( \Omega^2 k \) is not a direct summand of \( (M \otimes M^*) \downarrow_\Theta \) if \( Q \) is generalized quaternion), it follows that \( 2(\oplus \otimes^{\Pi}_{i=1} \Omega^2 k) \not\mid (L_n \otimes M \otimes M^*) \downarrow_\Theta \). This implies that the multiplicity of \( L_n \) in \( L_n \otimes M \otimes M^* \) is one.

Proof of Proposition 3.3. Let \( T: M=M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \rightarrow \cdots \) be a tree in \( \Theta \) with \( \Theta \cong \mathbb{Z}T \). We show that \( L_n \otimes M \cong M_n \) (mod projectives) for all \( n \geq 1 \) by induction on \( n \).

Clearly \( L_1 \otimes M = k \otimes M_1 \cong M_1 \). Let \( \mathcal{A}(k): 0 \rightarrow \Omega^2 k \rightarrow L_2 \oplus U \rightarrow k \rightarrow 0 \) be the AR-sequence terminating at \( k \), where \( U \) is projective or 0. Then the tensor sequence \( \mathcal{A}(k) \otimes M \) is the AR-sequence \( \mathcal{A}(M) \) modulo projectives by Theorem 1.3.

Hence \( L_1 \otimes M \cong M_1 \) (mod projectives).

Suppose then that \( L_i \otimes M \cong M_i \) (mod projectives) for all \( i \) with \( 1 \leq i \leq n-1 \). We claim that \( \mathcal{A}(M_{n-1}) \otimes M \) is the AR-sequence \( \mathcal{A}(M_{n-1}) \) modulo projectives: By Lemma 1.7(1), it suffices to show that \( (M_{n-1}, [\mathcal{A}(L_{n-1}) \otimes M]) = 1 \). Since \( L_n \) is a direct summand of \( L_{n-1} \otimes M \otimes M^* \) with multiplicity one by Lemma 3.4, we have \( (M_{n-1}, [\mathcal{A}(L_{n-1}) \otimes M]) = (L_{n-1} \otimes M \otimes M^*, [\mathcal{A}(L_{n-1})]) = 1 \) as desired.

Now \( \mathcal{A}(L_{n-1}) \otimes M: 0 \rightarrow \Omega^2 L_{n-1} \otimes M \rightarrow (\Omega^2 L_{n-2} \oplus L_n \oplus U') \otimes M \rightarrow L_{n-1} \otimes M \rightarrow 0 \) is the AR-sequence \( \mathcal{A}(M_{n-1}) \) modulo projectives, where \( U' \) is projective or 0. Thus we get \( L_n \otimes M \cong M_n \) (mod projectives).

**Corollary 3.5.** (1) Suppose that \( p \) is odd. Let \( M \) be an indecomposable \( kG \)-module with vertex \( P \) and \( S \) a \( P \)-source of \( M \). Suppose that \( \dim S = 2 \). Then tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto the connected component containing \( M \).

(2) Suppose that \( p = 2 \). Let \( M \) be an indecomposable \( kG \)-module with vertex \( P \) and \( S \) a \( P \)-source of \( M \). Suppose that \( \dim S = 3 \). Then tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto the connected component containing \( M \).

**Proof.** The result follows from Corollary 2.7 and Propositions 3.1 and 3.3.

**Proposition 3.6.** Assume (\#). Let \( M \) be an indecomposable \( kG \)-module with \( p \not\mid \dim M \), and let \( \Theta \) be the connected component containing \( M \). Suppose
that $M$ satisfies the following conditions.

(1) $M$ lies at the end of $\Theta$.

(2) $M \otimes M \approx k \oplus (\oplus_t W_t)$, where each $W_t$ is indecomposable and $p \mid \dim_k W_t$.

Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

In order to prove Proposition 3.6, we need the following.

**Lemma 3.7** ([22, p.16, Konstruktionslemma]). Let $M$ and $N$ be non-projective indecomposable $kG$-modules and

$$\begin{array}{ccc}
\alpha & N & \beta \\
\tau & M & 0 \\
N' & \\
\end{array}$$

an exact sequence. Suppose that $\alpha: \Omega^2 M \to N$ and $\beta: N \to M$ are irreducible maps and $N \not\cong N'$. Then $\tau$ is the AR-sequence $\mathcal{A}(M)$.

Proof of Proposition 3.6. Let $T: M = M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n$, be a tree in $\Theta$ with $\Theta \simeq ZT$. We will show that $L_n \otimes M \simeq M_n$ (mod projectives) and the tensor sequence $\mathcal{A}(L_n) \otimes M$ is the AR-sequence $\mathcal{A}(M_n)$ modulo projectives for all $n \geq 1$ by induction on $n$. Clearly $L_1 \otimes M = k \otimes M_1 \simeq M_1$. By Theorem 1.3, the tensor sequence $\mathcal{A}(k) \otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives. Hence $L_2 \otimes M \simeq M_2$ (mod projectives).

Suppose then that $L_i \otimes M \simeq M_i$ (mod projectives) for all $i$ with $1 \leq i \leq n-1$ and the tensor sequence $\mathcal{A}(L_i) \otimes M$ is the AR-sequence $\mathcal{A}(M_i)$ modulo projectives for all $i$ with $1 \leq i \leq n-2$. We will show that the tensor sequence $\mathcal{A}(L_{n-1}) \otimes M$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives.

Now $\mathcal{A}(L_{n-2}) \otimes M: 0 \to \Omega^2 L_{n-2} \otimes M \to \Omega^2 L_{n-3} \otimes M \oplus L_{n-1} \otimes M \to L_{n-2} \otimes M \to 0$ and $\mathcal{A}(\Omega^2 L_{n-2}) \otimes M: 0 \to \Omega^2 L_{n-2} \otimes M \to \Omega^2 L_{n-3} \otimes M \oplus \Omega^2 L_{n-1} \otimes M \to \Omega^2 L_{n-2} \otimes M \to 0$ are the AR-sequences $\mathcal{A}(M_{n-2})$ modulo projectives and $\mathcal{A}(\Omega^2 M_{n-2})$ modulo projectives respectively. Let $\alpha: \Omega^2 L_{n-1} \to \Omega^2 L_{n-2}$ and $\beta: \Omega^2 L_{n-2} \to L_{n-1}$ be irreducible maps. Then $\alpha \otimes \text{id}_M: \Omega^2 L_{n-1} \otimes M \to \Omega^2 L_{n-2} \otimes M$ is an irreducible map $\Omega^2 M_{n-1} \to \Omega^2 M_{n-2}$ plus some split map from the projective part of $\Omega^2 L_{n-1} \otimes M$ to the projective part of $\Omega^2 L_{n-2} \otimes M$, and $\beta \otimes \text{id}_M: \Omega^2 L_{n-2} \otimes M \to L_{n-1} \otimes M$ is an irreducible map $\Omega^2 M_{n-2} \to M_{n-1}$ plus some split map from the projective part of $\Omega^2 L_{n-2} \otimes M$ to the projective part of $L_{n-1} \otimes M$.

Consider the tensor sequence $\mathcal{A}(L_{n-1}) \otimes M$:

$$\begin{array}{ccc}
\alpha \otimes \text{id}_M & \Omega^2 L_{n-2} \otimes M & \beta \otimes \text{id}_M \\
0 \to \Omega^2 L_{n-1} \otimes M & L_{n-1} \otimes M \to 0.
\end{array}$$
Here $\Omega^2 M_{n-2} \not\cong L_n \otimes M$: Assume not. Then $\Omega^2 M_{n-2} \cong L_n \otimes M$ and $\Omega^2 M_{n-2} \otimes M^* \cong L_n \otimes M \otimes M^*$. Now by the inductive hypothesis $L_{n-2} \otimes M \cong M_{n-2} \pmod{\text{projectives}}$ and $\Omega^2 L_{n-2} \otimes M \cong \Omega^2 M_{n-2} \pmod{\text{projectives}}$. Thus the condition (2) implies that $\Omega^2 M_{n-2} \otimes M^* \cong \Omega^2 L_{n-2} \oplus (\bigoplus W'_i)$, where each $W'_i$ is indecomposable and $p \mid \dim_k W'_i$. Also the condition (2) implies that $L_n \otimes M \otimes M^* \cong L_n \oplus (\bigoplus W''_i)$, where each $W''_i$ is indecomposable and $p \mid \dim_k W''_i$. This implies that $L_n \cong \Omega^2 L_{n-2}$, a contradiction.

Now the tensor sequence $\mathcal{A}(L_{n-1}) \otimes M$ satisfies the assumption in Lemma 3.7. Thus $\mathcal{A}(L_{n-1}) \otimes M$ is the $\mathcal{A}$-sequence $\mathcal{A}(M_{n-1})$ modulo projectives. This implies that $L_n \otimes M \cong M_n \pmod{\text{projectives}}$.

**Corollary 3.8.** Assume (♯). Suppose that $M$ is an endotrivial $kG$-module. Let $\Theta$ be the connected component containing $M$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

**Proof.** Let $\mathcal{A}(k): 0 \to \Omega k \to L_2 \oplus U \to k \to 0$ be the AR-sequence. Here $L_2$ is non-projective indecomposable and $U$ is projective or $0$ by our assumption. By Theorem 1.3, the tensor sequence $\mathcal{A}(k) \otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives. Since tensoring with an endotrivial module preserves the number of non-projective indecomposable direct summands, the projective-free part of $L_2 \otimes M$ is indecomposable. This implies that $M$ lies at the end of $\Theta$. Hence $M$ satisfies the conditions in Proposition 3.6 and the result follows.

**Remark.** In [6], Bessenrodt studied endotrivial modules in the Auslander-Reiten quiver. She showed that without the hypothesis (♯), if $M$ is an endotrivial $kG$-module, then tensoring with $M$ induces a graph isomorphism from the connected component containing the trivial $kG$-module $k$ onto the connected component containing $M$ ([6, Theorem 2.3]).

### 4. $ZA_\infty$-Components of Dihedral 2-Groups

Throughout this section we assume that

$k$ is a field of characteristic 2 and a Sylow 2-subgroup $P$ of $G$ is dihedral of order at least 8.

Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$. Then $\Delta_0$ is isomorphic to $Z A_\infty$ by Theorem 1.2. It is known that all modules in $\Delta_0$ are endotrivial $kG$-modules (see, e.g., [6]).

**Proposition 4.1.** Let $M$ be an odd dimensional indecomposable $kG$-module. Let $\Theta$ be the connected component of $\Gamma'_s(kG)$ containing $M$ and $\Delta_0$ the connected
component containing \( k \). Then \( \Theta \) is isomorphic to \( \mathbb{Z}A_{\infty} \) and tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).

Proof. Let \( T_0: \cdots \rightarrow V_n \rightarrow \cdots \rightarrow V_2 \rightarrow k \leftarrow L_2 \leftarrow \cdots \leftarrow L_n \leftarrow \cdots \) be a tree in \( \Delta_0 \) with \( \Delta_0 \cong \mathbb{Z}T_0 \). Since tensoring with an endotrivial module preserves the number of non-projective indecomposable direct summands, the projective-free part \( M_n \) (resp. \( W_n \)) of \( L_n \otimes M \) (resp. \( V_n \otimes M \)) is indecomposable and odd dimensional. Therefore the tensor sequences \( \mathcal{A}(L_n) \otimes M \) and \( \mathcal{A}(V_n) \otimes M \) are the AR-sequences \( \mathcal{A}(M_n) \) and \( \mathcal{A}(W_n) \) modulo projectives respectively by Lemma 1.5. Thus we obtain a tree \( T: \cdots \rightarrow W_n \rightarrow \cdots \rightarrow W_2 \rightarrow M \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots \) with \( \Theta \cong \mathbb{Z}T \).

**Corollary 4.2.** Let \( M \) be an odd dimensional indecomposable \( kG \)-module and \( \Theta \) the connected component containing \( M \). Then all modules in \( \Theta \) have the same vertex \( P \).

Proof. By Proposition 4.1, the tree class of \( \Theta \) is \( A_{\infty} \). Therefore all modules in \( \Theta \) are odd dimensional by Lemma 1.9(1). This implies the result.

5. **\( \mathbb{Z}D_{\infty} \)-Components of Semidihedral 2-Groups**

Throughout this section, we assume that

\( k \) is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup \( P \) of \( G \) is semidihedral.

Let \( \Delta_0 \) be the connected component of \( \Gamma(kG) \) containing the trivial \( kG \)-module \( k \). Then \( \Delta_0 \) is isomorphic to \( \mathbb{Z}D_{\infty} \) by Theorem 1.2 (see [7, p 76 II. 10.7 Remark]). Thus a part of \( \Delta_0 \) is as follows for some indecomposable \( kG \)-modules \( L_2, L_3 \) and \( I \).

```
\begin{tikzcd}
\Omega^2 L_2 & \Omega^2 I \arrow[r] & L_2 \arrow[l] & I \arrow[r] & \Omega^{-2} L_2 \\
\Omega^2 k \arrow[u] & & & & \Omega^{-2} L_3 \arrow[u]
\end{tikzcd}
```

Let \( P = \langle x, y; x^2 = y^{2^k - 1} = 1, y^2 = y^{-1 + 2^i} \rangle \) and \( \mathcal{F} = \{ x \} \). Let \( 0 \rightarrow \Omega_2 k \rightarrow U \rightarrow k \rightarrow 0 \) be an \( \mathcal{F} \)-projective cover resolution of the trivial \( kG \)-module \( k \). Con-
cerning some basic facts on relative projective cover, we refer to [15], [19] and [18]. The following result is due to Okuyama.

**Theorem 5.1([18]).** With the same assumption and notation as above,

1. \( I \cong \Omega(k) \) and \( I \) is an endotrivial \( kG \)-module.
2. \( I \) is self-dual and odd dimensional.
3. If \( I' \) is self-dual, odd dimensional and indecomposable, then \( I' \cong k \) or \( I \).

**Lemma 5.2.** Let \( M \) be an odd dimensional indecomposable \( kG \)-module. Then \( M \not\cong M \otimes I \).

Proof. Assume contrary that \( M | M \otimes I \). Then \( M \otimes I \cong M \) (mod projectives), since tensoring with an endotrivial module preserves the number of non-projective indecomposable direct summands. Moreover it follows by Theorem 1.4 that \( k|M \otimes M^*|(M \otimes M^*) \otimes I \). This implies that \( I|M \otimes M^* \).

Since \( 2 \not\mid \dim_k M \), \( k \) is a direct summand of \( M \otimes M^* \) with multiplicity one. If an indecomposable \( kG \)-module \( W \) is a direct summand of \( M \otimes M^* \), then \( W^* \) is also a direct summand of \( M \otimes M^* \). Let \( M \otimes M^* \cong k \oplus I \oplus (\bigoplus_i (W_i \oplus W_i^*)) \oplus (\bigoplus_j T_j) \) be an indecomposable direct sum decomposition, where \( W_i \) is not self-dual and \( T_j \) is self-dual. Since \( M \otimes M^* \) is odd dimensional, some \( T_j \) is odd dimensional. By Theorem 5.1(3), this \( T_j \) must be isomorphic to \( I \). Hence we get \( I \oplus I | M \otimes M^* \) and \( k \oplus k | (I \oplus I) \otimes I | (M \otimes M^*) \otimes I \cong M \otimes M^* \) (mod projectives). But this contradicts that the multiplicity of \( k \) in \( M \otimes M^* \) is one.

**Theorem 5.3.** Let \( M \) be an odd dimensional indecomposable \( kG \)-module and \( \Theta \) the connected component of \( \Gamma_s(kG) \) containing \( M \). Then \( \Theta \) is isomorphic to \( \mathbb{Z}_{D_m} \) and \( M \) lies at the end of \( \Theta \).

Proof. We continue to use the same notation as above.

Let \( \mathcal{A}(k): 0 \rightarrow \Omega k \rightarrow m(k) \rightarrow k \rightarrow 0 \) and \( \mathcal{A}(I): 0 \rightarrow \Omega I \rightarrow m(I) \rightarrow I \rightarrow 0 \) be the AR-sequences terminating at \( k \) and \( I \) respectively. Note that \( L_n \cong m(k) \cong m(I) \) (mod projectives). By Theorem 1.3, the tensor sequence \( \mathcal{A}(k) \otimes M \) is the AR-sequence \( \mathcal{A}(M) \) modulo projectives. Since \( I \) is an endotrivial \( kG \)-module, the projective-free part \( M' \) of \( I \otimes M \) is indecomposable. Hence by Lemma 1.5, the tensor sequence \( \mathcal{A}(I) \otimes M \) is the AR-sequence \( \mathcal{A}(M') \) modulo projectives. Note that \( M' \) is not isomorphic to \( M \) by Lemma 5.2.

We claim that the projective-free part \( M_2 \) of \( L_2 \otimes M \) is indecomposable: Assume not. Then we have \( X_1 \oplus X_2 | L_2 \otimes M \) for some non-projective indecomposable \( kG \)-modules \( X_1 \) and \( X_2 \). Note that \( X_1 \) is not isomorphic to \( X_2 \) by Theorem 1.1. Since \( X_1 \oplus X_2 | m(M) \) and \( X_1 \oplus X_2 | m(M') \), where \( m(M) \) and \( m(M') \) are the middle terms of \( \mathcal{A}(M) \) and \( \mathcal{A}(M') \) respectively, we get a part of \( \Theta \) as follows.
But this is a contradiction since $\Theta$ can not have such a subquiver by Theorem 1.1.

Consequently we have $m(M) \cong M_2 \pmod{\text{projectives}}$ and $m(M') \cong M_2 \pmod{\text{projectives}}$. This implies that $\Theta \cong \mathbb{Z}D_\omega$ and $M$ lies at the end.

**Lemma 5.4.** Let $M$ be an odd dimensional indecomposable $kG$-module and $\Theta$ the connected component containing $M$. Then all modules in $\Theta$ have the same vertex $P$.

**Proof.** By Theorem 5.3 and Lemma 1.9(2), $\Theta$ is isomorphic to $\mathbb{Z}D_\omega$ and $M$ lies at the end of $\Theta$. Since $M$ is odd dimensional, a Sylow 2-subgroup $P$ of $G$ is a vertex of $M$. The result follows from [20, Theorem 4.3].

**Lemma 5.5.** Let $M$ be an odd dimensional indecomposable $kG$-module and $\Theta$ the connected component of $\Gamma_s(kG)$ containing $M$. Let $T: M \twoheadrightarrow M_2 \twoheadrightarrow M_3 \twoheadrightarrow \cdots \twoheadrightarrow M_n \twoheadrightarrow \cdots$ be a tree in $\Theta$ with $\Theta \cong \mathbb{Z}T$. Let $S$ be a $P$-source of $M$ and $\Xi$ the connected component of $\Gamma_s(kP)$ containing $S$. Then we have $P$-sources $S'$ and $S_n$ of $M'$ and $M_n$ ($n \geq 2$) respectively and a tree $T': S \twoheadrightarrow S_2 \twoheadrightarrow S_3 \twoheadrightarrow \cdots \twoheadrightarrow S_n \twoheadrightarrow \cdots$ with $\Xi \cong \mathbb{Z}T'$.

**Proof.** All modules in $\Theta$ have the same vertex $P$ by Lemma 5.4. Thus applying the similar argument in the proof of Proposition 2.3, Steps 1 and 2, we may assume that $P$ is a normal subgroup of $G$ and $G$ is the inertial group of $\Xi$. Since the order of $G/P$ is odd and $\Xi$ is isomorphic to $\mathbb{Z}D_\omega$, $G$ acts on $\Xi$ trivially. Therefore we may also assume that every module in $\Xi$ is $G$-invariant. Applying Lemma 1.11, we get the result.

In the rest we consider tensoring $\Delta_0$ with an odd dimensional indecomposable $kG$-module.

**Proposition 5.6.** Let $S$ be an odd dimensional indecomposable $kP$-module and $\Xi$ the connected component of $\Gamma_s(kP)$ containing $S$. Let $\Lambda_0$ be the connect-
ed component of $T_k(kP)$ containing the trivial $kP$-module $k$. Then tensoring with $S$ induces a graph isomorphism from $\Delta_0$ onto $\Xi$.

In order to prove Proposition 5.6, we need the following Lemmas 5.7 and 5.8. Let $T_0: k \to H_1 \to H_2 \to \cdots \to H_n \cdots$ be a tree in $\Delta_0$ with $\Delta_0 \cong \mathbb{Z}T_0$. Let $\downarrow I_0 P = \langle x, y; x^2 = y^{a-1} = 1, y^x = y^{-1+2^{-1}} \rangle$.

**Lemma 5.7.** $H_n \downarrow \langle x \rangle \cong k \oplus k$ (mod projectives) for all $n \geq 2$.

Proof. Use induction on $n$. Since all modules in $\Delta_0$ have the same vertex $P$, the AR-sequences $\mathcal{A}(k)$, $\mathcal{A}(I_0)$ and $\mathcal{A}(H_n)$ split on restriction to $\langle x \rangle$. Hence $(k \oplus \Omega^2 k) \downarrow \langle x \rangle \cong m(k) \downarrow \langle x \rangle \cong H_1 \downarrow \langle x \rangle \cong m(I_0) \downarrow \langle x \rangle \cong (I_0 \oplus \Omega^2 I_0) \downarrow \langle x \rangle$. Thus we get $I_0 \downarrow \langle x \rangle \cong k$ (mod projectives), $\Omega^2 I_0 \downarrow \langle x \rangle \cong k$ (mod projectives) and $H_2 \downarrow \langle x \rangle \cong k \oplus k$ (mod projectives). Also $\mathcal{A}(H_3): 0 \to \Omega^2 H_2 \to H_3 \to \Omega^2 I_1 \to H_2 \to 0$ splits on restriction to $\langle x \rangle$. So we have $(H_3 \oplus \Omega^2 H_2 \oplus \Omega^2 I_0) \downarrow \langle x \rangle \cong (\Omega^2 H_2 \oplus H_3) \downarrow \langle x \rangle$ and $H_3 \downarrow \langle x \rangle \cong k \oplus k$ (mod projectives).

Suppose then that $H_i \downarrow \langle x \rangle \cong k \oplus k$ (mod projectives) for all $i$ with $2 \leq i \leq n-1$. Since $\mathcal{A}(H_{n-1}): 0 \to \Omega^2 H_{n-1} \to H_n \oplus \Omega^2 H_{n-2} \to H_{n-1} \to 0$ splits on restriction to $\langle x \rangle$, we have $(H_n \oplus \Omega^2 H_{n-2}) \downarrow \langle x \rangle \cong (\Omega^2 H_{n-1} \oplus H_{n-1}) \downarrow \langle x \rangle$. This implies that $H_n \downarrow \langle x \rangle \cong k \oplus k$ (mod projectives).

**Lemma 5.8.** Let $S$ be an odd dimensional $kP$-module.

1. The trivial $k(x)$-module $k$ is a direct summand of $S \downarrow \langle x \rangle$ with multiplicity one.
2. $H_n$ is a direct summand of $H_n \otimes S \otimes S^*$ with multiplicity one for all $n \geq 2$.

Proof. (1) The statement follows from [7, p 73. Lemma II 10.5].

(2) From (1) we have $(S \otimes S^*) \downarrow \langle x \rangle \cong k$ (mod projectives). Hence $(H_n \otimes S \otimes S^*) \downarrow \langle x \rangle \cong k \oplus k$ (mod projectives) from Lemm 5.7. Thus we have $2H_n \downarrow \langle x \rangle / (H_n \otimes S \otimes S^*) \downarrow \langle x \rangle$, which implies the result.

Proof of Proposition 5.6. Let $T: S \to S_2 \to S_3 \to S_4 \to \cdots \to S_n \to \cdots$ be a tree in $\Xi$ with $\Xi \cong \mathbb{Z}T$. Since $k \otimes S \cong S$ and $I_0 \otimes S \cong S'$, it suffices to show that $H_n \otimes S \cong S_0$ (mod projectives) for all $n \geq 2$. We proceed by induction on $n$.

From the argument in the proof of Theorem 5.3, we have $H_2 \otimes S \cong S_2$ (mod projectives) and $\Omega^2 H_2 \otimes S \cong \Omega^2 S_2$ (mod projectives). Also we have $(S_2, [\mathcal{A}(H_2) \otimes S]) = (H_2 \otimes S, [\mathcal{A}(H_2) \otimes S]) = (H_2 \otimes S \otimes S^*, [\mathcal{A}(H_2)]) = 1$ since the multiplicity of $H_2$ in $H_2 \otimes S \otimes S^*$ is one by Lemma 5.8(2). This implies that the tensor sequence $\mathcal{A}(H_2) \otimes S: 0 \to \Omega^2 H_2 \otimes S \to (H_2 \otimes \Omega^2 k \oplus \Omega^2 I_0) \otimes S \to H_2 \otimes S \to 0$ is the AR-sequence $\mathcal{A}(S_2)$ modulo projectives by Lemma 1.7(1). Thus we get $H_2 \otimes S \cong S_2$. 


Suppose then that $H_i \otimes S \cong S_i$ (mod projectives) for all $i$ with $2 \leq i \leq n - 1$. Using Lemma 5.8(2) again, we have $(S_{n-1}, [\mathcal{A}(H_{n-1}) \otimes S]) = (H_{n-1} \otimes S \otimes S^*, [\mathcal{A}(H_{n-1})]) = 1$. Thus the tensor sequence $\mathcal{A}(H_{n-1}) \otimes S: 0 \to \Omega H_{n-1} \otimes S \to (H_n \oplus \Omega^2 H_{n-2} \otimes S \to H_{n-1} \otimes S \to 0$ is the AR-sequence $\mathcal{A}(S_{n-1})$ modulo projectives. Therefore we get $H_n \otimes S \cong S_n$ (mod projectives).

**Proposition 5.9.** Let $M$ be an odd dimensional indecomposable $kG$-module and $\Theta$ the connected component containing $M$. Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

**Proof.** Let $S$ be a $P$-source of $M$. Let $\Xi$ and $\Lambda_0$ be the connected components of $\Gamma_s(kP)$ containing $S$ and $k$ respectively. Then tensoring with $S$ induces a graph isomorphism from $\Lambda_0$ onto $\Xi$ by Proposition 5.6. Using an argument similar to the one in the proof of Proposition 3.1 (use Lemma 5.5 in place of Proposition 2.3), we get the result.

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**References**

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