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Harada, M.
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# ON QUASI-INJECTIVE MODULES WITH A CHAIN CONDITION OVER A COMMUTATIVE RING

MANABU HARADA

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In the previous paper [4] the author and T. Ishii studied the endomorphism rings of noetherian quasi-injective modules. As an application of it, we consider, in this paper, quasi-injective modules over a commutative ring $R$. If $R$ is noetherian, E. Matlis decided every indecomposable injective modules in [6].

Greatly making use of those results in [6], we shall decide all quasi-injective (resp. injective) modules which are either artinian or noetherian in §§2 and 3. Especially, we shall give necessary and sufficient conditions of $R$ for existence of quasi-injective (resp. injective) modules which are either artinian or noetherian (cf. [7], Theorem 5).

In this paper, a ring $R$ is always commutative unless otherwise stated and every $R$-module is unitary.

## 1. Preliminaries

Let $K$ be any ring (not necessarily commutative) and $M$ a right $K$-module. Put $S=\mathrm{Hom}_K(M, M)$, then we assume that $M$ is a left $S$-module. Let $N$ be a subset of $M$. Then we denote the annihilator ideal of $N$ in $S$ and in $K$ by $l(N)$ and ann $N$, respectively. Similarly, by $r(A)$ we denote the annihilator submodule of $M$ for a left ideal $A$ in $S$.

We call $M$ a *weakly distinguished* $K$-module if for any $K$-submodules $N_1 \supset N_2$ in $M$ such that $N_1/N_2$ is $K$-irreducible, $\mathrm{Hom}_K(N_1/N_2, M) \neq 0$. If $M$ is $K$-quasi-injective, then $M$ is weakly distinguished if and only if $rl(N)=N$ for any $K$-submodule $N$ in $M$, (see [1], Proposition 6).

Finally, we shall add here some direct consequences of [4]. From now on we shall assume that a ring $R$ is commutative.

**Proposition 1.** *Let $R$ be a commutative ring and $M$ a quasi-injective module. If $M$ is noetherian as an $R$-module, then $S=\mathrm{Hom}_R(M, M)$ is left and right artinian*, (see Theorem 1 below).

Proof. Since $R$ is commutative, $S$ is an $R$-submodule of a finite directsum of copies of $M$. Therefore, $S$ is artinian by [4], Theorem 1.

**Propostion 2,** *Let $R$ and $M$ be as above. We assume further that $M$ is weakly distinguished. Put $S=\mathrm{Hom}_R(M, M)$. Then $M$ is $R$-noetherian if and only if $S$ is left artinian. In this case, $M$ is $R$-artinian, $S$-injective and $R/A$ is artinian, where $A=\mathrm{ann}\, M$.*

Proof. If $M$ is $R$-noetherian, S is artinian by Proposition 1. Hence, $M$ is $S$-injective by [4], Theorem 2 and $M$ is $R$-artinian from the above remark, since $S$ is noetherian. Further, $R/A$ is an $R$-submodule of finite directsum of copies of $M$. Hence, $R/A$ is artinian. If $S$ is (left) artinian, then $M$ is $R$-noetherian as above.

## 2. Noetherian quasi-injective modules

We shall decide quasi-injective noetherian modules in this section.

**Lemma 1.** *Let $K$ be any ring and $M$ a quasi-injective and weakly distinguished right $K$-module. Put $S=\mathrm{Hom}_K(M, M)$ and $T=\mathrm{Hom}_S(M, M)$. Then every $K$-submodule of $M$ is a $T$-submodule of $M$.*

Proof. Let $N$ be a $K$-module of $M$. Then $rl(N)=N$ by the remark in §1. Hence, $N$ is a $T$-module.

Let $R$ be a commutative noetherian ring and $P$ a prime ideal in $R$. Let $\mathrm{E}(R/P)=E$ be an injective hull of $R/P$. Then Matlis showed in [6] that $E=\bigcup_i A_i$ and $\mathrm{Hom}_R(E, E)$ is a complete local noetherian ring, where $A_i=\{x \mid \in E, xP^i=0\}$.

**Lemma 2.** *Let $R$ be a commutative noetherian ring and $\{P_i\}$ a finite set of distinct maximal ideals in $R$. Then every $R$-submodule $N$ of $\Sigma\oplus\mathrm{E}(R/P_i)$ is weakly distinguished and quasi-injective.*

Proof. We may assume that $N$ is an essential submodule of $E=\Sigma\oplus E_i$, where $E_i=\mathrm{E}(R/P_i)$. Then $\mathrm{ann}\, x\supset\Pi P_i^n$ for any $x$ in $N$. Let $N_1$, $N_2$ be $R$-submodules of $N$ such that $N_1/N_2$ is $R$-irreducible, then $N_1/N_2\approx R/P_i$ for some $P_i$. Since $N\cap R/P_i\neq(0)$, $\mathrm{Hom}_R(N_1/N_2, N)\neq(0)$, which means that $N$ is weakly distinguished. Hence, $E$ is an $R$-weakly distinguished injective module. Moreover, if we put $S=\mathrm{Hom}_R(E, E)$, $S=\mathrm{Hom}_S(E, E)$. Hence, every $R$-submodule $M$ is an $S$-submodule by Lemma 1. Let $E'$ be an injective hull of $M$ contained in $E$. Then $E=E'\oplus E''$ and $E'\supset M$. $S'=\mathrm{Hom}_R(E', E')$ may be regarded as a subring of $S$. Hence, $M$ is also an $S'$-module. Therefore, $M$ is $R$-quasi-injective by [5], Theorem 1. 1.

We are interested in a noetherian or artinian quasi-injective module $M$ and hence, we may assume that $M$ is directly indecomposable.

**Theorem 1.** *Let $M$ be a directly indecomposable module over a commutative ring $R$. Then $M$ is quasi-injective and noetherian if and only if there exist an ideal*

*I such that $R/I$ is noetherian and a maximal ideal P containing I and M is contained in a submodule $A_n$ of $E_{R/I}(R/P)$. In this case, M is R-artinian, and hence $R/I$ is artinian*[1].

Proof. We assume that $M$ is $R$-noetherian and quasi-injective. Put $I = \mathrm{ann}\, M$. Then $\bar{R} = R/I$ is noetherian as the proof of Proposition 2. Hence, we may assume that $R$ is noetherian. Let $E$ be an injective hull of $M$. Then $E = E_R(R/P)$ with $P$ prime by [6], Proposition 3. 1. Put $S = \mathrm{Hom}_R(E, E)$. We know from [6], Theorem 3. 4 and its proof that $A_1 = S(R/P) \approx Sa \approx K$ for any non-zero element $a$ in $A_1$, where $K$ is the quotient field of $R/P$. Since $M \cap A_1 \neq (0)$ and $M$ is quasi-injective, $M$ contains a submodule which is isomorphic to $K$ by [5], Theorem 1. 1. Hence, $P$ is a maximal ideal in $R$, and $M$ is contained in some $A_n$, since $M$ is $R$-finitely generated and each $A_n$ has a composition length by [6], Theorem 3. 9. The remaining part is clear from the above and Lemma 2.

**Corollary.** *Let R be a commutative ring. Then there exists a noetherian injective module if and only if R contains a maximal ideal P such that $R_P$ is artinian,* (cf. [6], Theorem 3. 11).

Proof. It is an immediate consequence of Theorem 1 and [7], Theorem 5,

## 3. Artinian, quasi-injective modules

We shall decide quasi-injective, (resp. injective) artinian modules in this section.

**Theorem 2.** *Let R be a commutative ring and M a directly indecomposable R-module and $S = \mathrm{Hom}_R(M, M)$. If M is quasi-injective and artinian, then*

i. *There exists a maximal ideal P in R such that $M = \cup A_i$, where $A_i = \{x \mid \in M, xP^i = 0\}$, and M may be regarded as an $R_P$-module and $R_P$-quasi-injective.*

ii. *M is S-injective and S is a commutative $\mathfrak{P}$-adic complete local noetherian ring, where $\mathfrak{P}$ is a unique maximal ideal of S. Furthermore, the set of the S-submodules of M coincides with that of R-submodules of M.*

iii. *R is dense in S with repect to $\mathfrak{P}$-adic topology and hence, for any finite elements $m_i$ in M and an element s in S, there exists an element r in R such that $m_i s = m_i r$ for all i.*

*Conversely, if S satisfies the first parts of* ii *and* iii, *then M is a quasi-injective and artinian R-module.*

Proof. We assume that $M$ is a quasi-injective and artinian $R$-module. Let $m \neq 0$ be an element in $M$, then $mR \approx R/\mathrm{ann}\, m$ is an artinian ring. Hence, there

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Added in proof: 1) In this case $M$ is R/Ann $M$-injective by Theorem 1 of C. Faith *Modules finite over endomorphism ring,* Lecture Notes in Math., Springer, Heidelberg, 246.

exists a unique maximal ideal $P$ such that $P \supset \text{ann } m$ and $P^n \subset \text{ann } m$, since $M$ is indecomposable and quasi-injective. Therefore, $M$ contains a unique minimal $R$-module $R/P$ and $P$ does not depend on a choice of $m$. Let $s$ be in $R-P$ and $x \in l(s) \cap R/P$. Since $P$ is maximal, there exist $p \in P$, $r \in R$ such that $1=p+rs$. Hence, $x=xp+xrs=0$. Therefore, $l(s)=(0)$. Since $M$ is artinian, $s$ gives an automorphism of $M$. Hence, $M$ may be regarded as an $R_P$-module. It is clear that $M$ is $R_P$-quasi-injective.

ii. Put $S=\text{Hom}_R(M, M)$. Then $S$ is left noetherian by [3], Proposition 1. Furthermore, we know from [3], Theorem 2 that $M$ is $S$-injective, since $M$ is $R$-weakly distinguished (cf. the proof of Lemma 2 and i). On the other hand, we put $S'=\text{Hom}_S(M, M)$. Then $S' \subset S$ and hence, $S'$ is the center of $S$. Moreover, since $M$ is an artinian $S$-injective, $S'$ is noetherian as above. Let $N$ be the radical of $S$ then $S/N$ is a division ring by [2], Theorem 1 in p. 44 and Theorem 6 in p. 48, and $R/P \approx S/N$ as $S$-modules. Hence, $M$ is $S$-weakly distinguished. Thus, $M$ is also $S'$-injective as above. Since $S=\text{Hom}_{S'}(M, M)$, $S=S'$ is a complete local ring with respect to a $\mathfrak{P}$-adic topology by [6], Theorem 3. 7, where $\mathfrak{P}$ is a unique maximal ideal in $S'$ and $\mathfrak{P} \cap R=P$. The last part of ii is clear from the above and Lemma 1.

iii. The following argument is analogous to [6], Theorem 3. 7. Put $\bar{A}_i=\{x \mid \in M, x\mathfrak{P}^i=0\}$. We shall show for $s$ in $S$ that there exists $r_i$ in $R$ for each $\bar{A}_i$ such that $l(s-r_i) \supset \bar{A}_i$, Since $\bar{A}_1=R/P=S/\mathfrak{P}$, we have $r_1$. We assume that there exists $r_i$ in $R$ such that $l(s-r_i) \supset \bar{A}_i$. Let $\{m_1, m_2, \cdots, m_t\}$ be a system of minimal generators of $\bar{A}_{i+1}$ as an $S$-module (see Theorem 1), then we obtain elements $b_i$ in $R$ such that $m_ib_i \neq 0$, $m_jb_i=0$ if $i \neq j$ by [5], Theorem 2. 3. Put $g=s-r_i$, $g(\bar{A}_i)=0$ and hence, $g(m_i)\mathfrak{P}=g(m_i \mathfrak{P})=0$, which means $g(m_i) \subset \bar{A}_1$. Since $\bar{A}_1$ is essential in $M$ as an $R$-module and $R/P$ is irreducible, there exists $c_i$ in $R$ such that $m_ib_ic_i=g(m_i)$ for each $i$. Put $r'_{i+1}=\Sigma\, b_jc_j$, then $g(m_j)=m_jb_jc_j=m_jr'_{i+1}$ for all $j$. Hence, $(s-(r_i+r'_{i+1}))\bar{A}_{i+1}=(0)$. Since $r(\bar{A}_{j+1})=\mathfrak{P}^{j+1}$ by [6], Theorem 3. 4, $s=\lim r_j$, $r_j \in R$. Let $\{m_i\}$ be a finite elements in $M$, then there exists an $\bar{A}_n$ containing all $m_i$. Hence, if we take an element $r$ in $R$ such that $s-r \in \mathfrak{P}^n$, $m_ir=m_is$ for all $i$.

Conversely, we assume that $S$ satisfies the first parts of ii and iii. Then every $R$-submodule $N$ of $M$ is an $S$-module and every $R$-homomorphism of $N$ to $M$ is an $S$-homomorphism. Hence, $M$ is a quasi-injective and artinian by Lemma 2, since $M$ is $S$-artinan.

**Corollary.** *Let $M$, $R$ and $S$ be as above. If $M$ is a quasi-injective, artinian $R$-module, then for any intermediate ring $T$ between $R$ and $S$, $M$ is $T$-quasi-injective.*

REMARK. In Theorem 2 we have shown that $S$ is noetherian, however $R/A$ is not noetherian in general, where $A=\text{ann } M$. For example, let $Z$ be the ring of integers and $P$ a prime. $Z_{P^\infty}$ is $Z_P$-artinian, injective and indecomposable.

We can obtain a non-noetherian intermediate local ring $T$ between $Z_P$ and $\hat{Z}_P=\mathrm{Hom}_{Z_P}(Z_{P^\infty}, Z_{P^\infty})$ (see [3], Lemma 1) and $M$ is $T$-quasi-injective and $T$-artinian.

Next, we shall consider a case of injective modules.

**Theorem 3.** *Let $R$ be a commutative ring and $M$ an $R$-artinian, injective module. Then there exists a finite set of maximal ideals $P_1, P_2, \cdots, P_n$ such that $R_T$ is noetherian, where $T=R-(P_1\cup P_2\cup\cdots\cup P_n)$ and $n$ is the number of non-isomorphic indecomposable direct summands of $M$. Conversely, if $R_T$ is noetherian, there exists an $R$-artinian, injective module which is a directsum of $n$ non isomorphic indecomposable modules.*

Proof. Let $M=\sum_1^n\oplus M_i$ and the $M_i$ be directly indecomposable. We may assume $M_i\not\approx M_j$ if $i\neq j$. Each $M_i$ corresponds to a maximal ideal $P_i$ and $M_i$ may be regraded as $R_{P_i}$-module by Theorem 2. Further, $M_i$ is an injective hull of $R/P_i$ as an $R$-module. Put $T=R-(P_1\cup\cdots\cup P_n)$, then $R_T/P_iR_T\approx R/P$. Hence, $M$ is an $R_T$-cogenerator. Therefore, $R_T$ is noetherian by [8], Lemma 2. Conversely, we assume $R_T$ is noetherian and put $M_i=\mathrm{E}_R(R/P_i)$. Since $R/P_i$ is a unique minimal sub-module of $M_i$, $M_i=\mathrm{E}_{R_{P_i}}(R/P_i)$. Let $\varphi_i$; $R\to R_{P_i}$ be the canonical homomorphism. Then the operation of elements $r$ in $R$ on $M_i=\mathrm{E}_{R_{P_i}}(R/P_i)$ is given via $\varphi_i$. Hence, $M_i=\mathrm{E}_{R_T}(R_T/P_iR_T)$ and $\mathrm{Hom}_{R_{P_i}}(M_i, M_i)=\mathrm{Hom}_R(M_i, M_i)$ by the standard argument. Furthermore, since $R_{P_i}$ is noetherian, for any element $x$ in $M_i$ $\mathrm{ann}_{R_{P_i}} x\supseteq P_i^{n_i}R_{P_i}\supseteq\varphi_i(P_i^{n_i})$ for some $n_i$ and hence, $xP_i^{n_i}=(0)$. Put $M=\Sigma\oplus M_i$, then $M$ is an $R$-weakly distinguished module from the above, (cf. the proof of Lemma 2). Since $R_T$ is noetherian, $\mathrm{Hom}_{R_T}(M, M)=\Sigma\oplus\mathrm{Hom}_{R_{P_i}}(M_i, M_i)=\mathrm{Hom}_R(M, M)$ is noetherian by [6], Theorem 3.9. Therefore, $M$ is $R$-artinian, since $M$ is $R$-weakly distinguished.

**Lemma 3.** *Let $R$ be a local noetherian ring with maximal ideal $P$ and $M=\mathrm{E}_R(R/P)$. Let $S=\mathrm{Hom}_R(M, M)$ and $T$ be an intermediate ring between $R$ and $S$. If for any element $x$ in $\mathrm{E}_T(M)$, $xP^n=(0)$ for some $n$, then $M$ is $T$-injective.*

Proof. $\mathrm{E}_T(M)=M\oplus K$ as $R$-modules. If $K\neq(0)$, for any $k\neq 0$ in $K$, $kP^n=(0)$ by the assumption. Hence, $\mathrm{ann}\ k'=P$ for some $k'\in K$. Since $\mathrm{E}_T(M)$ is indecomposable, it contains a unique minimal $T$-module $R/P$. Which is a contradiction.

**Proposition 3.** *Let $R$, $M$ and $S$ be as in Lemma 3. Then for any intermediate local ring $T$ between $R$ and $S$, $M$ is $T$-injective if and only if $T$ is noetherian and $\mathfrak{P}\cap T=P'$, where $\mathfrak{P}$ and $P'$ are maximal ideals in $S$ and $T$, respectively.*

Proof. "Only if part" is an immediate consequence of Theorem 3. We assume that $T$ is noetherian as in the proposition. Since $M=\mathrm{E}_R(R/P)$ and

$(R/P)S=R/P$, $R/P\approx T/P'$ and $P'\cap R=P$. Let $M'=E_T(T/P')$, then for any $x$ in $M'$ $xP^n\subset xP'^n=(0)$ for some $n$. Hence, $M=M'$ by Lemma 3.

REMARK. Let $Z$, $P$ be as in the previous remark. Then there exists a tower of noetherian local rings $Z_P\subset R_1\subset R_2\subset\cdots$ such that $R_i$ dominates $R_{i-1}$ and $T=\cup R_i$ is not noetherian. Then $M$ is $R_i$-injective for each $i$, but not $T$-injective.

OSAKA CITY UNIVERSITY

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