



Title	On quasi-injective modules with a chain condition over a commutative ring
Author(s)	Harada, Manabu
Citation	Osaka Journal of Mathematics. 1972, 9(3), p. 421-426
Version Type	VoR
URL	<a href="https://doi.org/10.18910/12866">https://doi.org/10.18910/12866</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## ON QUASI-INJECTIVE MODULES WITH A CHAIN CONDITION OVER A COMMUTATIVE RING

MANABU HARADA

(Received November 25, 1971)

In the previous paper [4] the author and T. Ishii studied the endomorphism rings of noetherian quasi-injective modules. As an application of it, we consider, in this paper, quasi-injective modules over a commutative ring  $R$ . If  $R$  is noetherian, E. Matlis decided every indecomposable injective modules in [6].

Greatly making use of those results in [6], we shall decide all quasi-injective (resp. injective) modules which are either artinian or noetherian in §§2 and 3. Especially, we shall give necessary and sufficient conditions of  $R$  for existence of quasi-injective (resp. injective) modules which are either artinian or noetherian (cf. [7], Theorem 5).

In this paper, a ring  $R$  is always commutative unless otherwise stated and every  $R$ -module is unitary.

### 1. Preliminaries

Let  $K$  be any ring (not necessarily commutative) and  $M$  a right  $K$ -module. Put  $S = \text{Hom}_K(M, M)$ , then we assume that  $M$  is a left  $S$ -module. Let  $N$  be a subset of  $M$ . Then we denote the annihilator ideal of  $N$  in  $S$  and in  $K$  by  $l(N)$  and  $\text{ann } N$ , respectively. Similarly, by  $r(A)$  we denote the annihilator submodule of  $M$  for a left ideal  $A$  in  $S$ .

We call  $M$  a *weakly distinguished*  $K$ -module if for any  $K$ -submodules  $N_1 \supset N_2$  in  $M$  such that  $N_1/N_2$  is  $K$ -irreducible,  $\text{Hom}_K(N_1/N_2, M) \neq 0$ . If  $M$  is  $K$ -quasi-injective, then  $M$  is weakly distinguished if and only if  $r(l(N)) = N$  for any  $K$ -submodule  $N$  in  $M$ , (see [1], Proposition 6).

Finally, we shall add here some direct consequences of [4]. From now on we shall assume that a ring  $R$  is commutative.

**Proposition 1.** *Let  $R$  be a commutative ring and  $M$  a quasi-injective module. If  $M$  is noetherian as an  $R$ -module, then  $S = \text{Hom}_R(M, M)$  is left and right artinian, (see Theorem 1 below).*

**Proof.** Since  $R$  is commutative,  $S$  is an  $R$ -submodule of a finite directsum of copies of  $M$ . Therefore,  $S$  is artinian by [4], Theorem 1.

**Proposition 2.** *Let  $R$  and  $M$  be as above. We assume further that  $M$  is weakly distinguished. Put  $S = \text{Hom}_R(M, M)$ . Then  $M$  is  $R$ -noetherian if and only if  $S$  is left artinian. In this case,  $M$  is  $R$ -artinian,  $S$ -injective and  $R/A$  is artinian, where  $A = \text{ann } M$ .*

*Proof.* If  $M$  is  $R$ -noetherian,  $S$  is artinian by Proposition 1. Hence,  $M$  is  $S$ -injective by [4], Theorem 2 and  $M$  is  $R$ -artinian from the above remark, since  $S$  is noetherian. Further,  $R/A$  is an  $R$ -submodule of finite directsum of copies of  $M$ . Hence,  $R/A$  is artinian. If  $S$  is (left) artinian, then  $M$  is  $R$ -noetherian as above.

## 2. Noetherian quasi-injective modules

We shall decide quasi-injective noetherian modules in this section.

**Lemma 1.** *Let  $K$  be any ring and  $M$  a quasi-injective and weakly distinguished right  $K$ -module. Put  $S = \text{Hom}_K(M, M)$  and  $T = \text{Hom}_S(M, M)$ . Then every  $K$ -submodule of  $M$  is a  $T$ -submodule of  $M$ .*

*Proof.* Let  $N$  be a  $K$ -module of  $M$ . Then  $rl(N) = N$  by the remark in §1. Hence,  $N$  is a  $T$ -module.

Let  $R$  be a commutative noetherian ring and  $P$  a prime ideal in  $R$ . Let  $E(R/P) = E$  be an injective hull of  $R/P$ . Then Matlis showed in [6] that  $E = \bigcup_i A_i$  and  $\text{Hom}_R(E, E)$  is a complete local noetherian ring, where  $A_i = \{x \in E, xP^i = 0\}$ .

**Lemma 2.** *Let  $R$  be a commutative noetherian ring and  $\{P_i\}$  a finite set of distinct maximal ideals in  $R$ . Then every  $R$ -submodule  $N$  of  $\Sigma \oplus E(R/P_i)$  is weakly distinguished and quasi-injective.*

*Proof.* We may assume that  $N$  is an essential submodule of  $E = \Sigma \oplus E_i$ , where  $E_i = E(R/P_i)$ . Then  $\text{ann } x \supset \prod P_i^n$  for any  $x$  in  $N$ . Let  $N_1, N_2$  be  $R$ -submodules of  $N$  such that  $N_1/N_2$  is  $R$ -irreducible, then  $N_1/N_2 \approx R/P_i$  for some  $P_i$ . Since  $N \cap R/P_i \neq (0)$ ,  $\text{Hom}_R(N_1/N_2, N) \neq (0)$ , which means that  $N$  is weakly distinguished. Hence,  $E$  is an  $R$ -weakly distinguished injective module. Moreover, if we put  $S = \text{Hom}_R(E, E)$ ,  $S = \text{Hom}_S(E, E)$ . Hence, every  $R$ -submodule  $M$  is an  $S$ -submodule by Lemma 1. Let  $E'$  be an injective hull of  $M$  contained in  $E$ . Then  $E = E' \oplus E''$  and  $E' \supset M$ .  $S' = \text{Hom}_R(E', E')$  may be regarded as a subring of  $S$ . Hence,  $M$  is also an  $S'$ -module. Therefore,  $M$  is  $R$ -quasi-injective by [5], Theorem 1.1.

We are interested in a noetherian or artinian quasi-injective module  $M$  and hence, we may assume that  $M$  is directly indecomposable.

**Theorem 1.** *Let  $M$  be a directly indecomposable module over a commutative ring  $R$ . Then  $M$  is quasi-injective and noetherian if and only if there exist an ideal*

*I such that  $R/I$  is noetherian and a maximal ideal  $P$  containing  $I$  and  $M$  is contained in a submodule  $A_n$  of  $E_{R/I}(R/P)$ . In this case,  $M$  is  $R$ -artinian, and hence  $R/I$  is artinian<sup>1)</sup>.*

**Proof.** We assume that  $M$  is  $R$ -noetherian and quasi-injective. Put  $I = \text{ann } M$ . Then  $\bar{R} = R/I$  is noetherian as the proof of Proposition 2. Hence, we may assume that  $R$  is noetherian. Let  $E$  be an injective hull of  $M$ . Then  $E = E_R(R/P)$  with  $P$  prime by [6], Proposition 3. 1. Put  $S = \text{Hom}_R(E, E)$ . We know from [6], Theorem 3. 4 and its proof that  $A_1 = S(R/P) \approx Sa \approx K$  for any non-zero element  $a$  in  $A_1$ , where  $K$  is the quotient field of  $R/P$ . Since  $M \cap A_1 \neq (0)$  and  $M$  is quasi-injective,  $M$  contains a submodule which is isomorphic to  $K$  by [5], Theorem 1. 1. Hence,  $P$  is a maximal ideal in  $R$ , and  $M$  is contained in some  $A_n$ , since  $M$  is  $R$ -finitely generated and each  $A_n$  has a composition length by [6], Theorem 3. 9. The remaining part is clear from the above and Lemma 2.

**Corollary.** *Let  $R$  be a commutative ring. Then there exists a noetherian injective module if and only if  $R$  contains a maximal ideal  $P$  such that  $R_P$  is artinian, (cf. [6], Theorem 3. 11).*

**Proof.** It is an immediate consequence of Theorem 1 and [7], Theorem 5,

### 3. Artinian, quasi-injective modules

We shall decide quasi-injective, (resp. injective) artinian modules in this section.

**Theorem 2.** *Let  $R$  be a commutative ring and  $M$  a directly indecomposable  $R$ -module and  $S = \text{Hom}_R(M, M)$ . If  $M$  is quasi-injective and artinian, then*

i. *There exists a maximal ideal  $P$  in  $R$  such that  $M = \bigcup A_i$ , where  $A_i = \{x \in M, xP^i = 0\}$ , and  $M$  may be regarded as an  $R_P$ -module and  $R_P$ -quasi-injective.*

ii.  *$M$  is  $S$ -injective and  $S$  is a commutative  $\mathfrak{P}$ -adic complete local noetherian ring, where  $\mathfrak{P}$  is a unique maximal ideal of  $S$ . Furthermore, the set of the  $S$ -submodules of  $M$  coincides with that of  $R$ -submodules of  $M$ .*

iii.  *$R$  is dense in  $S$  with respect to  $\mathfrak{P}$ -adic topology and hence, for any finite elements  $m_i$  in  $M$  and an element  $s$  in  $S$ , there exists an element  $r$  in  $R$  such that  $m_i s = m_i r$  for all  $i$ .*

*Conversely, if  $S$  satisfies the first parts of ii and iii, then  $M$  is a quasi-injective and artinian  $R$ -module.*

**Proof.** We assume that  $M$  is a quasi-injective and artinian  $R$ -module. Let  $m \neq 0$  be an element in  $M$ , then  $mR \approx R/\text{ann } m$  is an artinian ring. Hence, there

---

Added in proof: 1) In this case  $M$  is  $R/\text{Ann } M$ -injective by Theorem 1 of C. Faith *Modules finite over endomorphism ring*, Lecture Notes in Math., Springer, Heidelberg, 246.

exists a unique maximal ideal  $P$  such that  $P \supset \text{ann } m$  and  $P^n \subset \text{ann } m$ , since  $M$  is indecomposable and quasi-injective. Therefore,  $M$  contains a unique minimal  $R$ -module  $R/P$  and  $P$  does not depend on a choice of  $m$ . Let  $s$  be in  $R-P$  and  $x \in l(s) \cap R/P$ . Since  $P$  is maximal, there exist  $p \in P, r \in R$  such that  $1 = p + rs$ . Hence,  $x = xp + xrs = 0$ . Therefore,  $l(s) = (0)$ . Since  $M$  is artinian,  $s$  gives an automorphism of  $M$ . Hence,  $M$  may be regarded as an  $R_P$ -module. It is clear that  $M$  is  $R_P$ -quasi-injective.

ii. Put  $S = \text{Hom}_R(M, M)$ . Then  $S$  is left noetherian by [3], Proposition 1. Furthermore, we know from [3], Theorem 2 that  $M$  is  $S$ -injective, since  $M$  is  $R$ -weakly distinguished (cf. the proof of Lemma 2 and i). On the other hand, we put  $S' = \text{Hom}_S(M, M)$ . Then  $S' \subset S$  and hence,  $S'$  is the center of  $S$ . Moreover, since  $M$  is an artinian  $S$ -injective,  $S'$  is noetherian as above. Let  $N$  be the radical of  $S$  then  $S/N$  is a division ring by [2], Theorem 1 in p. 44 and Theorem 6 in p. 48, and  $R/P \approx S/N$  as  $S$ -modules. Hence,  $M$  is  $S$ -weakly distinguished. Thus,  $M$  is also  $S'$ -injective as above. Since  $S = \text{Hom}_{S'}(M, M)$ ,  $S = S'$  is a complete local ring with respect to a  $\mathfrak{P}$ -adic topology by [6], Theorem 3.7, where  $\mathfrak{P}$  is a unique maximal ideal in  $S'$  and  $\mathfrak{P} \cap R = P$ . The last part of ii is clear from the above and Lemma 1.

iii. The following argument is analogous to [6], Theorem 3.7. Put  $\bar{A}_i = \{x \in M, x\mathfrak{P}^i = 0\}$ . We shall show for  $s$  in  $S$  that there exists  $r_i$  in  $R$  for each  $\bar{A}_i$  such that  $l(s - r_i) \supset \bar{A}_i$ . Since  $\bar{A}_1 = R/P = S/\mathfrak{P}$ , we have  $r_1$ . We assume that there exists  $r_i$  in  $R$  such that  $l(s - r_i) \supset \bar{A}_i$ . Let  $\{m_1, m_2, \dots, m_i\}$  be a system of minimal generators of  $\bar{A}_{i+1}$  as an  $S$ -module (see Theorem 1), then we obtain elements  $b_i$  in  $R$  such that  $m_i b_i \neq 0, m_j b_i = 0$  if  $i \neq j$  by [5], Theorem 2.3. Put  $g = s - r_i, g(\bar{A}_i) = 0$  and hence,  $g(m_i)\mathfrak{P} = g(m_i)\mathfrak{P} = 0$ , which means  $g(m_i) \subset \bar{A}_1$ . Since  $\bar{A}_1$  is essential in  $M$  as an  $R$ -module and  $R/P$  is irreducible, there exists  $c_i$  in  $R$  such that  $m_i b_i c_i = g(m_i)$  for each  $i$ . Put  $r'_{i+1} = \sum b_j c_j$ , then  $g(m_j) = m_j b_j c_j = m_j r'_{i+1}$  for all  $j$ . Hence,  $(s - (r_i + r'_{i+1}))\bar{A}_{i+1} = (0)$ . Since  $r(\bar{A}_{j+1}) = \mathfrak{P}^{j+1}$  by [6], Theorem 3.4,  $s = \lim r_j, r_j \in R$ . Let  $\{m_i\}$  be a finite elements in  $M$ , then there exists an  $\bar{A}_n$  containing all  $m_i$ . Hence, if we take an element  $r$  in  $R$  such that  $s - r \in \mathfrak{P}^n, m_i r = m_i s$  for all  $i$ .

Conversely, we assume that  $S$  satisfies the first parts of ii and iii. Then every  $R$ -submodule  $N$  of  $M$  is an  $S$ -module and every  $R$ -homomorphism of  $N$  to  $M$  is an  $S$ -homomorphism. Hence,  $M$  is a quasi-injective and artinian by Lemma 2, since  $M$  is  $S$ -artinian.

**Corollary.** *Let  $M, R$  and  $S$  be as above. If  $M$  is a quasi-injective, artinian  $R$ -module, then for any intermediate ring  $T$  between  $R$  and  $S$ ,  $M$  is  $T$ -quasi-injective.*

**REMARK.** In Theorem 2 we have shown that  $S$  is noetherian, however  $R/A$  is not noetherian in general, where  $A = \text{ann } M$ . For example, let  $Z$  be the ring of integers and  $P$  a prime.  $Z_{P^\infty}$  is  $Z_P$ -artinian, injective and indecomposable.

We can obtain a non-noetherian intermediate local ring  $T$  between  $Z_P$  and  $\hat{Z}_P = \text{Hom}_{Z_P}(Z_{P^\infty}, Z_{P^\infty})$  (see [3], Lemma 1) and  $M$  is  $T$ -quasi-injective and  $T$ -artinian.

Next, we shall consider a case of injective modules.

**Theorem 3.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -artinian, injective module. Then there exists a finite set of maximal ideals  $P_1, P_2, \dots, P_n$  such that  $R_T$  is noetherian, where  $T = R - (P_1 \cup P_2 \cup \dots \cup P_n)$  and  $n$  is the number of non-isomorphic indecomposable direct summands of  $M$ . Conversely, if  $R_T$  is noetherian, there exists an  $R$ -artinian, injective module which is a directsum of  $n$  non isomorphic indecomposable modules.*

Proof. Let  $M = \sum_i^n \oplus M_i$  and the  $M_i$  be directly indecomposable. We may assume  $M_i \not\approx M_j$  if  $i \neq j$ . Each  $M_i$  corresponds to a maximal ideal  $P_i$  and  $M_i$  may be regraded as  $R_{P_i}$ -module by Theorem 2. Further,  $M_i$  is an injective hull of  $R/P_i$  as an  $R$ -module. Put  $T = R - (P_1 \cup \dots \cup P_n)$ , then  $R_T/P_i R_T \approx R/P_i$ . Hence,  $M$  is an  $R_T$ -cogenerator. Therefore,  $R_T$  is noetherian by [8], Lemma 2. Conversely, we assume  $R_T$  is noetherian and put  $M_i = E_R(R/P_i)$ . Since  $R/P_i$  is a unique minimal sub-module of  $M_i$ ,  $M_i = E_{R_{P_i}}(R/P_i)$ . Let  $\varphi_i: R \rightarrow R_{P_i}$  be the canonical homomorphism. Then the operation of elements  $r$  in  $R$  on  $M_i = E_{R_{P_i}}(R/P_i)$  is given via  $\varphi_i$ . Hence,  $M_i = E_{R_T}(R_T/P_i R_T)$  and  $\text{Hom}_{R_{P_i}}(M_i, M_i) = \text{Hom}_R(M_i, M_i)$  by the standard argument. Furthermore, since  $R_{P_i}$  is noetherian, for any element  $x$  in  $M_i$   $\text{ann}_{R_{P_i}} x \supseteq P_i^{n_i} R_{P_i} \supseteq \varphi_i(P_i^{n_i})$  for some  $n_i$  and hence,  $x P_i^{n_i} = (0)$ . Put  $M = \sum \oplus M_i$ , then  $M$  is an  $R$ -weakly distinguished module from the above, (cf. the proof of Lemma 2). Since  $R_T$  is noetherian,  $\text{Hom}_{R_T}(M, M) = \sum \oplus \text{Hom}_{R_{P_i}}(M_i, M_i) = \text{Hom}_R(M, M)$  is noetherian by [6], Theorem 3.9. Therefore,  $M$  is  $R$ -artinian, since  $M$  is  $R$ -weakly distinguished.

**Lemma 3.** *Let  $R$  be a local noetherian ring with maximal ideal  $P$  and  $M = E_R(R/P)$ . Let  $S = \text{Hom}_R(M, M)$  and  $T$  be an intermediate ring between  $R$  and  $S$ . If for any element  $x$  in  $E_T(M)$ ,  $x P^n = (0)$  for some  $n$ , then  $M$  is  $T$ -injective.*

Proof.  $E_T(M) = M \oplus K$  as  $R$ -modules. If  $K \neq (0)$ , for any  $k \neq 0$  in  $K$ ,  $k P^n = (0)$  by the assumption. Hence,  $\text{ann } k' = P$  for some  $k' \in K$ . Since  $E_T(M)$  is indecomposable, it contains a unique minimal  $T$ -module  $R/P$ . Which is a contradiction.

**Proposition 3.** *Let  $R$ ,  $M$  and  $S$  be as in Lemma 3. Then for any intermediate local ring  $T$  between  $R$  and  $S$ ,  $M$  is  $T$ -injective if and only if  $T$  is noetherian and  $\mathfrak{P} \cap T = P'$ , where  $\mathfrak{P}$  and  $P'$  are maximal ideals in  $S$  and  $T$ , respectively.*

Proof. "Only if part" is an immediate consequence of Theorem 3. We assume that  $T$  is noetherian as in the proposition. Since  $M = E_R(R/P)$  and

$(R/P)S = R/P$ ,  $R/P \approx T/P'$  and  $P' \cap R = P$ . Let  $M' = E_T(T/P')$ , then for any  $x$  in  $M'$   $xP^n \subset xP'^n = (0)$  for some  $n$ . Hence,  $M = M'$  by Lemma 3.

REMARK. Let  $Z, P$  be as in the previous remark. Then there exists a tower of noetherian local rings  $Z_P \subset R_1 \subset R_2 \subset \cdots$  such that  $R_i$  dominates  $R_{i-1}$  and  $T = \bigcup R_i$  is not noetherian. Then  $M$  is  $R_i$ -injective for each  $i$ , but not  $T$ -injective.

OSAKA CITY UNIVERSITY

---

### References

- [1] G. Azumaya: *A duality theory for injective modules*, Amer. J. Math. **81** (1959), 249–278.
- [2] C. Faith: *Lectures on Injective Modules and Quotient Rings*, Springer, Heidelberg, 1967.
- [3] M. Harada: *On homological theorems of algebras*, J. Inst. Polyt. Osaka City Univ. **10** (1959), 123–127.
- [4] ——— and T. Ishii: *On endomorphism rings of noetherian quasi-injective modules*, Osaka J. Math. **9** (1972), 217–223.
- [5] R.E. Johnson and E. Wong: *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.
- [6] E. Matlis: *Injective modules over noetherian rings*, Pacific J. Math. **8** (1958), 511–528.
- [7] A. Rosenberg and D. Zelinsky: *On the finiteness of the injective hull*, Math. Z. **70** (1959), 372–380.
- [8] K. Sugano: *A note on Azumaya's theorem*, Osaka J. Math. **4** (1967), 157–160.