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ON QUASI-INJECTIVE MODULES WITH A CHAIN CONDITION OVER A COMMUTATIVE RING

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In the previous paper [4] the author and T. Ishii studied the endomorphism rings of noetherian quasi-injective modules. As an application of it, we consider, in this paper, quasi-injective modules over a commutative ring R . If R is noetherian, E. Matlis decided every indecomposable injective modules in [6].

Greatly making use of those results in [6], we shall decide all quasi-injective (resp. injective) modules which are either artinian or noetherian in §§2 and 3. Especially, we shall give necessary and sufficient conditions of R for existence of quasi-injective (resp. injective) modules which are either artinian or noetherian (cf. [7], Theorem 5).

In this paper, a ring R is always commutative unless otherwise stated and every R -module is unitary.

1. Preliminaries

Let K be any ring (not necessarily commutative) and M a right K -module. Put $S = \text{Hom}_K(M, M)$, then we assume that M is a left S -module. Let N be a subset of M . Then we denote the annihilator ideal of N in S and in K by $l(N)$ and $\text{ann } N$, respectively. Similarly, by $r(A)$ we denote the annihilator submodule of M for a left ideal A in S .

We call M a *weakly distinguished* K -module if for any K -submodules $N_1 \supset N_2$ in M such that N_1/N_2 is K -irreducible, $\text{Hom}_K(N_1/N_2, M) \neq 0$. If M is K -quasi-injective, then M is weakly distinguished if and only if $rl(N) = N$ for any K -submodule N in M , (see [1], Proposition 6).

Finally, we shall add here some direct consequences of [4]. From now on we shall assume that a ring R is commutative.

Proposition 1. *Let R be a commutative ring and M a quasi-injective module. If M is noetherian as an R -module, then $S = \text{Hom}_R(M, M)$ is left and right artinian, (see Theorem 1 below).*

Proof. Since R is commutative, S is an R -submodule of a finite directsum of copies of M . Therefore, S is artinian by [4], Theorem 1.

Proposition 2. *Let R and M be as above. We assume further that M is weakly distinguished. Put $S = \text{Hom}_R(M, M)$. Then M is R -noetherian if and only if S is left artinian. In this case, M is R -artinian, S -injective and R/A is artinian, where $A = \text{ann } M$.*

Proof. If M is R -noetherian, S is artinian by Proposition 1. Hence, M is S -injective by [4], Theorem 2 and M is R -artinian from the above remark, since S is noetherian. Further, R/A is an R -submodule of finite directsum of copies of M . Hence, R/A is artinian. If S is (left) artinian, then M is R -noetherian as above.

2. Noetherian quasi-injective modules

We shall decide quasi-injective noetherian modules in this section.

Lemma 1. *Let K be any ring and M a quasi-injective and weakly distinguished right K -module. Put $S = \text{Hom}_K(M, M)$ and $T = \text{Hom}_S(M, M)$. Then every K -submodule of M is a T -submodule of M .*

Proof. Let N be a K -module of M . Then $rl(N) = N$ by the remark in §1. Hence, N is a T -module.

Let R be a commutative noetherian ring and P a prime ideal in R . Let $E(R/P) = E$ be an injective hull of R/P . Then Matlis showed in [6] that $E = \bigcup_i A_i$ and $\text{Hom}_R(E, E)$ is a complete local noetherian ring, where $A_i = \{x \in E, xP^i = 0\}$.

Lemma 2. *Let R be a commutative noetherian ring and $\{P_i\}$ a finite set of distinct maximal ideals in R . Then every R -submodule N of $\Sigma \bigoplus E(R/P_i)$ is weakly distinguished and quasi-injective.*

Proof. We may assume that N is an essential submodule of $E = \Sigma \bigoplus E_i$, where $E_i = E(R/P_i)$. Then $\text{ann } x \supset \bigcap P_i$ for any x in N . Let N_1, N_2 be R -submodules of N such that N_1/N_2 is R -irreducible, then $N_1/N_2 \approx R/P_i$ for some P_i . Since $N \cap R/P_i \neq (0)$, $\text{Hom}_R(N_1/N_2, N) \neq (0)$, which means that N is weakly distinguished. Hence, E is an R -weakly distinguished injective module. Moreover, if we put $S = \text{Hom}_R(E, E)$, $S = \text{Hom}_S(E, E)$. Hence, every R -submodule M is an S -submodule by Lemma 1. Let E' be an injective hull of M contained in E . Then $E = E' \oplus E''$ and $E' \supset M$. $S' = \text{Hom}_R(E', E')$ may be regarded as a subring of S . Hence, M is also an S' -module. Therefore, M is R -quasi-injective by [5], Theorem 1.1.

We are interested in a noetherian or artinian quasi-injective module M and hence, we may assume that M is directly indecomposable.

Theorem 1. *Let M be a directly indecomposable module over a commutative ring R . Then M is quasi-injective and noetherian if and only if there exist an ideal*

I such that R/I is noetherian and a maximal ideal P containing I and M is contained in a submodule A_n of $E_{R/I}(R/P)$. In this case, M is R -artinian, and hence R/I is artinian¹⁾.

Proof. We assume that M is R -noetherian and quasi-injective. Put $I = \text{ann } M$. Then $\bar{R} = R/I$ is noetherian as the proof of Proposition 2. Hence, we may assume that R is noetherian. Let E be an injective hull of M . Then $E = E_R(R/P)$ with P prime by [6], Proposition 3.1. Put $S = \text{Hom}_R(E, E)$. We know from [6], Theorem 3.4 and its proof that $A_1 = S(R/P) \approx Sa \approx K$ for any non-zero element a in A_1 , where K is the quotient field of R/P . Since $M \cap A_1 \neq (0)$ and M is quasi-injective, M contains a submodule which is isomorphic to K by [5], Theorem 1.1. Hence, P is a maximal ideal in R , and M is contained in some A_n , since M is R -finitely generated and each A_n has a composition length by [6], Theorem 3.9. The remaining part is clear from the above and Lemma 2.

Corollary. *Let R be a commutative ring. Then there exists a noetherian injective module if and only if R contains a maximal ideal P such that R_P is artinian, (cf. [6], Theorem 3.11).*

Proof. It is an immediate consequence of Theorem 1 and [7], Theorem 5,

3. Artinian, quasi-injective modules

We shall decide quasi-injective, (resp. injective) artinian modules in this section.

Theorem 2. *Let R be a commutative ring and M a directly indecomposable R -module and $S = \text{Hom}_R(M, M)$. If M is quasi-injective and artinian, then*

i. *There exists a maximal ideal P in R such that $M = \bigcup A_i$, where $A_i = \{x \in M \mid xP^i = 0\}$, and M may be regarded as an R_P -module and R_P -quasi-injective.*

ii. *M is S -injective and S is a commutative \mathfrak{P} -adic complete local noetherian ring, where \mathfrak{P} is a unique maximal ideal of S . Furthermore, the set of the S -submodules of M coincides with that of R -submodules of M .*

iii. *R is dense in S with respect to \mathfrak{P} -adic topology and hence, for any finite elements m_i in M and an element s in S , there exists an element r in R such that $m_i s = m_i r$ for all i .*

Conversely, if S satisfies the first parts of ii and iii, then M is a quasi-injective and artinian R -module.

Proof. We assume that M is a quasi-injective and artinian R -module. Let $m \neq 0$ be an element in M , then $mR \approx R/\text{ann } m$ is an artinian ring. Hence, there

Added in proof: 1) In this case M is $R/\text{Ann } M$ -injective by Theorem 1 of C. Faith *Modules finite over endomorphism ring*, Lecture Notes in Math., Springer, Heidelberg, 246.

exists a unique maximal ideal P such that $P \supset \text{ann } m$ and $P^* \subset \text{ann } m$, since M is indecomposable and quasi-injective. Therefore, M contains a unique minimal R -module R/P and P does not depend on a choice of m . Let s be in $R - P$ and $x \in l(s) \cap R/P$. Since P is maximal, there exist $p \in P, r \in R$ such that $1 = p + rs$. Hence, $x = xp + xrs = 0$. Therefore, $l(s) = (0)$. Since M is artinian, s gives an automorphism of M . Hence, M may be regarded as an R_P -module. It is clear that M is R_P -quasi-injective.

ii. Put $S = \text{Hom}_R(M, M)$. Then S is left noetherian by [3], Proposition 1. Furthermore, we know from [3], Theorem 2 that M is S -injective, since M is R -weakly distinguished (cf. the proof of Lemma 2 and i). On the other hand, we put $S' = \text{Hom}_S(M, M)$. Then $S' \subset S$ and hence, S' is the center of S . Moreover, since M is an artinian S -injective, S' is noetherian as above. Let N be the radical of S then S/N is a division ring by [2], Theorem 1 in p. 44 and Theorem 6 in p. 48, and $R/P \approx S/N$ as S -modules. Hence, M is S -weakly distinguished. Thus, M is also S' -injective as above. Since $S = \text{Hom}_{S'}(M, M)$, $S = S'$ is a complete local ring with respect to a \mathfrak{P} -adic topology by [6], Theorem 3.7, where \mathfrak{P} is a unique maximal ideal in S' and $\mathfrak{P} \cap R = P$. The last part of ii is clear from the above and Lemma 1.

iii. The following argument is analogous to [6], Theorem 3.7. Put $\bar{A}_i = \{x \in M, x\mathfrak{P}^i = 0\}$. We shall show for s in S that there exists r_i in R for each \bar{A}_i such that $l(s - r_i) \supset \bar{A}_i$. Since $\bar{A}_1 = R/P = S/\mathfrak{P}$, we have r_1 . We assume that there exists r_i in R such that $l(s - r_i) \supset \bar{A}_i$. Let $\{m_1, m_2, \dots, m_t\}$ be a system of minimal generators of \bar{A}_{i+1} as an S -module (see Theorem 1), then we obtain elements b_i in R such that $m_i b_i \neq 0, m_j b_i = 0$ if $i \neq j$ by [5], Theorem 2.3. Put $g = s - r_i, g(\bar{A}_i) = 0$ and hence, $g(m_i)\mathfrak{P} = g(m_i, \mathfrak{P}) = 0$, which means $g(m_i) \subset \bar{A}_1$. Since \bar{A}_1 is essential in M as an R -module and R/P is irreducible, there exists c_i in R such that $m_i b_i c_i = g(m_i)$ for each i . Put $r'_{i+1} = \sum b_j c_j$, then $g(m_j) = m_j b_j c_j = m_j r'_{i+1}$ for all j . Hence, $(s - (r_i + r'_{i+1}))\bar{A}_{i+1} = (0)$. Since $r(\bar{A}_{i+1}) = \mathfrak{P}^{i+1}$ by [6], Theorem 3.4, $s = \lim r_j, r_j \in R$. Let $\{m_i\}$ be a finite elements in M , then there exists an \bar{A}_n containing all m_i . Hence, if we take an element r in R such that $s - r \in \mathfrak{P}^n, m_i r = m_i s$ for all i .

Conversely, we assume that S satisfies the first parts of ii and iii. Then every R -submodule N of M is an S -module and every R -homomorphism of N to M is an S -homomorphism. Hence, M is a quasi-injective and artinian by Lemma 2, since M is S -artinian.

Corollary. *Let M, R and S be as above. If M is a quasi-injective, artinian R -module, then for any intermediate ring T between R and S , M is T -quasi-injective.*

REMARK. In Theorem 2 we have shown that S is noetherian, however R/A is not noetherian in general, where $A = \text{ann } M$. For example, let Z be the ring of integers and P a prime. Z_{P^∞} is Z_P -artinian, injective and indecomposable.

We can obtain a non-noetherian intermediate local ring T between Z_P and $\hat{Z}_P = \text{Hom}_{Z_P}(Z_{P^\infty}, Z_{P^\infty})$ (see [3], Lemma 1) and M is T -quasi-injective and T -artinian.

Next, we shall consider a case of injective modules.

Theorem 3. *Let R be a commutative ring and M an R -artinian, injective module. Then there exists a finite set of maximal ideals P_1, P_2, \dots, P_n such that R_T is noetherian, where $T = R - (P_1 \cup P_2 \cup \dots \cup P_n)$ and n is the number of non-isomorphic indecomposable direct summands of M . Conversely, if R_T is noetherian, there exists an R -artinian, injective module which is a directsum of n non isomorphic indecomposable modules.*

Proof. Let $M = \sum_1^n \oplus M_i$ and the M_i be directly indecomposable. We may assume $M_i \not\approx M_j$ if $i \neq j$. Each M_i corresponds to a maximal ideal P_i and M_i may be regraded as R_{P_i} -module by Theorem 2. Further, M_i is an injective hull of R/P_i as an R -module. Put $T = R - (P_1 \cup \dots \cup P_n)$, then $R_T/P_i R_T \approx R/P_i$. Hence, M is an R_T -cogenerator. Therefore, R_T is noetherian by [8], Lemma 2. Conversely, we assume R_T is noetherian and put $M_i = E_R(R/P_i)$. Since R/P_i is a unique minimal sub-module of M_i , $M_i = E_{R_{P_i}}(R/P_i)$. Let $\varphi_i: R \rightarrow R_{P_i}$ be the canonical homomorphism. Then the operation of elements r in R on $M_i = E_{R_{P_i}}(R/P_i)$ is given via φ_i . Hence, $M_i = E_{R_T}(R_T/P_i R_T)$ and $\text{Hom}_{R_{P_i}}(M_i, M_i) = \text{Hom}_R(M_i, M_i)$ by the standard argument. Furthermore, since R_{P_i} is noetherian, for any element x in M_i $\text{ann}_{R_{P_i}} x \supseteq P_i^{n_i} R_{P_i} \supseteq \varphi_i(P_i^{n_i})$ for some n_i and hence, $x P_i^{n_i} = (0)$. Put $M = \sum \oplus M_i$, then M is an R -weakly distinguished module from the above, (cf. the proof of Lemma 2). Since R_T is noetherian, $\text{Hom}_{R_T}(M, M) = \sum \oplus \text{Hom}_{R_{P_i}}(M_i, M_i) = \text{Hom}_R(M, M)$ is noetherian by [6], Theorem 3.9. Therefore, M is R -artinian, since M is R -weakly distinguished.

Lemma 3. *Let R be a local noetherian ring with maximal ideal P and $M = E_R(R/P)$. Let $S = \text{Hom}_R(M, M)$ and T be an intermediate ring between R and S . If for any element x in $E_T(M)$, $x P^n = (0)$ for some n , then M is T -injective.*

Proof. $E_T(M) = M \oplus K$ as R -modules. If $K \neq (0)$, for any $k \neq 0$ in K , $k P^n = (0)$ by the assumption. Hence, $\text{ann } k' = P$ for some $k' \in K$. Since $E_T(M)$ is indecomposable, it contains a unique minimal T -module R/P . Which is a contradiction.

Proposition 3. *Let R , M and S be as in Lemma 3. Then for any intermediate local ring T between R and S , M is T -injective if and only if T is noetherian and $\mathfrak{P} \cap T = P'$, where \mathfrak{P} and P' are maximal ideals in S and T , respectively.*

Proof. “Only if part” is an immediate consequence of Theorem 3. We assume that T is noetherian as in the proposition. Since $M = E_R(R/P)$ and

$(R/P)S=R/P$, $R/P \approx T/P'$ and $P' \cap R=P$. Let $M'=\mathbf{E}_T(T/P')$, then for any x in M' $xP^n \subset xP'^n=(0)$ for some n . Hence, $M=M'$ by Lemma 3.

REMARK. Let Z, P be as in the previous remark. Then there exists a tower of noetherian local rings $Z_P \subset R_1 \subset R_2 \subset \dots$ such that R_i dominates R_{i-1} and $T=\bigcup R_i$ is not noetherian. Then M is R_i -injective for each i , but not T -injective.

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