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Congruence Classes of Knots

By R. H. Fox

Consider a solid torus V in 3-dimensional euclidean space S . An oriented simple closed curve on the boundary T of V is a *meridian* of V if it bounds in V but not on T , and it is called a *longitude* if it bounds in $S-V$ but not on T . The fundamental group of T is a free abelian group of rank 2, and it has a basis consisting of elements a and b represented respectively by a meridian and a longitude. Any orientation preserving autohomeomorphism τ of V induces an automorphism $\tau_*: a \rightarrow a^{\pm 1}, b \rightarrow a^m b^{\pm 1}$ of $\pi(T)$ and τ is described up to homotopy by the automorphism τ_* that it induces. I shall call an autohomeomorphism τ a *simple twist* if the automorphism that it induces is $\tau_*: a \rightarrow a, b \rightarrow ab$. The automorphism induced by τ^m is $\tau_*^m: a \rightarrow a, b \rightarrow a^m b$.

If k is any simple closed curve in the interior of V then $\tau^m(k)$ is also a simple closed curve in the interior of V . Moreover if k is oriented and $\tau^m(k)$ is given its inherited orientation then the linking numbers $L(k, a)$ and $L(\tau^m(k), a)$ are equal.

Let n and q be non-negative integers and κ and λ knot types. I shall say that κ and λ are *congruent modulo n, q* , and write $\kappa \equiv \lambda \pmod{n, q}$, if there are simple closed curves k_0, k_1, \dots, k_l , integers c_1, \dots, c_l , and solid tori V_1, \dots, V_l such that

- (1) V_i contains $k_{i-1} \cup k_i$ in its interior;
- (2) $\tau_i^{c_i}(k_{i-1}) = k_i$, where τ_i is a simple twist of V_i ;
- (3) $L(k_{i-1}, a_i) = L(k_i, a_i) \equiv 0 \pmod{q}$, where a_i is represented by a meridian of V_i ;
- (4) k_0 represents κ and k_l represents λ .

Congruence modulo n, q is symmetric, reflexive and transitive. Congruence modulo $0, q$ is just equivalence, i.e. $\kappa \equiv \lambda \pmod{0, q}$ iff $\kappa = \lambda$. The well-known fact [1] that any knot projection may be normed so as to be the diagram of a trivial knot shows that any two knot types are congruent modulo $1, 0$ and also congruent modulo $1, 2$. It is not

difficult to find distinct knot types that are congruent modulo 1, 1; on the other hand, to find knot types that can be shown to be incongruent modulo 1, 1 does not seem to be easy.

It would be interesting to know whether equivalence could be replaced by a set of congruences. The question is: Do there exist distinct knot types κ, λ such that $\kappa \equiv \lambda \pmod{n, q}$ for every $n > 0$ and $q \geq 0$?

The object of this note is to give a necessary condition for congruence modulo n, q . The condition, which is rather effective if $n > 1$, involves the Alexander polynomial $\Delta_\kappa(t)$ of a knot type κ , i.e. the Alexander polynomial [2] of the fundamental group of the complement of a simple closed curve k that represents κ . An Alexander matrix of κ , i.e. an Alexander matrix [2] of $\pi(S-k)$, is denoted by $A_\kappa(t)$, and $\sigma_n(t)$ denotes $(t^n - 1)/(t - 1) = 1 + t + t^2 + \dots + t^{n-1}$.

Theorem¹⁾. *If $\kappa \equiv \lambda \pmod{n, q}$ then, for properly chosen $A_\kappa(t)$ and $A_\lambda(t)$,*

$$A_\kappa(t) \equiv A_\lambda(t) \pmod{\sigma_n(t^q)},$$

hence

$$\Delta_\kappa(t) \equiv \pm t^r \Delta_\lambda(t) \pmod{\sigma_n(t^q)},$$

(and similarly for the elementary ideals [2] of deficiency greater than 1).

Proof: We need consider only the case $l=1, c_1=1$, so that k represents κ and $\tau^n(k)$ represents λ . Presentations of the fundamental groups $\pi(S-k)$ and $\pi(S-\tau^n(k))$ may be obtained from the fundamental groups $\pi(S-V)$ and $\pi(V-k)$ and $\pi(V-\tau^n(k))$ by application of the van Kampen theorem [3]. This procedure yields presentations that are almost the same:

$$\begin{aligned} \pi(S-k) &= (a, b, A, B, x_1, x_2, \dots : a=A, b=B, r_1=1, r_2=1, \dots) \\ \pi(S-\tau^n(k)) &= (a, b, A, B, x_1, x_2, \dots : a=A, b=A^n B, r_1=1, r_2=1, \dots) \end{aligned}$$

where a and b are represented by meridian and longitude of V , and A and B are represented by the same curves in (the closure of) $S-V$. The 1-dimensional homology groups of $S-k$ and $S-\tau^n(k)$ are infinite cyclic; we denote ambiguously by t a generator of either group (selected so that one is carried into the other by τ^n). Then abelianization of $\pi(S-k)$ or $\pi(S-\tau^n(k))$ maps A into t^q (and B into 1). Hence

1) The theory can be generalized to links (which must be ordered and oriented). For links of multiplicity μ we replace q by (q_1, \dots, q_μ) where q_i is the linking number of the i th component with a . Instead of $\sigma_n(t^q)$ we have $\sigma_n(t_1^{q_1} t_2^{q_2} \dots t_\mu^{q_\mu})$.

	a	b	A	B	\cdot	\dots	
$A_\kappa(t) =$	$a = A$	-1	0	1	0	0	\dots
	$b = B$	0	-1	0	1	0	\dots
	\cdot	*	*	*	*	*	\dots
	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\dots
	\dots	\cdot	\cdot	\cdot	\cdot	\cdot	\dots
	a	b	A	B	\cdot	\dots	
$A_\lambda(t) =$	$a = A$	-1	0	1	0	0	\dots
	$b = A^n B$	0	-1	$\frac{t^{nq}-1}{t^q-1}$	t^{nq}	0	\dots
	\cdot	*	*	*	*	*	\dots
	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\dots
	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\dots

Therefore

$A_\lambda(t) - A_\kappa(t) =$	0	0	0	0	0	\dots
	0	0	$\frac{t^{nq}-1}{t^q-1}$	$t^{nq}-1$	0	\dots
	0	0	0	0	0	\dots
	\cdot	\cdot	\cdot	\cdot	\cdot	\dots
	\cdot	\cdot	\cdot	\cdot	\cdot	\dots

$\equiv 0 \pmod{\sigma_n(t^q)}.$

The only congruence classes that are fairly easy to deal with experimentally are those for which $n \equiv 0 \pmod{2}$ and $q = 0$ or 2 . Let us consider the repartition of the fifteen prime knots of not more than seven crossings (cf. the knot table [4]) into congruence classes modulo 2, 0 and modulo 2, 2.

The polynomial character of $\kappa \pmod{2, 0}$ is $\Delta_\kappa(t) \pmod{2}$. Each residue class has as principal representative a polynomial whose coefficients are all either 0 or 1. By experiment I find the following congruence classes mod 2, 0:

$$\begin{aligned}
 0 &\equiv 5_2 \equiv 6_1 \equiv 7_4 \equiv 7_5, & \Delta(t) &\equiv 1 & \pmod{2} \\
 3_1 &\equiv 4_1 \equiv 7_2 \equiv 7_3, & \Delta(t) &\equiv 1 + t + t^2 & \pmod{2} \\
 5_1 &\equiv 6_2 \equiv 6_3 \equiv 7_6 \equiv 7_7, & \Delta(t) &\equiv 1 + t + t^2 + t^3 + t^4 & \pmod{2} \\
 7_1, & & \Delta(t) &\equiv 1 + t + t^2 + t^3 + t^4 + t^5 + t^6 & \pmod{2}
 \end{aligned}$$

It was observed by Kinoshita that the non-amphicheiral knot 3_1 must be congruent mod 2, 0 to its reflexion, since it is congruent mod 2, 0 to the amphicheiral knot 4_1 .

The polynomial character of κ mod 2, 2 is $\Delta_\kappa(t)$ mod $(1+t^2)$; its principal representative is a positive odd integer (since $\Delta(t)$ is symmetric and $\Delta(1) = 1$). By experiment I find the following residue classes mod 2, 2.

$$\begin{array}{ll} 0 \equiv 3_1 \equiv 5_1 \equiv 6_2 \equiv 7_1 \equiv 7_3 \equiv 7_5, & \Delta(t) \equiv 1 \\ 4_1 \equiv 5_2 \equiv 6_3, & \Delta(t) \equiv 3 \\ 6_1 \equiv 7_2 \equiv 7_6, & \Delta(t) \equiv 5 \\ 7_4 \equiv 7_7, & \Delta(t) \equiv 7 \end{array}$$

The congruence mod 2, 2 of 7_4 and 7_7 was discovered by F. Hosokawa. I conclude with some examples of the experimental work involved.

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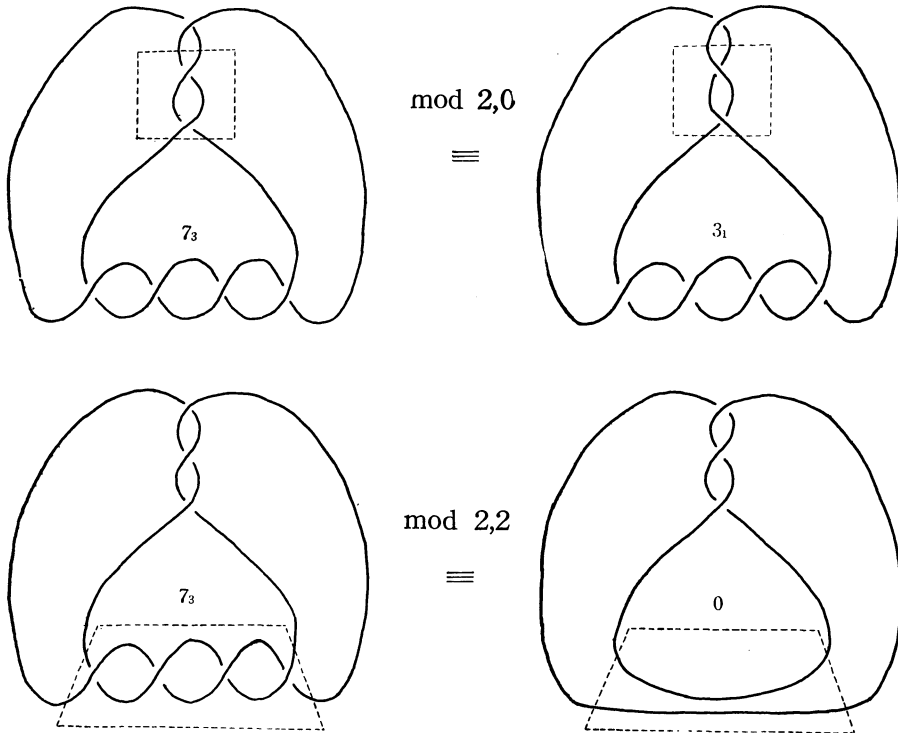


Fig. 1

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