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THE GONALITY CONJECTURE FOR CURVES ON CERTAIN TORIC SURFACES

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Abstract

The gonality is one of important invariants in the study of linear systems on curves. The gonality conjecture which was posed by Green and Lazarsfeld predicts that we can read off the gonality of a curve from any one line bundle of sufficiently large degree on the curve. This conjecture had been proved for curves on Hirzebruch surfaces by Aprodu. In this artlcle, we will extend this result for curves on certain toric surfaces.

Introduction

In this article, a *curve* will always mean a smooth irreducible complex projective curve unless otherwise stated. For a curve X, the gonality of X is defined as

$$\operatorname{gon}(X) = \min\{k \mid X \text{ carries a } g_k^1\},$$

where g_k^1 denotes a 1-dimensional linear system of degree k on X. A curve of gonality k is called k-gonal. The gonality is an important invariant in the study of linear systems on curves, although it is often difficult to determine it for a given curve. It is well-known that a plane curve of degree d is $(d-1)$-gonal. Martens determined the gonality of curves on Hirzebruch surfaces in [5].

One of the central problems around the gonality is the so-called *gonality conjecture* (Conjecture 0.2 below) posed by Green and Lazarsfeld in [3]. Let us fix the notation in order to state the gonality conjecture and for the later use. Let V be a finite dimensional complex vector space, SV the symmetric algebra of V, and $B = \bigoplus_{q \in \mathbb{Z}} B_q$ a graded SV-module. Then, as in [4], one has the Koszul complex

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} V \otimes B_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} V \otimes B_{q-1} \to \cdots,$$

which yields the Koszul cohomology group $K_{p,q}(B, V) = \operatorname{Ker} d_{p,q} / \operatorname{Im} d_{p+1,q-1}$. In par-

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ticular, for an irreducible complex projective variety Z and a line bundle L on Z, we put

$$K_{p,q}(Z, L) = K_{p,q}\left(\bigoplus_{i\in\mathbb{Z}} H^0(Z, iL), H^0(Z, L)\right).$$

DEFINITION 0.1 ([3]). Let L be a line bundle on a curve X, and l a non-negative integer. We say that the pair (X, L) satisfies the property (M_l) (or, simply, L satisfies the property (M_l)) if $K_{p,1}(X, L) = 0$ for any integer $p \geq h^0(X, L) - l - 1$.

It is closely related to the minimal free resolution of $\bigoplus H^0(X, iL)$ when L is projectively normal. See [3] for the detail. If X is a k-gonal curve of genus g, then it is well-known that any line bundle of degree not less than $2g + k$ cannot satisfy (M_k). The gonality conjecture predicts a converse of this fact:

Conjecture 0.2 (The gonality conjecture). *Let X be a curve of genus g and k a positive integer. If the property (M_k) fails for any line bundle L on X with $\deg L \gg 2g$, then X carries a g^1_k.*

Hence we can read off the gonality of a curve from any one line bundle of sufficiently large degree on it if the conjecture is true. As for curves on the Hirzebruch surfaces, we have not only Martens' result referred above but also an affirmative answer to the gonality conjecture. This was done by Aprodu in [1]. So it is a natural question to extend their results to curves on more general surfaces, e.g., toric surfaces obtained from a Hirzebruch surface by a finite succession of equivariant blowing-ups. Such toric sufaces have finite $\mathbb{P}^1$-fibrations by toric morphisms. In this paper, we restrict ourselves to a class of toric surfaces admitting a unique $\mathbb{P}^1$-fibration by a toric morphism (see §1 for the precise description). We determine the gonality of curves on such surfaces, and also show that the gonality conjecture holds for them. Namely, we shall show the following:

Theorem 0.3 (Main Theorem). *Let S be a toric surface which has a unique $\mathbb{P}^1$-fibration $\psi\colon S \to \mathbb{P}^1$ by a toric morphism, and denote by F a fiber of ψ. Let X be a curve on S and put $X.F = k$. Then one of the following holds.*

(a) *X is a rational curve,*

(b) *X is isomorphic to a non-singlar plane curve of degree k,*

(c) *X is k-gonal, and the gonality conjecture is valid for X.*

The proof owes much to [1] and will go with the induction on the sum of k and the Picard number of the surface.

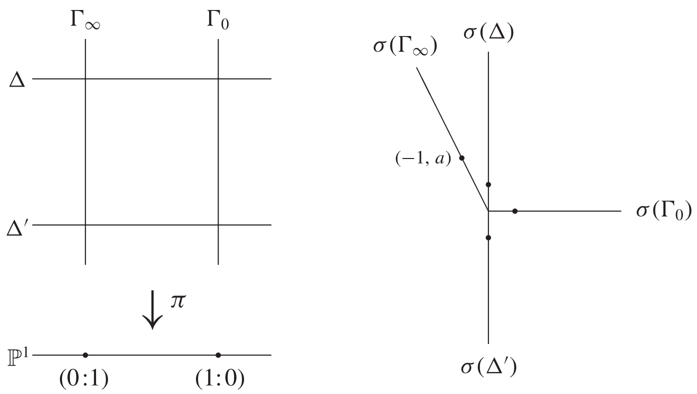

Fig. 1.

1. Notation and set-up

For many of the theoretical facts about toric surfaces included in this section, we refer to [7] without further mention. For a non-negative integer a, we let Σ_a be a Hirzebruch surface of degree a:

$$\Sigma_a = \{((X_0 : X_1 : X_2), (Y_0 : Y_1)) \mid X_1 Y_1^a = X_2 Y_0^a\} \subset \mathbb{P}^2 \times \mathbb{P}^1.$$

The ruling map π of Σ_a is defined as the projection to the second factor:

$$\begin{array}{rccc} \pi : & \Sigma_a & \to & \mathbb{P}^1 \\ & ((X_0 : X_1 : X_2), (Y_0 : Y_1)) & \mapsto & (Y_0 : Y_1). \end{array}$$

We denote by Δ a minimal section of π ($\Delta^2 = -a$) and by Δ' a section of π which does not meet Δ, and put $\Gamma_0 = \pi^{-1}((1 : 0))$, $\Gamma_\infty = \pi^{-1}((0 : 1))$. Recall that Σ_a is a typical example of a toric surface. As is well-known, a non-singular toric surface can be obtained by a division of $\mathbb{R}^2$. In the case of Σ_a, it is as in Fig. 1.

By definition, a toric surface Σ contains an algebraic torus T as a non-empty Zariski open set, and it acts on Σ. Divisors on Σ are called T-invariant if they are T-stable. When we express Σ by a division of $\mathbb{R}^2$, they correspond to half-lines starting from (0, 0). These half-lines are called (1-dimentional) cones. A point on a cone is called a primitive element if it is the $\mathbb{Z}$-lattice point closest to (0, 0). For instance, let us consider the case of Fig. 1. The T-invariant divisors of Σ_a are Δ, Δ', Γ_0 and Γ_∞. We put $n = (1, 0)$. Then the cone corresponding to Γ_0 is

$$\sigma(\Gamma_0) = \mathbb{R}_{\geq 0} n = \{cn \mid c \in \mathbb{R}_{\geq 0}\},$$

and n is the primitive element of $\sigma(\Gamma_0)$. Similarly, $n' = (0, 1)$ is the primitive element of the cone $\sigma(\Delta) = \mathbb{R}_{\geq 0} n' = \{cn' \mid c \in \mathbb{R}_{\geq 0}\}$ which corresponds to Δ.

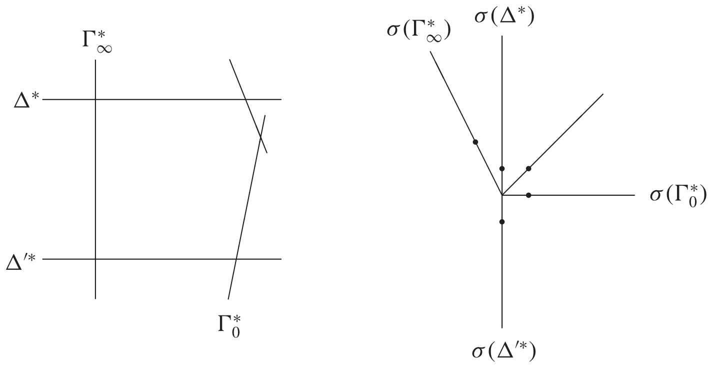

Fig. 2.

Intersections of T-invariant divisors are called T-fixed points. A blowing-up of Σ with center a T-fixed point can be expressed as a subdivision of the original division: Let D_1 and D_2 be T-invariant divisors on Σ. We denote by n_1 and n_2 the primitive elements of the cones corresponding to D_1 and D_2, respectively. The blowing-up of Σ with center $D_1 \cap D_2$ corresponds to the subdivision obtained by adding the cone $\mathbb{R}_{\geq 0}(n_1 + n_2)$ to the original division. For instance, in the case of Fig. 1, the blowing-up of Σ_a with center $\Delta \cap \Gamma_0$ corresponds to the subdivision as in Fig. 2.

We henceforth assume $a \geq 1$, and let S be a surface obtained from Σ_a by a finite succession of blowing-ups with T-fixed points as centers. We assume that such T-fixed points do not lie on Δ'. We denote by $\varphi\colon S \to \Sigma_a$ this blowing-ups, and call $\psi = \pi \circ \varphi$ the ruling map of S. This surface is expressed by the division of $\mathbb{R}^2$ as in Fig. 3.

Let C and C' be the proper transforms of Δ and Δ' by φ, respectively. Since $(\pi \circ \varphi)^{-1}((1:0)) = \bigcup_{i=1}^{d} D_i$ is a simple chain of non-singular rational curves, we can label them in the following way:

$$\begin{cases} D_1.C = 1, \\ D_i.D_{i+1} = 1 \quad (1 \leq i \leq d-1), \\ D_d.C' = 1. \end{cases}$$

Similarly, we denote by $E_1, \ldots, E_e$ all the irreducible components contained in $(\pi \circ \varphi)^{-1}((0:1))$, where we define their order as:

$$\begin{cases} E_1.C = 1, \\ E_j.E_{j+1} = 1 \quad (1 \leq j \leq e-1), \\ E_e.C' = 1. \end{cases}$$

We denote by $n_i = (x_i, y_i)$ the primitive elements of $\sigma(D_i)$. Similarly, we denote by

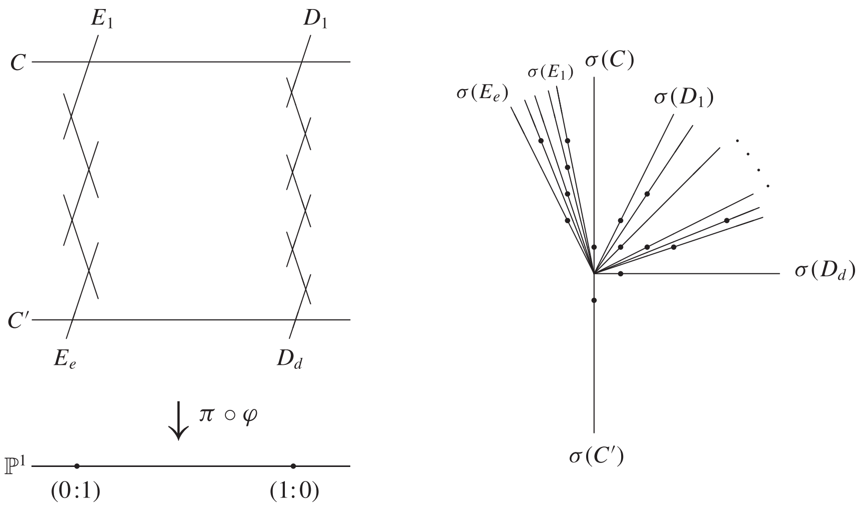

Fig. 3.

$m_j = (z_j, w_j)$ the primitive element of $\sigma(E_j)$. Then they satisfy the following properties.

$$(1)\qquad \begin{cases} x_1 = x_d = 1, \\ x_i \geq 1 \quad (2 \leq i \leq d-1), \\ y_i \geq 1 \quad (1 \leq i \leq d-1), \\ y_d = 0, \\ z_1 = z_e = -1, \\ z_j \leq -1 \quad (2 \leq j \leq e-1), \\ w_j \geq -z_j + 1 \quad (1 \leq j \leq e-1), \\ w_e = C'^2. \end{cases}$$

Note that we have $C'^2 = \Delta'^2 = a \geq 1$, since the center of the blowing-up φ lie outside Δ'. Furthermore, we have

$$(2)\qquad \begin{cases} D_i^2 = -\dfrac{x_{i-1} + x_{i+1}}{x_i} \quad (1 \leq i \leq d), \\ E_j^2 = -\dfrac{z_{j-1} + z_{j+1}}{z_j} \quad (1 \leq j \leq e), \end{cases}$$

where we put $x_0 = x_{d+1} = z_0 = z_{e+1} = 0$.

The Picard group of S is generated (not freely) by the classes of C', D_i ($1 \leq i \leq d$), and E_j ($1 \leq j \leq e$). When we take a divisor D on S, the linear equivalence class of D can be expressed with integers l, b_i, c_j as

$$D \sim lC' + \sum_{i=1}^{d} b_i D_i + \sum_{j=1}^{e} c_j E_j,$$

where "$\sim$" means linear equivalence. In particular, a computation using (2) shows that we can take non-negative integers l, b_i, and c_j if D is nef on S. A canonical divisor K_S of S is

$$K_S \sim -C - C' - \sum_{i=1}^{d} D_i - \sum_{j=1}^{e} E_j.$$

A general fiber F of ψ is

$$F \sim \sum_{i=1}^{d} x_i D_i \sim \sum_{j=1}^{e} -z_j E_j.$$

Moreover, we have

$$C \sim C' - \sum_{i=1}^{d-1} y_i D_i - \sum_{j=1}^{e} w_j E_j.$$

2. Key proposition

We keep the notation in the previous section. Let S be a toric surface as in the previous section, and X a curve of genus g on S. We put $k = X.F$. We say that the pair (S, X) satisfies the property ($\sharp$) (or, simply, X satisfies ($\sharp$)) if $C'^2 = 1$ and $X \sim kC'$. In this section, we shall prove the following proposition.

Proposition 2.1. *If $k \geq 2$ and X is nef but does not satisfy ($\sharp$), then $\mathcal{O}_S(X)|_X$ satisfies (M_{k-1}).*

To prove this proposition, we need several lemmas. We express the linear equivalence class of X as

$$X \sim kC' + \sum_{i=1}^{d} p_i D_i + \sum_{j=1}^{e} q_j E_j \tag{3}$$

with some integers p_i, q_j.

Lemma 2.2. *Suppose that $k \geq 2$ and X is nef on S. Then X is a rational curve if and only if $C'^2 = 1$, $k = 2$ and $X \sim 2C'$.*

Proof. The sufficency is easy: If $C'^2 = 1$ and $X \sim 2C'$, by a computation, we have $X.(X + K_S) = -2$. Then $g = (1/2)X.(X + K_S) + 1 = 0$. To prove the necessity, we assume $g = 0$. Then $h^0(S, X + K_S) = 0$ because $h^0(S, K_S) = h^0(X, K_X) = 0$. On the other hand, we have

$$X + K_S \sim (k-2)C + \sum_{i=1}^{d}((k-1)y_i + p_i - 1)D_i + \sum_{j=1}^{e}((k-1)w_j + q_j - 1)E_j.$$

Since X is nef, we can take non-negative integers p_i, q_j in the expression in (3). Furthermore, we have $C'^2 \geq 1$. Hence (1) shows that

$$\begin{cases}(k-1)y_i + p_i - 1 \geq 0 & (1 \leq i \leq d-1),\\ (k-1)w_j + q_j - 1 \geq 0 & (1 \leq j \leq e).\end{cases}$$

The equation $h^0(S, X + K_S) = 0$ implies that $X + K_S$ is not linearly equivalent to an effective divisor. Then p_d must be zero, and we have $X.D_d = p_{d-1} + k \geq k$. Since $X.F = X.\left(\sum_{i=1}^{d} x_i D_i\right) = k$, we obtain

$$X.D_i = \begin{cases}0 & (1 \leq i \leq d-1),\\ k & (i = d).\end{cases}$$

On the other hand, we have

$$\begin{aligned} X + K_S = X + K_S + F - F &\sim X + K_S + \sum_{i=1}^{d} x_i D_i + \sum_{j=1}^{e} z_j E_j \\ &\sim (k-2)C + \sum_{i=1}^{d}((k-1)y_i + x_i + p_i - 1)D_i \\ &\quad + \sum_{j=1}^{e}((k-1)w_j + z_j + q_j - 1)E_j. \end{aligned}$$

Since this is not an effective divisor, $(k-1)w_e + z_e + q_e - 1 = (k-1)w_e + q_e - 2$ must be less than zero. Noting that $w_e = C'^2 \geq 1$, we have $k = 2$, $w_e = 1$, and $q_e = 0$. Hence $X.E_e = q_{e-1} + 2 \geq 2$. Then, by the equation $X.F = X.\left(\sum_{j=1}^{e} -z_j E_j\right) = 2$, we obtain

$$X.E_j = \begin{cases}0 & (1 \leq j \leq e-1),\\ 2 & (j = e).\end{cases}$$

Moreover, we have $X.C' = p_d + kC'^2 + q_e = kw_e = 2$, and $X.C = X.\left(C' - \sum_{i=1}^{d-1} y_i D_i - \sum_{j=1}^{e} w_j E_j\right) = X.C' - X.E_e = 0$. In sum, we obtain that $C'^2 = 1$, $k = 2$, and X is numerically equivalent to $2C'$. Since S is simply connected, we also have $X \sim 2C'$. $\square$

Let I be a non-zero effective divisor on S, and put $H = X - I$.

Lemma 2.3. *If $H^1(S, -I) = 0$, then for any integer $p \geq h^0(S, H - I) + 1$,*

$$K_{p,1}(S, H) \simeq K_{p,1}(X, H|_X).$$

Proof. The short exact sequence of sheaves $0 \to \mathcal{O}_S(H-X) = \mathcal{O}_S(-I) \to \mathcal{O}_S(H) \to \mathcal{O}_S(H)|_X \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, -I) \to H^0(S, H) \to H^0(X, H|_X) \to H^1(S, -I) \to \cdots.$$

Since $H^0(S, -I) = H^1(S, -I) = 0$, we have $H^0(S, H) \simeq H^0(X, H|_X)$. We put $V = H^0(S, H)$, $B = \bigoplus_{q\geq 0} H^0(S, qH)$, $B' = \bigoplus_{q\geq 0} H^0(S, qH - X)$, and $A = B/B'$. By considering the short exact sequence $0 \to B' \to B \to A \to 0$, we obtain the Koszul cohomology long exact sequence

$$\cdots \to K_{p,1}(B', V) \to K_{p,1}(B, V) \to K_{p,1}(A, V) \to K_{p-1,2}(B', V) \to \cdots.$$

It is shown in [4, Theorem (3.a.1)] that

$$\begin{cases} K_{p,1}(B', V) = 0 & \text{if } p \geq h^0(S, H - X) = 0, \\ K_{p-1,2}(B', V) = 0 & \text{if } p \geq h^0(S, 2H - X) + 1 = h^0(S, H - I) + 1. \end{cases}$$

We thus have $K_{p,1}(S, H) \simeq K_{p,1}(A, V)$ for any integer $p \geq h^0(S, H - I) + 1$. On the other hand, let us consider the short exact sequence of graded SV-modules

$$0 \to A \to \bigoplus_{q\geq 0} H^0(X, qH|_X) \to C := \left(\bigoplus_{q\geq 0} H^0(X, qH|_X)\right)\Big/ A \to 0.$$

The isomorphisms $A_0 \simeq \mathbb{C}$ and $A_1 \simeq H^0(X, H|_X)$ imply $C_0 = C_1 = 0$. Thus we can apply [1, Remark 1.1] to obtain

$$K_{p,1}(A, V) \simeq K_{p,1}\left(\bigoplus_{q\geq 0} H^0(X, (qH)|_X), H^0(X, H|_X)\right) = K_{p,1}(X, H|_X)$$

for any integer p. □

For the proof of Lemma 2.6 below, we need the following two theorems.

Theorem 2.4 ([4, Theorem 3.c.1]). *Let L be a line bundle on a curve X and put $m = \dim \varphi_{|L|}(X)$. Then, for any integer $p \geq h^0(X, L) - m$,*

$$K_{p,1}(X, L) = 0.$$

Theorem 2.5 ([1, Theorem 1]). *Let X be a curve of genus $g \geq 1$, L a non-special and globally generated line bundle on X, and $k \geq 0$ an integer such that L satisfies (M_k). Then, for any effective divisor D on X, $L + D$ also satisfies the property (M_k).*

Lemma 2.6. *Suppose X is nef on S and $g \geq 1$. If all of the following* (i)–(v) *hold, then $\mathcal{O}_S(X)|_X$ satisfies (M_1).*
(i) *H is globally generated,*
(ii) $H^2 > 0$,
(iii) *$H|_X$ is non-special,*
(iv) $h^0(S, H) - h^0(S, H - I) \geq 3$,
(v) $H^1(S, -I) = 0$.

Proof. Since H is globally generated and $H^2 > 0$, by Bertini's theorem, we can take a non-singular irreducible curve $Y \in |H|$. Then Theorem 2.4 shows that $K_{p,1}(Y, H|_Y) = 0$ for any integer $p \geq h^0(Y, H|_Y) - \dim \varphi_{|H|_Y|}(Y) = h^0(Y, H|_Y) - 1$. The short exact sequence of sheaves $0 \to \mathcal{O}_S(H - Y) \simeq \mathcal{O}_S \to \mathcal{O}_S(H) \to \mathcal{O}_S(H)|_Y \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, H) \to H^0(Y, H|_Y) \to H^1(S, \mathcal{O}_S) \to \cdots.$$

Since $H^0(S, \mathcal{O}_S) = \mathbb{C}$ and $H^1(S, \mathcal{O}_S) = 0$, we get $h^0(Y, H|_Y) = H^0(S, H) - 1$. We thus have

$$K_{p,1}(Y, H|_Y) = 0 \tag{4}$$

for any integer $p \geq h^0(S, H) - 2$. On the other hand, by [1, Remark 1.3], we have $K_{p,1}(Y, H|_Y) \simeq K_{p,1}(S, H)$ for any integer p. Besides, we obtain that $K_{p,1}(S, H) \simeq K_{p,1}(X, H|_X)$ for any integer $p \geq h^0(S, H - I) + 1$ by Lemma 2.3. Hence, by combining these facts with (4) and (iv), we have

$$K_{p,1}(X, H|_X) = 0$$

for any integer $p \geq h^0(S, H) - 2$.

The short exact sequence $0 \to \mathcal{O}_S(H - X) = \mathcal{O}_S(-I) \to \mathcal{O}_S(H) \to \mathcal{O}_S(H)|_X \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, -I) \to H^0(S, H) \to H^0(X, H|_X) \to H^1(S, -I) \to \cdots.$$

The equality $H^0(S, -I) = H^1(S, -I) = 0$ implies $h^0(S, H) = h^0(X, H|_X)$. In sum, we conclude

$$K_{p,1}(X, H|_X) = 0$$

for any integer $p \geq h^0(X, H|_X) - 2$, that is, $H|_X$ satisfies (M_1). Now, $H|_X$ is non-special and globally generated. Moreover, $h^0(X, I|_X) \geq h^0(S, I) > 0$. Therefore, by Theorem 2.5, $\mathcal{O}_S(X)|_X$ also satisfies (M_1). □

In the rest of this section, we suppose $k \geq 2$ and X is nef, and put

$$\begin{aligned} d' &= \min\{i \mid D_i^2 \geq -1\}, \\ e' &= \min\{j \mid E_j^2 \geq -1\}, \\ I &= C + \sum_{i=1}^{d'-1} D_i + \sum_{j=1}^{e'-1} E_j + F, \\ H &= X - I. \end{aligned}$$

Lemma 2.7. *If $X.D_{d'} \geq 1$ and $X.E_{e'} \geq 1$, then the following hold.*
(i) *H is globally generated,*
(ii) *$H^2 > 0$,*
(iii) *$H|_X$ is non-special and globally generated.*

Proof. By [6], it is sufficient for (i) to verify that H has non-negative intersection numbers with C, C', D_i, and E_j. Firstly, for $1 \leq i \leq d' - 2$, we have

$$H.D_i = X.D_i - I.D_i = X.D_i - D_i^2 - 2 \geq X.D_i \geq 0.$$

Next, we see $H.D_{d'-1} = X.D_{d'-1} - D_{d'-1}^2 - 1 \geq -D_{d'-1}^2 - 1 \geq 1$, and $H.D_{d'} = X.D_{d'} - 1 \geq 0$. Moreover, for $d' + 1 \leq i \leq d$, we have $H.D_i = X.D_i \geq 0$. In sum, we obtain

$$H.D_i \geq \begin{cases} 0 & (i \neq d' - 1), \\ 1 & (i = d' - 1). \end{cases}$$

Similarly,

$$H.E_j \geq \begin{cases} 0 & (j \neq e' - 1), \\ 1 & (j = e' - 1). \end{cases}$$

We have $H.C' = X.C' - I.C' = p_d + kC'^2 + q_e - 1 \geq k - 1 \geq 1$. Finally, let us consider $H.C$. We have

$$I.C = C^2 + \begin{cases} 1 & (d' = e' = 1), \\ 3 & (d' \geq 2 \text{ and } e' \geq 2), \\ 2 & (\text{otherwise}), \end{cases}$$

$$C^2 \leq \begin{cases} -1 & (d' = e' = 1), \\ -3 & (d' \geq 2 \text{ and } e' \geq 2), \\ -2 & (\text{otherwise}) \end{cases}$$

to obtain $I.C \leq 0$. Hence we have $H.C \geq X.C \geq 0$ since X is nef. (ii) Since H is globally generated, it is nef and we can find non-negative integers b_i, c_j such that

$$H \sim (k-1)C' + \sum_{i=1}^{d} b_i D_i + \sum_{j=1}^{e} c_j E_j.$$

Then we have $H^2 \geq (k-1)H.C' \geq k-1 \geq 1$. (iii) is verified by a simple computation:

$$\begin{aligned}\deg H|_X - 2g = X.(-I-K_S) - 2 &= X.\left(C' + \sum_{i=d'}^{d} D_i + \sum_{j=e'}^{e} E_j - F\right) - 2\\ &\geq X.(C' + D_{d'} + E_{e'} - F) - 2 \geq X.(C' - F)\\ &= p_d + kC'^2 + q_e - k \geq 0.\end{aligned}$$

□

Lemma 2.8. *Suppose that X does not satisfy (♯). If $X.D_{d'} \geq 1$ and $X.E_{e'} \geq 1$, then $h^0(S, H) - h^0(S, H-I) \geq k+1$.*

Proof. By Lemma 2.7, H is globally generated and $H^2 > 0$. Then, by Bertini's theorem, we can take a non-singular irreducible curve $Y \in |H|$. We denote by $g(Y)$ its genus. As we saw in the proof of Lemma 2.6, we have $h^0(S, H) = h^0(Y, H|_Y) + 1$, and $h^0(S, H-I) = h^0(Y, (H-I)|_Y)$. Hence it is sufficient for the claim to verify $h^0(Y, H|_Y) - h^0(Y, (H-I)|_Y) \geq k$. Since

$$\begin{aligned}\deg H|_Y - 2g(Y) = Y.(-K_S) - 2 &= H.\left(C + C' + \sum_{i=1}^{d} D_i + \sum_{j=1}^{e} E_j\right) - 2\\ &\geq H.C' - 2 \geq -1,\end{aligned}$$

$H|_Y$ is non-special. On the other hand, we have

$$\begin{aligned}\deg(H-I)|_Y - 2g(Y) &= Y.(-I-K_S) - 2\\ &= H.\left(C' + \sum_{i=d'}^{d} D_i + \sum_{j=e'}^{e} E_j - F\right) - 2\\ &\geq H.(C'-F) - 2 = p_d + kC'^2 + q_e - k - 2.\end{aligned}$$

If $p_d = q_e = 0$ and $C'^2 = 1$, then we can show that X satisfies (♯) by the same argument as in the proof of Lemma 2.2. Hence we can assume that $p_d \geq 1$ or $q_e \geq 1$ or $C'^2 \geq 2$. It follows that $\deg(H-I)|_Y - 2g(Y) \geq -1$. Thus $(H-I)|_Y$ is also non-special. Hence

$$\begin{aligned}&h^0(Y, H|_Y) - h^0(Y, (H-I)|_Y)\\ &= \deg H|_Y + 1 - g(Y) - (\deg(H-I)|_Y + 1 - g(Y))\\ &= Y.H - Y.(H-I) = H.I.\end{aligned}$$

We consider the case of $d' \geq 2$. Then, as we saw in the proof of Lemma 2.7, $H.D_{d'-1} \geq 1$. We thus have

$$H.I = H.\left(C + \sum_{i=1}^{d'-1} D_i + \sum_{j=1}^{e'-1} E_j + F\right) = k - 1 + H.\left(C + \sum_{i=1}^{d'-1} D_i + \sum_{j=1}^{e'-1} E_j\right)$$
$$\geq k - 1 + H.D_{d'-1} \geq k.$$

Hence the claim is true if $d' \geq 2$. We can argue similarly in the case of $e' \geq 2$. Let us assume $d' = e' = 1$. Then $H.I = k - 1 + H.C = k + p_1 + q_1 - 2 - C^2$. If $p_1 \geq 1$ or $q_1 \geq 1$ or $C^2 \leq -2$, then we obtain $H.I \geq k$. On the other hand, if $p_1 = q_1 = 0$ and $C^2 = -1$, then $C'^2 = 1$, and X would saisfy (♯). □

Now, we show Proposition 2.1.

Proof of Proposition 2.1. We have $g \geq 1$ by Lemma 2.2. We denote by $\rho(S)$ (≥ 2) the Picard number of S. We will show the claim by the induction on $k + \rho(S)$. If $k = \rho(S) = 2$, then we have $X.D_{d'} \geq 1$ and $X.E_{e'} \geq 1$. Hence Lemma 2.7 and Lemma 2.8 allow us to apply Lemma 2.6 to X. Therefore, the claim is true in this case. Then, let us consider the case of $k + \rho(S) \geq 5$. Assume that $(X', \mathcal{O}_S(X')|_{X'})$ satisfies $(M_{k'-1})$ if $k' + \rho(S') < k + \rho(S)$, when we take S' and X', and define k' in the similar way as in the case of S and X.

(i) Suppose $X.D_{d'} \geq 1$ and $X.E_{e'} \geq 1$. If $k = 2$, then the claim is verified by Lemma 2.6. Assume that $k \geq 3$. We take a non-singular irreducible curve $Y \in |H|$. Then Y is nef, and $Y.F = k - 1$. Now, let us assume that Y satisfies (♯), that is, $C'^2 = 1$ and $Y \sim (k-1)C'$. Then we have

$$X \sim Y + I \sim (k-1)C' + C + \sum_{i=1}^{d'-1} D_i + \sum_{j=1}^{e'-1} E_j + F.$$

If $d' \geq 2$, then $X.D_{d'-1} = D_{d'-1}^2 + 1 < 0$. It contradicts the fact that X is nef. Hence $d' = 1$. Similarly, we obtain $e' = 1$. Hence, $X \sim (k-1)C' + C + F$. Since $X.C = C^2 + 1 \geq 0$, we see $C^2 = -1$. Then we have $C' \sim C + F$ and $X \sim kC'$. It contradicts the assumption that X does not satisfy (♯). Hence Y does not satisfy (♯). Then, by the hypothesis of the induction, $(Y, H|_Y)$ satisfies (M_{k-2}). That is,

$$K_{p,1}(Y, H|_Y) = 0$$

for any integer $p \geq h^0(Y, H|_Y) - k + 1$. Now, $h^0(Y, H|_Y) = h^0(S, H) - 1 = h^0(X, H|_X) - 1$. Moreover, by [1, Remark1.3], we have $K_{p,1}(Y, H|_Y) \simeq K_{p,1}(S, H)$ for any integer p. Then we have

(5) $$K_{p,1}(S, H) = 0$$

for any integer $p \geq h^0(X, H|_X) - k$.

On the other hand, the short exact sequence of sheaves $0 \to \mathcal{O}_S(-I) \to \mathcal{O}_S \to \mathcal{O}_I \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, -I) \to H^0(S, \mathcal{O}_S) \to H^0(I, \mathcal{O}_I) \to H^1(S, -I) \to H^1(S, \mathcal{O}_S) \to \cdots.$$

Since $H^0(S,-I) = H^1(S,\mathcal{O}_S) = 0$ and $H^0(S,\mathcal{O}_S) = H^0(I,\mathcal{O}_I) = \mathbb{C}$, we have $H^1(S,-I) = 0$. Hence, by Lemma 2.3, we have $K_{p,1}(S, H) \simeq K_{p,1}(X, H|_X)$ for any integer $p \geq h^0(S, H - I) + 1$. We remark that $h^0(S, H) - h^0(S, H - I) \geq k + 1$ holds by Lemma 2.8. In sum, combining these facts with (5), we obtain

$$K_{p,1}(X, H|_X) = 0$$

for any integer $p \geq h^0(X, H|_X) - k$, that is, $(X, H|_X)$ satisfies (M_{k-1}). Now, $H|_X$ is non-special and globally generated by Lemma 2.7. Hence, by Theorem 2.5, $(X, \mathcal{O}_S(X)|_X)$ also satisfies (M_{k-1}).

(ii) Suppose $X.D_{d'} = 0$. In this case, it is obvious that $d \geq 2$, $\rho(S) \geq 3$, and $D_{d'}^2 = -1$. Let S' be a surface obtained from S by blowing $D_{d'}$ down, and F' be a general fiber of the map $S' \to \mathbb{P}^1$. Then we can regard $X \subset S'$. We denote by $\rho(S')$ the Picard number of S'. We have $\rho(S') = \rho(S) - 1$ and $X.F' = k$. Hence, by the hypothesis of the induction, $(X, \mathcal{O}_{S'}(X)|_X)$ satisfies (M_{k-1}). Therefore, the claim is verified.

(iii) If $X.E_{e'} = 0$, then we can show the claim by the same argument as in (ii). □

3. Proof of Main Theorem

For the proof of the Main Theorem, we need the following result.

Theorem 3.1 ([1, Corollary 2]). *Let X be a curve of genus $g \geq 1$, which carries a g_k^1. If there is a non-special and globally generated line bundle on X satisfing (M_{k-1}), then X is k-gonal, and the gonality conjecture is valid for X.*

Proof of Theorem 0.3. (a) If $k = 0$, then X is contained in a fiber. Hence X is rational. If $k = 1$, then ψ induces a morphism from X to $\mathbb{P}^1$ of degree 1. Hence X is rational. If X is not nef on S, then $X^2 < 0$. We thus have

$$X.(X + K_S) = X^2 + X.\left(-C - C' - \sum_{i=1}^{d} D_i - \sum_{j=1}^{e} E_j\right) < 0.$$

It follows that X is rational. So we may assume that $k \geq 2$ and X is nef on S. Since $X.F = k$, then gonality of X is at most k.

(b) Suppose X satisfies ($\sharp$). Then we can regard X as a curve which is linearly equivalent to $k\Delta'$ on Σ_1 by a finite succession of blowing-downs along D_i or E_j which has the self-intersection number -1 and disjoint from X. Then, by blowing-down along the minimal section Δ, X can be regarded as a plane curve of degree k.

(c) Suppose X does not satisfy ($\sharp$). Then we have $g \geq 1$ by Lemma 2.2. Moreover, Proposition 2.1 shows that $\mathcal{O}_S(X)|_X$ satisfies (M_{k-1}). On the other hand, since

$$\deg \mathcal{O}_S(X)|_X - 2g = X.(-K_S) - 2 \geq X.C' - 2 = p_d + kC'^2 + q_e - 2 \geq 0,$$

$\mathcal{O}_S(X)|_X$ is non-special and globally generated. Therefore, it follows from Theorem 3.1 that X is k-gonal and the gonality conjecture is valid for X. □

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