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ON G-ANR'S AND THEIR G-HOMOTOPY TYPES

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Introduction

It is well-known that ANR's (absolute neighbourhood retracts) for metric spaces have various good properties (c.f., [7]), and all paracompact topological manifolds are ANR's. From a homotopic point of view, they have the homotopy types of CW-complexes (c.f., [14]).

Throughout this paper, G will be assumed to be a *compact Lie group*.

In the present work we study G-equivariant ANR's (abbreviated to G-ANR hereafter) for metrizable G-spaces (defined in § 4) and mainly discuss parallel properties to ANR's as in Hu [7] (§§ 4–7) and show that G-ANR's have the G-homotopy types of G-CW complexes (§§ 13–14).

For a finite G, the G-homotopy types of G-ANR's were discussed in [16].

This paper is devided as follows:

1. Paracompactness and G-coverings
2. TN G-coverings and G-nerves
3. G-CW complexes and metrizability
4. G-ANR's and G-ANE's
5. An equivariant version of Dugundji's extension theorem
6. Relation between G-ANR's and G-ANE's
7. Union of G-ANR's
8. Relation for subgroups and G-manifolds
9. Small G-homotopies and G-homotopy extension property
10. G-domination
11. Mapping spaces
12. Small G-deformation and adjunction spaces
13. G-homotopy types of G-ANR's
14. G-homotopy types of countable G-CW complexes.

First we discuss topological properties of G-spaces as preliminaries (§§1–3): In § 1 we consider paracompactness of G-spaces to prepare for the construction of TN G-coverings. In § 2 we introduce the notion of TN G-coverings

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of G-spaces and their G-nerves which corresponds to numerable coverings and their nerves and plays a basic role to develop our discussion on G-ANR's. The exsistence of TN G-coverings is based on paracompactness and the exsistence of tubes. In §3 we see that G-CW complexes are paracompact and perfectly normal, and consider metrizability.

G-ANR's and G-ANE's are defined in §4. In §5 we state an equivariant version of Dugundji's extension theorem (Theorem 5.3), which implies that every locally convex topological linear G-space is a G-ANE. In fact, the Dugundji's argument [3] ([7]) can be applied if we replace a nerve by a G-nerve. In §6, using this, we show that the following three conditions are mutually equivalent: 1) G-ANR, 2) metrizable G-ANE, and 3) G-neighbourhood retract of a convex G-subset of a Banach G-space. Also we look at the elementary properties of G-ANR's and G-ANE's, e.g., the following G-spaces being G-ANR's (resp. G-ANE's): open G-subspaces and G-neighbourhood retracts of G-ANR's (resp. G-ANE's), finite products of G-ANR's (resp. G-ANE's), G-ENR's, etc.; the consideration for G-contractibility, etc..

In §7, we see that local G-ANR's (resp. local G-ANE's) are G-ANR's (resp. G-ANE's) and G-spaces having the weak topology with respect to closed coverings by G-ANR (resp. G-ANE) subspaces are G-ANR's (resp. G-ANE's) under a suitable restriction. In §8, first we consider functors such as the restriction functor, the H-fixed-point functor and the functor $G\underset{H}{\times}-$, where H is a closed subgroup of G, and we see the invariance of equivariant ANR under operations of these functors, i.e., $\mathrm{Res}_H^G X$, X^H, or $G\underset{H}{\times}X$ for a G-ANR or an H-ANR X is an H-ANR, an NH/H-ANR or a G-ANR respectively, etc.. Next we show that every (locally) smooth G-manifold is a G-ANR (Theorem 8.8). We also see that a certain kind of G-bundles are G-ANR's.

In §9 we examine characterizations of G-ANR's by small G-homotopies and G-homotopy extension properties in a parallel way to [5] and [7]. In §10, we show that every G-ANR is G-deminated by the G-nerve of a certain TN G-covering of it (Proposition 10.1). This result is used in §13.

In §11 we see that the mapping space from a compact G-space to a G-ANR (e.g., a path space, a loop space, etc.) is a G-ANR. In §12, we treat characterizations of G-ANR's by small G-deformations and G-dominations. Also we see that the adjunction space of a G-ANR pair and a G-ANR is a G-ANR under the metrizability, in particular, that every (locally) finite G-CW complex is a G-ANR. (Every G-CW complex is a G-ANE.)

In §13 we prove that every G-ANR has the G-homotopy type of a G-CW complex (Theorem 13.3, c.f., [14]). And we see that the converse holds for a countable G-CW complex in §14.

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1. Paracompactness and G-coverings

All (G-)spaces considered in this paper will be **Hausdorff** (G-)spaces.

In this section we consider paracompactness of G-spaces.
Let $\mathcal{U}$ be a covering of a space X. The star $St(A, \mathcal{U})$ of a subset $A \subset X$ with respect to $\mathcal{U}$ denotes the subset

$$St(A, \mathcal{U}) = \cup \{U \in \mathcal{U}: U \cap A \neq \emptyset\}$$

of X. If $x \in X$, then $St(\{x\}, \mathcal{U})$ is denoted by $St(x, \mathcal{U})$. We call that a covering $\mathcal{U}$ of a space X is a *star-refinement* of another covering $\mathcal{V}$ of X if the covering $\{St(U, \mathcal{U})\}_{U \in \mathcal{U}}$ is a refinement of $\mathcal{V}$. Then $\{St(x, \mathcal{U})\}_{x \in X}$ is a refinement of $\{St(U, \mathcal{U})\}_{U \in \mathcal{U}}$ and $\mathcal{V}$.

We use the following abbreviations:

(G-)*nbd* = (G-invariant) neighbourhood,
(G-)map=(G-equivariant) continuous map.

Lemma 1.1. *If X is a G-space and A is a G-subset of X, then every nbd V of A contains a G-nbd U of A.*

The proof is obtained by putting $U = X - G(X - V)$.

Lemma 1.2. *Let X be a G-space and $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ a locally finite covering of X. Then*:

(1) *Every point $x \in X$ has a G-nbd V such that the set $\{\lambda \in \Lambda \mid U_\lambda \cap V \neq \emptyset\}$ is finite.*

(2) *The covering $\{GU_\lambda\}_{\lambda \in \Lambda}$ is locally finite.*

Proof. (1) follows from compactness of orbits and Lemma 1.1.

(2): Consider the above G-nbd V for each $x \in X$. Then $GU_\lambda \cap V \neq \emptyset$ iff $U_\lambda \cap V \neq \emptyset$. Thus

$$\text{Card } \{\lambda \in \Lambda \mid GU_\lambda \cap V \neq \emptyset\} = \text{Card } \{\lambda \in \Lambda \mid U_\lambda \cap V \neq \emptyset\} < \infty . \qquad \text{q.e.d.}$$

DEFINITION 1.3. Let X be a G-space.

(1) A covering $\mathcal{U} = \{U\}$ of X is called to be (G–)*invariant* iff each $U \in \mathcal{U}$ is G-invariant.

(2) A covering $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of X is called a *G-covering* iff

i) $gU_\lambda \in \mathcal{U}$ for every $U_\lambda \in \mathcal{U}$ and every $g \in G$,

ii) the index set Λ is a G-set satisfying

$$gU_\lambda = U_{g\lambda} \tag{I}$$

for each $\lambda \in \Lambda$ and $g \in G$.

(3) Let $\mathcal{U}=\{U_\lambda\}_{\lambda\in\Lambda}$ be a G-covering of X. We define a G-subspace $\tilde{U}_\alpha$ for each $\alpha\in\Lambda/G$ by

$$\tilde{U}_\alpha=\bigcup_{\lambda\in\alpha}U_\lambda \qquad (=GU_\lambda,\ \lambda\in\alpha)\,.$$

The invariant covering $\tilde{\mathcal{U}}=\{\tilde{U}_\alpha\}_{\alpha\in\Lambda/G}$ is called the *saturation* of $\mathcal{U}$.

REMARK. 1) When a covering $\mathcal{U}=\{U_\lambda\}_{\lambda\in\Lambda}$ of a G-space X satisfies the above (2). i), we may regard $\mathcal{U}$ as a G-covering by endowing the index set Λ with a G-action by the formula (I).

2) For any covering $\mathcal{V}$ of a G-space we can form a G-covering $\mathcal{V}^t=\{gV\,|\,g\in G,\ V\in\mathcal{V}\}$ by adding all g-transform gV, $g\in G$ and $V\in\mathcal{V}$, to $\mathcal{V}$ and indexing as above.

3) An invariant covering $\mathcal{U}$ is also a G-covering and $\mathcal{U}=\tilde{\mathcal{U}}$. In particular, $\tilde{\mathcal{U}}=\tilde{\tilde{\mathcal{U}}}$.

Proposition 1.4. *Let X be a paracompact G-space. Then the followings hold*:

(1) *Every open G-covering $\mathcal{V}$ of X has an open refinement $\mathcal{U}$ which is a G-covering and of which the saturation $\tilde{\mathcal{U}}$ is locally finite. In particular, every open invariant covering of X has a locally finite open invariant refinement.*

(2) *Every open invariant covering $\mathcal{V}=\{V_\alpha\}_{\alpha\in A}$ of X has an invariant partition of unity $\{p_\alpha\}_{\alpha\in A}$ such that*

$$\overline{p_\alpha^{-1}((0,1])}\subset V_\alpha$$

for each $\alpha\in A$. (A partition of unity is assumed to be locally finite.)

(3) *The orbit space X/G is paracompact.*

Proof. (1): Since X is paracompact, there is a locally finite open refinement $\mathcal{W}$ of $\mathcal{V}$. Put $\mathcal{U}=\mathcal{W}^t=\{gW\,|\,g\in G,\ W\in\mathcal{W}\}$. Then $\tilde{\mathcal{U}}$ is locally finite by Lemma 1.2 and $\mathcal{U}$ is the required G-covering. If $\mathcal{V}$ is invariant, then $\tilde{\mathcal{U}}$ is a refinement of $\mathcal{V}$.

(2): By [2], Chap. 9, §4, Corollary, p. 91, there is a partition of unity $\{p'_\alpha\}_{\alpha\in A}$ such that $\overline{p'^{-1}_\alpha((0,1])}\subset V_\alpha$ for each $\alpha\in A$. By averaging p'_α over G we get an invariant partition of unity $\{p_\alpha\}_{\alpha\in A}$ such that $\overline{p_\alpha^{-1}((0,1])}\subset V_\alpha$ for each $\alpha\in A$.

(3): Let $\mathcal{V}$ be an open covering of X/G. Applying the above (1) to $\{\Pi_X^{-1}(V)\}_{V\in\mathcal{V}}$, we get a locally finite invariant open refinement $\mathcal{U}$ of $\{\Pi_X^{-1}(V)\}_{V\in\mathcal{V}}$, where $\Pi_X\colon X\to X/G$ is the projection. Then $\{\Pi_X(U)\}_{U\in\mathcal{U}}$ is a locally finite open refinement of $\mathcal{V}$. q.e.d.

Lemma 1.5. *Every open covering $\mathcal{V}$ of a paracompact G-space X has an* *open refinement* $\mathcal{U}$ *which is a G-covering of* X.

Proof. Since X is paracompact, $\mathcal{V}$ is even by [8], Chap. 5,28. Hence there is an open *nbd* W of the diagonal of $X\times X$ such that $\{W[x]\}_{x\in X}$ is a refinement of $\mathcal{V}$, where $W[x]=\{y\in X\,|\,(x,y)\in W\}$. By Lemma 1.1 there is an open G-*nbd* U of the diagonal such that $U\subset W$, where $X\times X$ has the diagonal G-action. Put $\mathcal{U}=\{U[x]\}_{x\in X}$. Then $\mathcal{U}$ is an open G-covering and a refinement of $\mathcal{V}$. q.e.d.

Proposition 1.6. *Every open G-covering* $\mathcal{V}$ *of a paracompact G-space* X *has an open star-refinement* $\mathcal{U}$ *which is a G-covering.*

Proof. Since X is paracompact and hence fully normal, there exists an open star-refinement $\mathcal{U}'$ of $\mathcal{V}$. By Lemma 1.5 we have an open G-covering $\mathcal{U}$ which is a refinement of $\mathcal{U}'$ and hence a star-refinement of $\mathcal{V}$. q.e.d.

2. TN G-coverings and G-nerves

Let X be a G-space and $O=Gx\subset X$ a G-orbit (of type G/H). By a (G-) *tube about* O we mean a pair $T=(T, r)$ of an open G-*nbd* T of O and a G-retraction $r\colon T\to O$ (instead of a G-embedding $\phi\colon G\times_H A\to X$ onto T as in [1], II, 4.2). Then the orbit O is called the *central orbit of* T.

When U is a (open) subset of O, the (open) subset $S=r^{-1}(U)$ of X is called the (open) *tube segment of* T *generated by* U. Then, for each $g\in G$, gS is a (open) tube segment of T generated by gU and $gS\cap O=gU$.

Clearly any open G-*nbd* T' of O in T is also a tube about O with the G-retraction $r'=r|_{T'}$. The open tube segments in a completely regular G-space form a base for the topology by the existence of tubes (the Mostow theorem), see [1], II, 5.4, [17], 1.7,19.

By a (open) *tube-segmental G-covering* we mean a (open) G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ such that $\tilde{S}_\alpha=\bigcup_{\lambda\in\alpha}S_\lambda$ is a tube T_α with a G-retraction $r_\alpha\colon T_\alpha\to O_\alpha$ for each $\alpha\in\Lambda/G$ and S_λ is a (open) tube segment of T_α for each $\lambda\in\alpha$. Then the saturation $\tilde{\mathcal{S}}=\{T_\alpha\}_{\alpha\in\Lambda/G}$ is an invariant covering by tubes.

Proposition 2.1. *Every open G-covering* $\mathcal{V}$ *of a completely regular G-space* X *has a refinement* $\mathcal{S}$ *which is an open tube-segmental G-covering of* X.

Proof. For each orbit $Gx\subset X$ we select a point $x\in Gx$ and choose a tube segment S_x of a tube T_{Gx} generated by an open *nbd* of x in Gx such that S_x is contained in some $V\in\mathcal{V}$. Put $S_{gx}=gS_x$ for each $gx\in Gx$. Then $\mathcal{S}=\{S_x\}_{x\in X}$ is the required one. q.e.d.

DEFINITION 2.2. An open tube-segmental G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ of a G-space X is called a *TN G-covering* (*tubular numerable G-covering*) iff there exists an invariant partition of unity $\{p_\alpha\}_{\alpha\in\Lambda/G}$ such that

$$p_\alpha^{-1}((0, 1])\subset T_\alpha=\bigcup_{\lambda\in\alpha} S_\lambda$$

for every $\alpha\in\Lambda/G$. ($\{p_\alpha\}_{\alpha\in\Lambda/G}$ is assumed to be locally finite.)

The saturation $\tilde{\mathcal{S}}=\{T_\alpha\}_{\alpha\in\Lambda/G}$ of a TN G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ with $\{p_\alpha\}_{\alpha\in\Lambda/G}$ is also a TN G-covering with $\{p_\alpha\}_{\alpha\in\Lambda/G}$. If an open tube-segmental G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ has a partition of unity $\{p'_\beta\}_{\beta\in B}$ subordinate to $\tilde{\mathcal{S}}$, then $\mathcal{S}$ is a TN G-covering, since we can replace $\{p'_\beta\}_{\beta\in B}$ with $\{p_\alpha\}_{\alpha\in\Lambda/G}$ as in the proof of [2], Chap. 9, § 4, Corollary.

Proposition 2.3. *Every open G-covering $\mathcal{V}$ of a paracompact G-space X has a TN G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ of X which is a star-refinement of $\mathcal{V}$.*

The proof follows from Propositions 1.4. (2), 1.6, and 2.1.

Here we construct the *G-nerve* of a *TN* G-covering. Let $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ be a TN G-covering of a G-space X, $\tilde{\mathcal{S}}=\{T_\alpha\}_{\alpha\in\Lambda/G}$ the saturation of $\mathcal{S}$, and r_α: $T_\alpha\to O_\alpha$ the G-retraction of the tube T_α to the central orbit O_α for each $\alpha\in\Lambda/G$.

Let $N=N(\tilde{\mathcal{S}})$ denote the nerve of $\tilde{\mathcal{S}}$ and $N_n=N_n(\tilde{\mathcal{S}})$ the set of n-simplexes of N. We assume that Λ/G is (partially) ordered such that the induced order on the set of vertices of each simplex of N is linear. (e.g., Λ/G is well-ordered.)

For each n-simplex $\sigma=\{\alpha_0<\cdots<\alpha_n\}$ of N we define an open G-subspace $K_\sigma=K_\sigma(\mathcal{S})$ of $O_{\alpha_0}\times\cdots\times O_{\alpha_n}$ by

$$K_\sigma=\cup\{O_{\lambda_0}\times\cdots\times O_{\lambda_n}:\ \lambda_i\in\alpha_i\,,\ i=0,\ \cdots, n\,,\ \bigcap_{i=0}^{n} S_{\lambda_i}\neq\emptyset\}\,,$$

where O_λ denotes the open set $r_\alpha(S_\lambda)$ $(=S_\lambda\cap O_\alpha)$ of O_α for each $\lambda\in\alpha$ and $\alpha\in\Lambda/G$. (Note that $K_\sigma(\tilde{\mathcal{S}})=O_{\alpha_0}\times\cdots\times O_{\alpha_n}$.)

We define a simplicial G-space $K_*=K_*(\mathcal{S})$ (without degeneracy) (called the *simplicial G-nerve* of $\mathcal{S}$) as follows: The n-th space K_n, $n\geqq 0$, of K_* is given by

$$K_n=\sum_{\sigma\in N_n} K_\sigma\,,$$

and the ith face operator ∂_i: $K_n\to K_{n-1}$ is given by omitting the ith term, where $\sum$ denotes disjoint union. (Note that $K_*(\mathcal{S})$ is an open sub-simplicial G-space of $K_*(\tilde{\mathcal{S}})$.)

The geometric realization $|K_*(\mathcal{S})|=\sum_{n\geqq 0} K_n(\mathcal{S})\times\Delta^n/\sim$ of the simplicial G-space $K_*(\mathcal{S})$ is called the (*geometric*) *G-nerve* of $\mathcal{S}$ and denoted by $K(\mathcal{S})$. ($K(\mathcal{S})$ is an open G-subspace of $K(\tilde{\mathcal{S}})$.)

The image of $\sum_{i=0}^{n} K_i(\mathcal{S})\times\Delta^i$ in $K(\mathcal{S})$ is called the *n-skeleton* of $K(\mathcal{S})$ and denoted by $K^n(\mathcal{S})$.

Since $K_*(\mathcal{S})$ has no degeneracy, the inclusions $K^n(\mathcal{S})\to K^{n+1}(\mathcal{S})$ are G-cofibrations and hence $K(\mathcal{S})$ is a Hausdorff G-space. Let $|x, t|$ denote the image of $(x, t)\in K_n\times\Delta^n$ in $K(\mathcal{S})$, and $|x, t|$ is also denoted by $|x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n|$, where t_i denotes the ith barycentric coordinate of $t\in\Delta^n$ and $x=(x_{\alpha_0}, \cdots, x_{\alpha_n})\in K_\sigma\subset O_{\alpha_0}\times\cdots\times O_{\alpha_n}$.

Proposition 2.4. *Let $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ be a TN G-covering of a G-space X with an invariant partition of unity $\{p_\alpha\}_{\alpha\in\Lambda/G}$. Define $P: X\to K(\mathcal{S})$ by*

$$P(x) = |r_{\alpha_0}(x), \cdots, r_{\alpha_n}(x); p_{\alpha_0}(x), \cdots, p_{\alpha_n}(x)|$$

for $x\in X$, where $\{\alpha_0, \cdots, \alpha_n\}=\{\alpha\in\Lambda/G\,|\,p_\alpha(x)\neq 0\}$. Then P is a G-map.

Proof. Let $x\in X$, $\{\alpha_0, \cdots, \alpha_n\}=\{\alpha\in\Lambda/G\,|\,p_\alpha(x)>0\}$, and $\sigma=\{\alpha_0<\cdots<\alpha_n\}\in N(\tilde{\mathcal{S}})$. First we show that $(r_{\alpha_0}(x), \cdots, r_{\alpha_n}(x))\in K_\sigma$. Since $p_{\alpha_i}(x)>0$, we see that $x\in T_{\alpha_i}$ and there is $\lambda_i\in\alpha_i$ with $x\in S_{\lambda_i}$ for $i=0, \cdots, n$. Then $x\in\bigcap_{i=0}^{n} S_{\lambda_i}\neq\emptyset$ and hence $(r_{\alpha_0}(x), \cdots, r_{\alpha_n}(x))\in K_\sigma$ by definition. Thus P is well-defined.

To show the continuity of P, we consider the Segal's classifying space $BX_{\tilde{\mathcal{S}}}$ for $\tilde{\mathcal{S}}$ and the map $P': X\to BX_{\tilde{\mathcal{S}}}$, [18]. The G-space $BX_{\tilde{\mathcal{S}}}$ is regarded as the geometric realization of the simplicial G-space $X_{\tilde{\mathcal{S}}*}$ whose nth space $X_{\tilde{\mathcal{S}}n}$ is $\sum_{\sigma\in N_n(\tilde{\mathcal{S}})} T_\sigma$, $T_\sigma=\bigcap_{\alpha\in\sigma} T_\alpha$, and whose face operators are given by the inclusions. (There is no degeneracy. Segal's $BX_{\mathcal{S}}$ is the barycentric subdivision of ours.) The map $P': X\to BX_{\tilde{\mathcal{S}}}$ is given by $P'(x)=|x; p_{\alpha_0}(x), \cdots, p_{\alpha_n}(x)|$. Clearly P' is a G-map. We define a simplicial G-map $f_*: X_{\tilde{\mathcal{S}}*}\to K_*(\mathcal{S})$ by

$$f_n|_{T_\sigma} = r_{\alpha_0}\times\cdots\times r_{\alpha_n}$$

for $\sigma\in N_n(\tilde{\mathcal{S}})$. Then, clearly, $P=P'\circ|f_*|$. Since P' and $|f_*|$ are G-maps, P is also a G-map. q.e.d.

Let $|N(\tilde{\mathcal{S}})|$ be the geometric realization of the nerve $N(\tilde{\mathcal{S}})$ with the trivial G-action. Let $\pi=\pi_{K(\mathcal{S})}: K(\mathcal{S})\to|N(\tilde{\mathcal{S}})|$ be the projection induced by sending $O_\alpha\subset K^0(\mathcal{S})$ to the vertex α of $|(\tilde{\mathcal{S}})|$. That is, for $x=|x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n|\in K(\mathcal{S})$, $\pi(x)$ is the point with the αth barycentric coordinate $\pi(x)(\alpha)$,

$$\pi(x)(\alpha) = \begin{cases} t_i & \text{for } \alpha=\alpha_i \\ 0 & \text{for } \alpha\in\Lambda/G-\{\alpha_0, \cdots, \alpha_n\}\,. \end{cases}$$

Clearly π is a G-(invariant)map and $\pi^{-1}(\alpha)=O_\alpha$. The inverse image π^{-1} (the open star of α) is called the *open star* of O_α in $K(\mathcal{S})$ and denoted by

$O(O^{\varphi}, K(\mathcal{S}))$. Then

$$O(O_\alpha, K(\mathcal{S})) = \{x \in K(\mathcal{S}) \mid \pi(x)(\alpha) \neq 0\}$$
$$= \{|x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n| \mid \alpha_i = \alpha,\ t_i \neq 0 \text{ for some } \alpha_i\} .$$

Let ρ_α: $O(O_\alpha, K(\mathcal{S})) \to O_\alpha$ be the G-retraction defined by

$$\rho_\alpha(|x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n|) = x_{\alpha_i}$$

for $\alpha_i = \alpha$. For $O_\lambda = S_\lambda \cap O_\alpha$, $\lambda \in \alpha$, $\rho_\alpha^{-1}(O_\lambda)$ is called the *open star* of O_λ in $K(\mathcal{S})$ and denoted by $O(O_\lambda, K(\mathcal{S}))$. Then

$$O(O_\lambda, K(\mathcal{S})) = \{x \in K(\mathcal{S}) \mid \pi(x)(\alpha) \neq 0,\ \rho_\alpha(x) \in O_\lambda\}$$
$$= \{|x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n| \mid \alpha_i = \alpha,\ x_{\alpha_i} \in O_\lambda,\ t_i \neq 0 \text{ for some } \alpha_i\} .$$

Clearly open stars in $K(\mathcal{S})$ are open subsets of $K(\mathcal{S})$.

Every point x of $K(\mathcal{S})$ is presented by the barycentric coordinates $\pi(x)(\alpha)$ and "O_α-coordinates" $\rho_\alpha(x)$ for $\pi(x)(\alpha) \neq 0$. For $\sigma = \{\alpha_0, \cdots, \alpha_n\} \in N_n(\tilde{\mathcal{S}})$ and $\pi_{K(\tilde{\mathcal{S}})}: K(\tilde{\mathcal{S}}) \to |N(\tilde{\mathcal{S}})|$, $\pi_{K(\tilde{\mathcal{S}})}^{-1}(|\sigma|)$ is the join $O_{\alpha_0} * \cdots * O_{\alpha_n}$ of the orbits.

3. *G-CW* complexes and metrizability

G-CW complexes are defined and studied in [10]. We will show that they have the same topological properties as *CW*-complexes.

First we quote the following result from [4], 4.4.15.

Proposition 3.1. *Let f: $Y \to X$ be a closed map from a metrizable space Y onto a space X. If $f^{-1}(x)$ is compact for every $x \in X$, then X is metrizable.*

Let X be a space and $\{X_\lambda\}_{\lambda \in \Lambda}$ a closed covering of X. Then X is called to have the *weak topology with respect to* $\{X_\lambda\}_{\lambda \in \Lambda}$ iff, for any subset Ω of Λ, 1) $\bigcup_{\omega \in \Omega} X_\omega$ is closed in X and 2) a subset A of $\bigcup_{\omega \in \Omega} X_\omega$ is closed iff $A \cap X_\omega$ is closed in X for every $\omega \in \Omega$. If $\{X_\lambda\}$ is a locally finite closed covering of X then X has the weak topology with respect to $\{X_\lambda\}$. If X is a *G-CW* complex, then X has the weak topology with respect to the closed covering by closed G-cells or by finite G-subcomplexes.

Theorem 3.2. *Every G-CW complex is paracompact and perfectly normal.*

Proof. By Proposition 3.1 and by induction on the number of G-cells every finite *G-CW* complex is metrizable and hence paracompact and perfectly normal. As every *G-CW* complex has the weak topology with respect to the covering by finite G-subcomplexes, the theorem follows from [13], [15].

Lemma 3.3. *Let X be a space having the weak topology with respect to* *a closed covering* $\{X_\lambda\}_{\lambda\in\Lambda}$ *and* Λ *well-ordered. Let* Y *be a space,* $f\colon Y\to X$ *a map, and* Y_λ *a closed subspace of* Y *defined by*

$$Y_\lambda=\overline{f^{-1}(X_\lambda)-\bigcup_{\mu<\lambda}f^{-1}(X_\mu)}$$

for each $\lambda\in\Lambda$. *If* Y *is metrizable, then the closed covering* $\{Y_\lambda\}_{\lambda\in\Lambda}$ *of* Y *is locally finite.*

For the proof see [9], Lemma 1.

Proposition 3.4. *A G-CW complex* X *is metrizable iff* X *is locally finite.*

Proof. Let X be a locally finite G-CW complex, $E(X)=\sum_\alpha G/H_\alpha\times\Delta^n$ disjoint union of all G-cells of X, and $q\colon E(X)\to X$ the quotient map. Then q is clearly a closed surjective map. Compactness of $q^{-1}(x)$, $x\in X$, follows from local finiteness of X and compactness of G/H_α and Δ^n. Since $E(X)$ is disjoint union of metrizable spaces $G/H_\alpha\times\Delta^n$, $E(X)$ is also metrizable by [4], 4.2.1. Thus X is metrizable by Proposition 3.1.

Let X be a metrizable G-CW complex and $\{G\bar{e}_\alpha\}_{\alpha\in A}$ the closed covering by all closed G-cells of X. Choose a well-ordering relation $<$ on A such that $\alpha<\beta$ whenever $\dim G\bar{e}_\alpha<\dim G\bar{e}_\beta$. Then

$$\overline{G\bar{e}_\alpha-\bigcup_{\gamma<\alpha}G\bar{e}_\gamma}=G\bar{e}_\alpha\,.$$

Applying Lemma 3.3 to $f=Id_X$, we see that $\{G\bar{e}\}$ is locally finite. Thus X is locally finite. q.e.d.

4. *G*-ANR's and *G*-ANE's

By a (metrizable) *G-pair* we mean a pair (X, A) of a (metrizable) G-space X and a closed G-subspace A of X. By a *metric G-space* we mean a G-space with an (G-)invariant metric. We always assume that a *metric* of a metrizable G-space is *(G-)invariant*, for we can choose an invariant metric by averaging any metric over G.

Let (Y, B) be a G-pair and $f\colon B\to X$ a G-map. A G-map $\bar{f}\colon U\to X$ is called a *G-nbd extension* of f iff U is a G-nbd of B and $\bar{f}|_B=f$.

A G-space X is called a *G-ANE* (=*G-absolute nbd extensor*) (resp. *G-AE* (=*G-absolute extensor*)) iff, for every metrizable G-pair (Y, B) and every G-map $f\colon B\to X$, there exists a G-nbd extension $\bar{f}\colon U\to X$ (resp. a G-extension $\bar{f}\colon Y\to X$) of f.

Let (Y, X) be a G-pair. By a *G-nbd retraction to X in Y* we mean a G-retraction $r\colon U\to X$ from a G-nbd U of X in Y. A G-space X is called a *G-nbd* *retract* (resp. *G-retract*) of Y iff X is a closed G-subspace of Y and there exists a G-nbd retraction $r: U \to X$ (resp. a G-retraction $r: Y \to X$).

A G-space X is called a *G-ANR* (=*G-absolute nbd retract*) (resp. *G-AR* (=*G-absolute retract*)) iff X is metrizable and, whenever X is a closed G-subspace of a metrizable G-space Y, X is a G-nbd retract (resp. a G-retract) of Y.

Proposition 4.1. *If X is a G-AE or a G-AR, then $X^G \neq \emptyset$.*

Proof. Let X be a G-AE (G-AR). Let $f: B \to X$ be a G-map from a metrizable G-space B ($B=X, f=Id_X$). Let Y be the G-space $B \cup \{*\}$, disjoint union of B and the one-point G-set $\{*\}$. Then Y is metrizable and there is a G-extension $\bar{f}: Y \to X$ of f (a G-retraction $\bar{f}$). Since $\bar{f}(*) \in X^G$, we see $X^G \neq \emptyset$. q.e.d.

REMARK. G-ANR's for normal G-spaces ([17], 1.6.1) are G-ANE's in our sense, (for metrizable G-spaces are normal,) and they are G-ANR's in our sense if they are metrizable by Theorem 6.4.

5. An equivariant version of Dugundji's extension theorem

A G-space L is called a *topological linear G-space* iff L is a topological linear space (over $\boldsymbol{R}$) on which G acts linearly.

By a *normed linear G-space* we mean a topological linear G-space L together with an invariant norm $\| \ \|$. L is a metric G-space with the invariant metric d given by $d(x, y)=\|x-y\|$. If a topological linear G-space L is normable, then L has an invariant norm by averaging any norm over G.

If a normed linear G-space B is complete under the above norm topology, then B is called a *Banach G-space.* When a topological linear G-space L is a Banach space, we can make L a Banach G-space by replacing the norm with the "average" over G.

Banach G-spaces and normable linear G-spaces are locally convex.

EXAMPLE 5.1. Let X be a G-space. Let $B(X)$ denote the Banach space of all bounded continuous functions $X \to \boldsymbol{R}$, where the norm $\| \ \|$ is defined by

$$\|f\| = \sup_{x \in X} |f(x)| .$$

We define a G-action on $B(X)$ by

$$(gf)(x) = f(g^{-1}x)$$

for $f \in B(X)$, $g \in G$, and $x \in X$. Clearly the norm is invariant. Then $B(X)$ is a Banach G-space.

We remark that the convex hull C of any G-subset A in a topological linear G-space L is a G-subset of L.

Lemma 5.2. *Let (X, A) be a completely regular G-pair and C a G-space. Let $U(C)$ be the subspace $\{x \in X \mid C^{G_x} \neq \emptyset\}$. Then:*

(1) *$U(C)$ is an open G-subspace of X.*

(2) *If there is a G-map $f: A \to C$, then $U(C)$ is an open G-nbd of A. In particular, $U(A)$ is an open G-nbd of A.*

(3) *If $C^G \neq \emptyset$, then $U(C) = X$.*

Proof. Now $U(C) = \bigcup_{y \in C} \bigcup_{(H) \subset (G_y)} X_{(H)}$, where (H) denotes the conjugacy class of H and $X_{(H)} = \{x \in X \mid (G_x) = (H)\}$. Since $\bigcup_{(H) \subset (G_y)} X_{(H)}$ is an open G-subspace of X for each $y \in C$ by [17], 1.7.2, so is $U(C)$. If there is a G-map $f: A \to C$, then $G_a \subset G_{f(a)}$ and $f(a) \in C^{G_a} \neq \emptyset$ for every $a \in A$, hence $A \subset U(C)$. The remainder is clear. q.e.d.

REMARK. For G-spaces X and C, there is no G-map from any G-subspace of $X - U(C)$ to C.

The main purpose of this section is to show the following equivariant version of Dugundji's extension theorem, [3], 4.1, [7], (II, 14.1).

Theorem 5.3. *Let C be a convex G-subset of a locally convex topolegical linear G-space L, (X, A) a metrizable G-pair, and $f: A \to C$ a G-map. Let $U(C)$ denote the open G-nbd $\{x \in X \mid C^{G_x} \neq \emptyset\}$ of A. Then there exists a G-extension $\bar{f}: U(C) \to C$ of f. Hence C is a G-ANE. If $C^G \neq \emptyset$, then $U(C) = X$ and C is a G-AE.*

REMARK. 1) Any G-map cannot be extended equivariantly to any larger G-subspace than $U(C)$.

2) If G is a finite group, then every convex G-set C has a G-fixed point (i.e., $C^G \neq \emptyset$). In fact, $1/|G| \sum_{g \in G} gx$, $x \in C$, is a G-fixed point. Hence every convex G-subset of a locally convex topological linear G-space is a G-AE for a finite G.

3) When G is a compact Lie group, a convex G-subset of a Banach G-space does not always have a G-fixed point. Indeed, when $G = S^1$, there exists an example of a convex G-subset in $B(S^1)$ which has no G-fixed point.

Corollary 5.4. *Every convex G-subset C of a normable linear G-space is a G-ANE. If $C^G \neq \emptyset$, then C is a G-AE.*

Theorem 5.5. *Let L be a locally convex topological linear G-space, (X, A) a metrizable G-pair, and $f: A \to L$ a G-map. Then:*

(1) *There exists a G-extension $\bar{f}: X \to L$ of f such that the image $f(X)$ is contained in the convex hull of $f(A) \cup \{0\}$.*

(2) *There exists a G-extension $\bar{f}: U(A) \to L$ of f such that the image $\bar{f}(U(A))$*

is contained in the convex hull of $f(A)$, *where* $U(A)$ *is the open* G-*nbd* $\{x\in X\mid A^{G_x}\neq\emptyset\}$ *of* A.

The theorem is a corollary to Theorem 5.3.

Let (X, A) be a metrizable G-pair and d an invariant metric of X. For $x\in X-A$ let V_x denote the open *nbd* of x in $X-A$ defined by

$$V_x=\{y\in X\mid d(x, y)<\frac{1}{2}d(x, A)\}\ .$$

Consider an open G-covering $\{V_x\}_{x\in X-A}$ of $X-A$. By Proposition 2.3, there is a *TN* G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ which is a star-refinement of $\{V_x\}_{x\in X-A}$. Then we obtain the following equivariant version of [3], 2.1, [7], (II, 11.1) similar to [7].

Lemma 5.6. *Let* (X, A) *be a metrizable* G-*pair. Then there exists a* *TN* G-*covering* $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ *of* $X-A$ *such that*

1) *for any nbd* V *of* $a\in A$ *in* X, *there exists a nbd* W, $a\in W\subset V$, *such that* $W\cap S_\lambda\neq\emptyset$ *implies* $S_\lambda\subset V$ *for* $S_\lambda\in\mathcal{S}$,

2) *any nbd of* $a\in\partial A$ *contains infinitely many elements of* $\mathcal{S}$.

A *TN* G-covering of $X-A$ satisfying the above conditions 1) and 2) is called a *canonical TN* G-*covering* of $X-A$. We state an equivariant version of [3], 3.1, [7], (II, 12.1).

Proposition 5.7. *Let* (X, A) *be a metrizable* G-*pair. Then there exists a* G-*space* Y *and a* G-*map* $\mu\colon X\to Y$ *with the properties*:

1) $\mu|_A$ is a G-homeomorphism and $\mu(A)$ is a closed G-subspace of Y.

2) $Y-\mu(A)$ is the G-nerve of a canonical *TN* G-covering of $X-A$ and $\mu(X-A)\subset Y-\mu(A)$.

Proof. Let $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ be a canonical *TN* G-covering of $X-A$ with $\{P_\alpha\}_{\alpha\in\Lambda/G}$, $K=K(\mathcal{S})$ the G-nerve of $\mathcal{S}$, and $O(O_\lambda, K)$ the open star of $O_\lambda(=r_\alpha(S_\lambda)=O_\alpha\cap S_\lambda)$. Put $Y=A\cup K$ as G-sets. For a *nbd* V of $a\in A$ in X we define a subset V^* of $A\cup K$ by

$$V^*=(A\cap V)\cup(\cup\{O(O_\lambda, K)\colon \lambda\in\Lambda,\ S_\lambda\subset V\})\ .$$

Clearly $a\in V^*$, $V^*\cap A=V\cap A$, and $V^*\cap K$ is open in K.

Topologize $A\cup K$ as follows:

i) A basis for *nbd*s of $a\in A$ in $A\cup K$ is taken to be the totality of the sets V^* determined by *nbd*s V of a in X.

ii) A basis for *nbd*s of $y\in K$ in $A\cup K$ is taken to be the totality of *nbd*s of y in K.

It is easy to verify that $A\cup K$ with this topology is a Hausdorff G-space, and that both A and K, as subspaces, preserve their original topologies. Clearly K is an open G-subspace and hence A is a closed G-subspace.

Using the G-map $P: (X-A)\to K$ defined by $\{p_\alpha\}_{\alpha\in\Lambda/G}$, we define $\mu: X\to A\cup K$ by

$$\mu(x) = \begin{cases} x & \text{if } x\in A \\ P(x) & \text{if } x\in X-A . \end{cases}$$

Then μ is clearly well-defined, equivariant, and both $\mu|_A$ and $\mu|_{X-A}$ are continuous. To prove the continuity of μ, it suffices to show the continuity at points of $\partial A=A\cap\overline{(X-A)}$ in X. Let $a\in\partial A$ and V^* a basic *nbd* of $a\,(=\mu(a))$ in $A\cup K$ which is given by a *nbd* V of a in X. By Lemma 5.6 there is a *nbd* W, $a\in W\subset V$, such that $S_\lambda\cap W\neq\emptyset$ implies $S_\lambda\subset V$. We assert $\mu(W)\subset V^*$. In fact, let $x\in W$. If $x\in W-A$, then $\mu(x)=P(x)\in K$. There is $S_\lambda\in\mathcal{S}$ such that $p_\alpha(x)\neq 0$, $\alpha=G\lambda$ and $x\in S_\lambda$. Since $x\in S_\lambda\cap W\neq\emptyset$, we see $r_\alpha(x)\in O_\lambda\subset S_\lambda\subset V$. This shows

$$\mu(x) = P(x)\in O(O_\lambda, K)\subset V^* .$$

If $x\in W\cap A$, then

$$\mu(x) = x\in W\cap A\subset V\cap A = V^*\cap A\subset V^* .$$

This proves that μ is continuous. The properties 1) and 2) now follows at once. q.e.d.

We will construct a G-*nbd* retraction to A in $A\cup K^0$ in order to extend a G-map of A to the 0-skeleton K^0 of K. As for the G-*nbd*, we choose $U(A)^*\cap(A\cup K^0)$ for the open G-*nbd* $U(A)=\{x\in X\,|\,A^{G_x}\neq\emptyset\}$.

Lemma 5.8. *There exists a G-nbd retraction*

$$\phi: U(A)^*\cap(A\cup K^0)\to A$$

in $A\cup K^0$.

Proof. Now K^0 is disjoint union of the orbits O_α and

$$K^0\cap U(A)^* = \sum_{T_\alpha\subset U(A)} O_\alpha .$$

Select a point $x_\alpha\in O_\alpha$ for each $O_\alpha\subset U(A)^*\cap K^0$, then $A^{G_{x_\alpha}}\neq\emptyset$. Choose a point $a_\alpha\in A^{G_{x_\alpha}}$ such that $d(x_\alpha, a_\alpha)<2d(x_\alpha, A^{G_{x_\alpha}})$. Define a G-map ϕ_α: $O_\alpha\to Ga_\alpha\subset A$ by $\phi_\alpha(gx_\alpha)=ga_\alpha$. Then

$$d(x, \phi_\alpha(x))<2d(x, A^{G_x})$$

for $x\in O_\alpha$ $(A^{G_x}\neq\emptyset)$ by the G-invariance of d. Define ϕ: $U(A)^*\cap(A\cup K^0)\to A$ by

$$\phi(x) = \begin{cases} x & \text{if } x\in A \\ \phi_\alpha(x) & \text{if } x\in O_\alpha \subset U(A)^* \cap K^0 . \end{cases}$$

Then ϕ is clearly equivariant and both $\phi|_A$ and $\phi|_{O_\alpha}=\phi_\alpha$ are continuous. To prove the continuity, it suffices to show the continuity at points of ∂A in $U(A)^* \cap (A\cup K^0)$. Let $a\in\partial A$ and $V_\varepsilon=\{x\in A\,|\,d(x, a)<\varepsilon\}$, the ε-*nbd* of a in A, for an arbitrary $\varepsilon>0$. Let $V=\{x\in X\,|\,d(x, a)<\varepsilon/5\}$. Choose an open tube segment S in V generated by an open *nbd* of a in Ga with a G-retraction $r: GS\to Ga$. Then $S\subset U(A)$. We complete the continuity proof by showing

$$\phi(S^* \cap (A\cup K^0))\subset V_\varepsilon .$$

Since $S^*\cap A=S\cap A$, we see $\phi(S^*\cap A)=\phi(S\cap A)=S\cap A\subset V\cap A\subset V_\varepsilon$. Let $x\in S^*\cap K^0$. Then there is $S_\lambda\in\mathcal{S}$ such that $S_\lambda\subset S$ and $x\in O(O_\lambda, K)\cap K^0=O_\lambda$. Hence $x\in O_\lambda\subset S_\lambda\subset S$. Now $r(x)\in A^{G_x}\cap S\neq\emptyset$. Put $b=\phi(x)\in A^{G_x}$ and let $c\in A^{G_x}\cap S$. Since $\{x, c\}\subset S\subset V$, we see $d(x, a)<\varepsilon/5$ and $d(c, a)<\varepsilon/5$. Thus

$$\begin{aligned} d(a, \phi(x)) = d(a, b) &\leqq d(a, x)+d(x, b) \\ &< d(a, x)+2d(x, A^{G_x}) \\ &\leqq d(a, x)+2(d(x, a)+d(a, c)) \\ &= 3d(a, x)+2d(a, c) \\ &< 3\varepsilon/5+2\varepsilon/5 = \varepsilon . \end{aligned}$$

Hence $\phi(x)=b\in V_\varepsilon$, which completes the proof. q.e.d.

Corollary 5.9. *Let $f'\colon A\cup(K^0-U(A)^*)\to C$ be a G-map. Then the extension $\bar{f}'\colon A\cup K^0\to C$ defined by*

$$\bar{f}'(x) = \begin{cases} f'(x) & \text{if } x\in A\cup(K^0-U(A)^*) \\ f'(\phi(x)) & \text{if } x\in K^0\cap U(A)^* \end{cases}$$

is a G-map.

Lemma 5.10. *Let C be a convex G-subset of a locally convex topological linear G-space L and $f^0\colon A\cup K^0\to C$ a map. Define an extension $f^\infty\colon A\cup K\to C$ of f^0 by*

$$f^\infty(y) = \begin{cases} \sum_{i=0}^{n} t_i f^0(x_{\alpha_i}) & \text{if } y=|x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n|\in K \\ f^0(y) & \text{if } y\in A . \end{cases}$$

Then f^∞ is continuous. If f^0 is a G-map, so is f^∞.

Proof. Since C is convex, f^∞ is well-defined, and equivariant if f^0 is so. Clearly both $f^\infty|_A$ and $f^\infty|_K$ are continuous. Thus the continuity of f^∞ need only be checked at points of ∂A in $A\cup K$.

Let $a\in\partial A$. Since L is locally convex and so is C, any given *nbd* of $f^{\infty}(a)=f^{0}(a)$ in C contains a convex *nbd* M of $f^{\infty}(a)$ in C. As f^0 is continuous at a, there is a basic *nbd* V^* of a with

$$f^{0}(V^*\cap(A\cup K^{0}))\subset M.$$

By Lemma 5.6 there is a *nbd* W, $a\in W\subset V$, in X such that $S_\lambda\cap W\neq\emptyset$ implies $S_\lambda\subset V$. We will complete the proof by showing

$$f^{\infty}(W^*)\subset M.$$

Since $W^*\cap A=W\cap A\subset V^*\cap A$, we have $f^{\infty}(W^*\cap A)=f^{0}(W^*\cap A)\subset M$. Let $y=|x_{\alpha_0}\cdots x_{\alpha_n};t_0,\cdots,t_n|\in W^*\cap K$. From $y\in W^*$ there is $S_\lambda\in\mathcal{S}$ such that $S_\lambda\subset W$ and $y\in O(O_\lambda,K)$, i.e., $\alpha_i=G\lambda$ and $x_{\alpha_i}\in O_\lambda$ for some i. Put $\lambda_i=\lambda$. From $y\in K$ there are $\lambda_j\in\alpha_j$, $j=0,\cdots,n$, such that $x_{\alpha_j}\in O_{\lambda_j}$ and $\bigcap_{j=0}^{n}S_{\lambda_j}\neq\emptyset$. Then $S_\lambda\cap S_{\lambda_j}\subset W\cap S_{\lambda_j}\neq\emptyset$ and hence $x_{\alpha_j}\in S_{\lambda_j}\subset V$ for $j=0,\cdots,n$. This implies $x_{\alpha_j}\in V^*\cap K^0$ and $f^0(x_{\alpha_j})\in M$ for $j=0,\cdots,n$. Since M is convex, we have

$$f^{\infty}(y)=\sum_{j=0}^{n}t_jf^{0}(x_{\alpha_j})\in M.$$

Thus $f^{\infty}(W^*)\subset M$ and the lemma is proved. q.e.d.

Proof of Theorem 5.3. Let (X,A) be a metrizable G-pair, C a convex G-subset of a locally convex topological linear G-space, and $f\colon A\to C$ a G-map. Applying Proposition 5.7 to $(U(C),A)$, we have a canonical *TN* G-covering $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ of $U(C)-A$, a G-space $A\cup K=A\cup K(\mathcal{S})$, and a G-map $\mu\colon U(C)\to A\cup K$. We extend f to $A\cup K^0$ as follows: Now $K^0-U(A)^*$ is disjoint union of orbits. We select a point $x_\alpha\in O_\alpha$ for each $O_\alpha\subset K^0-U(A)^*$, choose a point $z_\alpha\in C^{G_{x_\alpha}}$, and define a G-map $f_\alpha\colon O_\alpha\to Gz_\alpha$ by $f_\alpha(gx_\alpha)=gz_\alpha$. We define a G-map $f^0\colon A\cup K^0\to C$ by

$$f^{0}(x)=\begin{cases}f(\phi(x)) & \text{if } x\in U(A)^*\cap(A\cup K^0)\\ f_\alpha(x) & \text{if } x\in O_\alpha\subset K^0-U(A)^*,\end{cases}$$

where ϕ is the G-*nbd* retraction of Lemma 5.8. Then f^0 is a G-map. Applying Lemma 5.10 to f^0, we have a G-extension $f^{\infty}\colon A\cup K\to C$ of f^0. Finally we put $\bar{f}=f^{\infty}\circ\mu$, which is the required G-extension. q.e.d.

Remark. The obtained G-extension $\bar{f}\colon U(C)\to C$ of f is explicitly exhibited by

$$\bar{f}(x)=\sum_{T_\alpha\subset U(A)}p_\alpha(x)f(\phi(r_\alpha(x)))+\sum_{T_\alpha\not\subset U(A)}p_\alpha(x)f_\alpha(r_\alpha(x))$$

for $x\in U(C)-A$ (and $\bar{f}|_A=f$), where $f_\alpha\colon O_\alpha\to C$ is any G-map for $T_\alpha\not\subset U(A)$, and $\phi|_{O_\alpha}: O_\alpha \to A$ is a G-map for $T_\alpha \subset U(A)$ such that

$$d(a, \phi(x)) \leqq 3d(a, x) + 2d(a, A^{G_x})$$

for every $a \in A$ and $x \in O_\alpha$.

Theorem 5.11. *Let (X, A) be a metrizable G-pair. Then there exists an isometric linear G-embedding $\psi: B(A) \to B(X)$ such that $\psi(f)$ is an extension of f for every $f \in B(A)$.*

Proof. Note that $U(\boldsymbol{R}) = X$. With the notation of Theorem 5.3 we have a canonical TN G-covering $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ of $X - A$ with an invariant partition of unity $\{p_\alpha\}_{\alpha \in \Lambda/G}$ and G-retractions $r_\alpha: T_\alpha \to O_\alpha$ for $T_\alpha \in \tilde{\mathcal{S}}$, and a G-map $\phi: A \cup (\sum_{T_\alpha \subset U(A)} O_\alpha) \to A$ with $\phi|_A = Id_A$ once at all. Define, for each $f \in B(A)$,

$$\psi(f)(x) = \sum_{T_\alpha \subset U(A)} p_\alpha(x) f(\phi(r_\alpha(x))) \qquad (+0) .$$

Then ψ is the required one. q.e.d.

6. Relation between a G-ANR and a G-ANE

In this section we study the relation between a G-ANR and a G-ANE, and the elementary properties of them parallel to the non-equivariant case as in [7].

Proposition 6.1.

(1) *Every open G-subspace of a G-ANE is a G-ANE.*

(2) *Every G-retract of a G-AE is a G-AE and every G-nbd retract of a G-ANE is a G-ANE.*

The proof is obtained by routine translation of [7], (II, 5.1), (II, 5.2), and (II, 6.1) into the terminology of the category Top^G of (Hausdorff) G-spaces and G-maps.

Here we state an equivariant version of Wojdyslawski's embedding theorem.

Theorem 6.2. *Let X be a metric G-space with a bounded invariant metric d. Let $B(X)$ denote the Banach G-space of all bounded continuous functions on X (Example 5.1). Define $i: X \to B(X)$ by*

$$i(x)(y) = d(x, y)$$

for $x, y \in X$. Then i is an isometric G-embedding and the image $i(X)$ is a closed G-subspace of the convex hull C of $i(X)$ in $B(X)$. If X is separable, so is C.

Proof. Since d is G-invariant, we see

$$i(gx)(y) = d(gx, y) = d(x, g^{-1}y) = i(x)(g^{-1}y) = (gi(x))(y)$$

for $g \in G$ and $x, y \in X$. Thus i is equivariant. The remainder is the same as [7], (III, 2.1) (the Wojdyslawski theorem). q.e.d.

For any given metric d' of X we can form a bounded metric d of X by defining $d(x, y) = d'(x, y)/(1 + d'(x, y))$, $x, y \in X$. Thus, by Theorem 6.2, we get the following

Corollary 6.3. *Every metrizable G-space can be embedded as a closed G-subspace in a convex G-subset of a Banach G-space.*

Theorem 6.4. *A metrizable G-space is a G-ANR (resp. G-AR) iff it is a G-ANE (resp. G-AE).*

Proof. Let X be a metrizable G-AE (G-ANE) embedded as a closed G-subspace of a metrizable G-space Y. Consider the identity map Id_X of X. Since X is a G-AE (G-ANE), Id_X has a G-(nbd) extension, which is a G-(nbd) retraction.

Embedding X into a convex G-subset of a Banach G-space as a closed G-subspace by Corollary 6.3, the converse follows from Corollary 5.4 and Propositions 4.1, 6.1. q.e.d.

As corollaries to Theorem 6.4 the following Propositions 6.5–6.7 are obvious by the above results.

Proposition 6.5. *Every convex G-subset C of a locally convex metrizable topological linear G-space is a G-ANR. If $C^G \neq \emptyset$, then C is a G-AR.*

Proposition 6.6. *A G-space X is a G-ANR (resp. G-AR) iff X is G-homeomorphic to a G-nbd retract (resp. G-retract) of a convex G-subset C of a Banach G-space. (resp. with $C^G \neq \emptyset$).*

Proposition 6.7.

(1) *Every open G-subspace of a G-ANR is a G-ANR.*

(2) *Every G-nbd retract of a G-ANR is a G-ANR and every G-retract of a G-AR is a G-AR.*

Proposition 6.8. *Every topological product of a finite collection of G-ANE's (resp. G-ANR's) is a G-ANE (resp. G-ANR).*

The proof is obvious.

Applying Propositions 6.1 and 6.7 to tubes, we obtain

Proposition 6.9. *Every tube in a G-ANR (resp. G-ANE) is a G-ANR* (*resp. G-ANE*).

A G-nbd retract of a Euclidian G-space (=finite dimensional orthogonal G-representation space) is called a G-ENR (G-Euclidian nbd retract). By Propositions 6.5 and 6.7 we see the followings (, though it is a circular argument.)

Proposition 6.10. *Every Euclidian G-space is a G-AR and every G-ENR is a G-ANR. In particular, every G-orbit is a G-ANR.*

REMARK. This proposition also follows from the Tietze-Gleason theorem and Proposition 6.7, or [17], 1.6.2 and 1.6.4.

A G-space X is G-contractible iff the identity map of X is G-homotopic to a constant map (into $X^G \neq \emptyset$).

A G-space X is said to be locally G_x-contractible at a point $x \in X$ iff every G_x-nbd U contains a G_x-nbd V which is G_x-contractible in U.

A G-space X is called to be locally equivariantly contractible iff X is locally G_x-contractible at every point $x \in X$. For example, a convex G-subset of a Banach G-space is locally equivariantly contractible and G-contractible.

Proposition 6.11. *Every G-AR is G-contractible and every G-ANR is locally equivariantly contractible.*

Proof. Let X be a G-ANR and embedded as a G-nbd retract of a convex G-subset C of a Banach G-space with a G-nbd retraction $r: U \to X$, $X \subset U \subset C$. Let $x \in X$ and V be a given G_x-nbd of x in X. There is an ε-nbd W' of x in C such that $W' \subset r^{-1}(V) \subset U$. Put $W = W' \cap X$. Define a G-homotopy h_t: $W \to V$, $t \in I$, by

$$h_t(y) = r((1-t)y + tx)$$

for $y \in W$. This shows that X is locally equivariantly contractible.

If X is a G-AR, there is a G-retraction $r: C \to X$ and $X^G \neq \emptyset$. Choose a point $x \in X^G$. Define a G-contraction $h_t: X \to X$ by $h_t(y) = r((1-t)y + tx)$ for $y \in X$. This shows the G-contractibility of X. q.e.d.

Lemma 6.12 (G-Urysohn). *If A and B are disjoint closed G-subsets of a normal G-space X, then there is an invariant continuous function f: X→I such that* $f(A)=0$ *and* $f(B)=1$.

The proof is obtained by averaging a function given by Urysohn's lemma over G.

Proposition 6.13. *Every G-contractible G-ANR (resp. G-ANE) is a G-AR (resp. G-AE).*

The proof is obtained by routine translation of [7], (II, 7.1) and (III, 7.2) into the terminology of Top^G.

7. Union of G-ANR's

DEFINITION 7.1.

(1) A G-space X is called a *local G-ANE* iff every point of X has a G-nbd which is a G-ANE.

(2) A G-space X is called a *local G-ANR* iff X is metrizable and every point of X has a G-nbd which is a G-ANR.

REMARK. A metrizable G-space X which is a local G-ANE is a local G-ANR by Theorem 6.4.

The purpose of this section is to show the following equivariant versions of [6], Theorem 19.2 and [9], theorem, or [7], (II, 17.1).

Theorem 7.2.

(1) *Every local G-ANE is a G-ANE.*

(2) *Every local G-ANR is a G-ANR.*

Theorem 7.3. *Let X be a G-space having the weak topology with respect to a closed invariant covering $\{X_\lambda\}_{\lambda\in\Lambda}$. Assume that, for each finite subcollection $\{X_{\lambda_0}, \cdots, X_{\lambda_n}\}$ of $\{X_\lambda\}_{\lambda\in\Lambda}$ with non-void intersection, $\bigcap_{i=0}^{n} X_{\lambda_i}$ is a G-ANE. Then X is a G-ANE. If X is metrizable, then X is a G-ANR.*

Corollary 7.4. *Let X be a G-space (resp. metrizable G-space). Suppose one of the following two conditions:*

(1) *X is the union of open G-ANE subspaces.*

(2) *X is the union of two closed G-subspaces X_1 and X_2 such that both X_1, X_2, and $X_1\cap X_2$ are G-ANE's.*

Then X is a G-ANE (resp. G-ANR).

Lemma 7.5. *Let (Y, B) be a metrizable G-pair and $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ a locally finite closed invariant covering of B. Then there exist a G-nbd F of B in Y and a locally finite closed invariant covering $\mathcal{F}=\{F_\lambda\}_{\lambda\in\Lambda}$ of F such that*

(1) *$F_\lambda\cap B=B_\lambda$ for each $\lambda\in\Lambda$ and*

(2) *the nerve $N(\mathcal{F})$ of $\mathcal{F}$ is isomorphic to the nerve $N(\mathcal{B})$ of $\mathcal{B}$.*

This is an equivariant version of [9], Lemma 2 and proved by applying [9] to $(Y/G, B/G)$ and $\mathcal{B}/G=\{B_\lambda/G\}$, and pulling up the obtained ones to Y by Π_Y: $Y\to Y/G$.

Lemma 7.6. *Let (Y, B) be a G-pair and $\mathcal{F}=\{F_\lambda\}_{\lambda\in\Lambda}$ a locally finite closed invariant covering of a closed G-nbd F of B. Suppose that, for each $\lambda\in\Lambda$, there is a G-nbd C_λ of $F_\lambda\cap B$ in F_λ. Then*

$$C=\bigcup_{\lambda\in\Lambda}C_\lambda$$

is a closed G-nbd of B in Y.

Clearly C is a G-subspace. The rest of the proof is similar to [6], Lemma 20.2.

Lemma 7.7. *Let X be a G-space and $\{X_0,\cdots,X_n\}$ a closed invariant covering of X such that $\bigcap_{j=0}^{p}X_{i_j}$ is a G-ANE for each $\{X_{i_0},\cdots,X_{i_p}\}\subset\{X_0,\cdots,X_n\}$ with $\bigcap_{j=0}^{p}X_{i_j}\neq\emptyset$. Let (Y,B) be a metrizable G-pair, $f\colon B\to X$ a G-map, and $\{Y_0,\cdots,Y_n\}$ a closed invariant covering of Y such that*

$$f(Y_i\cap B)\subset X_i$$

for $i=0,\cdots,n$. Then there exist a closed G-nbd S of B in Y and a G-extension $\bar{f}\colon S\to X$ such that $\bar{f}(S\cap Y_i)\subset X_i$ for $i=0,\cdots,n$.

The proof is obtained by formal translation of [9], Lemma 4 into the terminology of Top^G.

Lemma 7.8. *Let X be a G-space and $\{X_\lambda\}_{\lambda\in\Lambda}$ an invariant covering such that $\bigcap_{j=0}^{n}X_{\lambda_j}$ is a G-ANE for each finite subcollection $\{X_{\lambda_0},\cdots,X_{\lambda_n}\}\subset\{X_\lambda\}_{\lambda\in\Lambda}$ with $\bigcap_{j=0}^{n}X_{\lambda_j}\neq\emptyset$. Let (F,B) be a metrizable G-pair, $f\colon B\to X$ a G-map, and $\mathcal{F}=\{F_\lambda\}_{\lambda\in\Lambda}$ a locally finite closed invariant covering of F such that the nerve $N(\mathcal{F})$ of $\mathcal{F}$ is isomorphic to the nerve $N(\mathcal{B})$ of $\mathcal{B}$ and $f(B_\lambda)\subset X_\lambda$ for each $\lambda\in\Lambda$, where $B_\lambda=F_\lambda\cap B$ for $\lambda\in\Lambda$ and $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$. Then there exist a closed G-nbd M of B in F and a G-extension $\bar{f}\colon M\to X$ of f such that $\bar{f}(M\cap F_\lambda)\subset X_\lambda$ for each $\lambda\in\Lambda$.*

The lemma is proved in a parallel way to [9], §3 by using Lemmas 7.6 and 7.7.

Proof of Theorems 7.2 and 7.3.

We prove the G-ANE parts. The G-ANR parts follow from Theorem 6.4.

Let (Y,B) be a metrizable G-pair and $f\colon B\to X$ a G-map. In order to prove the theorems, we will construct a locally finite invariant closed coverings $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ of B and $\mathcal{F}=\{F_\lambda\}_{\lambda\in\Lambda}$ of a closed G-nbd F of B in Y, and an invariant covering $\{X_\lambda\}_{\lambda\in\Lambda}$ of X for Theorem 7.2, satisfying the assumption of Lemma 7.8. Then we obtain a G-nbd extension by Lemma 7.8, which shows that X is a G-ANE.

To prove Theorem 7.2, first we construct $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ and $\{X_\lambda\}_{\lambda\in\Lambda}$: Since X is a local G-ANE, there is a covering $\mathcal{U}$ of X by open G-ANE subspaces.

As B and B/G are paracompact, there is a locally finite closed invariant refinement $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ of $\{f^{-1}(U)\}_{U\in\mathcal{U}}$. For each $\lambda\in\Lambda$ we choose $U\in\mathcal{U}$ with $f(B_\lambda)\subset U$ and call it X_λ. (Then $f(B_\lambda)\subset X_\lambda$.) Since every X_λ is an open G-ANE subspace, each $\bigcap_{i=0}^{n} X_{\lambda_i}(\neq\emptyset)$, $\{\lambda_0, \cdots, \lambda_n\}\subset\Lambda$, is an open G-ANE subspace by Proposition 6.1. Then $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ and $\{X_\lambda\}_{\lambda\in\Lambda}$ satisfy the assumption of lemma 7.8. (Replace X by $\bigcup_{\lambda\in\Lambda} X_\lambda$ if necessary.)

To prove Theorem 7.3, we construct $\mathcal{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ for a given $\{X_\lambda\}_{\lambda\in\Lambda}$ by well-ordering the index set Λ and by Lemma 3.3 ($B_\lambda=\overline{f^{-1}(X_\lambda)-\bigcup_{\mu<\lambda} f^{-1}(X_\mu)}$).

Finally, constructing F and $\mathcal{F}=\{F_\lambda\}_{\lambda\in\Lambda}$ from the obtained $\mathcal{B}$ by Lemma 7.5, we complete the proof. q.e.d.

Proposition 7.9. *If X_1 and X_2 are two closed G-subspaces of a G-ANE (resp. G-ANR) X such that $X_1\cup X_2=X$ and $X_1\cap X_2$ is a G-ANE (resp. G-ANR), then both X_1 and X_2 are G-ANE's (resp. G-ANR's).*

The proof is similar to [7], (II, 9.1).

8. Relation for subgroups and G-manifolds

Let $\alpha: G'\to G$ be a continuous homomorphism of compact Lie groups. Let H be a closed subgroup of G, $NH=N_GH$, the normalizer of H in G, and $WH=NH/H$. Let $X^H=\{x\in X\,|\,G_x\supset H\}$ (the H-fixed point set), $X_H=\{x\in X\,|\,G_x=H\}$, $X^{(H)}=GX^H$, and $X_{(H)}=GX_H$.

We consider functors such as the restriction functor $\mathrm{Res}^G_{G'}$, : $\mathrm{Top}^G\to\mathrm{Top}^{G'}$ (Res^G_H), the H-fixed point functor: $\mathrm{Top}^G\to\mathrm{Top}^H$, and the functor $G\times_H -$: $\mathrm{Top}^H\to\mathrm{Top}^G$.

Proposition 8.1. *Let X be a G-ANR. Then:*

(1) *$\mathrm{Res}^G_{G'}X$ is a G'-ANR for any homomorphism $\alpha: G'\to G$ from any compact Lie group G'.*

(2) *X^H is a WH-ANR and X_H is an open WH-ANR subspace of X^H.*

Proof. By Proposition 6.6 X is regarded as a G-nbd retract of a convex G-subset C of a Banach G-space B with a G-nbd retraction $r: U\to X$. Then, clearly, $\mathrm{Res}^G_{G'}r$: $\mathrm{Res}^G_{G'}U\to\mathrm{Res}^G_{G'}X$ is a G'-nbd retraction in the convex G'-subset $\mathrm{Res}^G_{G'}C$ of the Banach G'-space $\mathrm{Res}^G_{G'}B$, and r^H: $U^H\to X^H$ is a WH-nbd retraction in the convex WH-subset C^H of the Banach WH-space B^H. X_H is an open WH-subspace of X^H. The proof follows from Propositions 6.6 and 6.7. q.e.d.

Corollary 8.2.

(1) *Every ANR is a G-ANR with the trivial G-action.*

(2) *Every G-ANR is an ANR.*

Proposition 8.3. *If X is a G-ANE, then $Res_H^G X$ is an H-ANE for any closed subgroup H.*

Proof. Let (Y, B) be a metrizable H-pair and $f: B \to \mathrm{Res}_H^G X$ an H-map. Consider the G-map $f' = \phi \circ G \times_H f$: $G \times_H B \to X$, where ϕ: $G \times_H \mathrm{Res}_H^G X \to X$ is defined by $\phi(g, x) = gx$. Since X is a G-ANE, there is a G-nbd extension $\bar{f}$: $U \to X$ of f', $G \times_H B \subset U \subset G \times_H Y$. Put $V = U \cap Y$ $(Y = H \times_H Y \subset G \times_H Y)$. Then $\bar{f}|_V$ is an H-extension of f. q.e.d.

Proposition 8.4. *If X is a G-ANE (resp. G-ANR), then X^H is an NH-ANE (resp. NH-ANR) and X_H is an open NH-ANE (resp. NH-ANR) subspace.*

Proof. Let (Y, B) be a metrizable NH-pair and $f: B \to X^H$ an NH-map. Consider the G-map $f' = \phi \circ G \times_{NH} f$: $G \times_{NH} B \to X$, where ϕ: $G \times_{NH} X^H \to X$, $\phi(g, x) = gx$. Since X is a G-ANE, there is a G-nbd extension $\bar{f}$: $U \to X$ of f', $G \times_{NH} B \subset U \subset G \times_{NH} Y$. Then $\bar{f}^H$: $U^H \to X^H$ is an NH-nbd extension of f. Thus X^H is an NH-ANE. If X is a G-ANR, then X^H is metrizable and hence an NH-ANR. q.e.d.

Proposition 8.5. *Let X be an H-space. Then $G \times_H X$ is a G-ANR (resp. G-ANE) iff X is an H-ANR (resp. H-ANE).*

Proof. $G \times_H X$ is metrizable iff X is so by Proposition 3.1. We prove for a G-ANE. The proof for a G-ANR follows from Theorem 6.4.

Sufficiency: Let p: $G \times_H X \to G/H$ be the projection. Let (Y, B) be a metrizable G-pair and f: $B \to G \times_H X$ a G-map. Since G/H is a G-ANR, there is a G-nbd extension q: $U \to G/H$ of $p \circ f$. Put $V = q^{-1}([H])$ and $B_0 = V \cap B$. V is an H-space, $G \times_H V = U$, and B_0 is a closed H-subspace of V. If X is an H-ANE, there is an H-nbd extension $\bar{f}$: $W \to X$ of $f|_{B_0}$, $B_0 \subset W \subset V$. Then the G-map $G\bar{f}$: $GW \to G \times_H X$ defined by $G\bar{f}(gy) = (g, \bar{f}(y))$, $g \in G$, $y \in W$, is a G-nbd extension of f, which shows the sufficiency.

Necessity: As H is an H-ANR, H is an H-nbd retract of G with an H-nbd retraction r: $U \to H$, $H \subset U \subset G$. Then $r \times_H Id_X$: $U \times_H X \to H \times_H X = X$ is an H-nbd retraction in $G \times_H X$. The necessity follows from Proposition 6.1 and 8.3. q.e.d.

By Propositions 6.9 and 8.5, we obtain the following

Proposition 8.6. *Let X be a G-space.*

(1) *Let S_x be a slice at $x \in X$ and $T=GS_x$ the tube about Gx. Then S_x is a G_x-ANR (resp. G_x-ANE) iff T is a G-ANR (resp. G-ANE).*
(2) *If X is a G-ANR (resp. G-ANE), then every slice at $x \in X$ is a G_x-ANR (resp. G_x-ANE).*

Proposition 8.7. *Let X be a G-ANR (resp. completely regular G-ANE). Then:*
(1) *$X_{(H)}$ is a G-ANR (resp. G-ANE).*
(2) *$X^{(H)}$ is a G-ANR (resp. G-ANE).*

Proof. Since every slice S_x at $x \in X^H$ in X^H is a G_x-ANE and GS_x is a tube about Gx in $X^{(H)}$, $X^{(H)}$ is a local G-ANE and hence a G-ANE by Theorem 7.2. This shows (2). As $X_{(H)}$ is an open G-subspace of $X^{(H)}$, (1) follows from Proposition 6.1. The proof for a G-ANR follows from Theorem 6.4. q.e.d.

By a *G-manifold* we mean a G-space which is a paracompact topological manifold.

A G-manifold M is called to be *locally smooth* iff, for each $x \in M$, there exists a slice at x which is G_x-homeomorphic to a Euclidian G_x-space.

By the smooth slice theorem every smooth G-manifold is locally smooth.

Every paracompact manifold is metrizable. By Theorem 7.2, Propositions 6.10 and 8.6, we obtain the following

Theorem 8.8. *Every locally smooth G-manifold is a G-ANR. In particular, every smooth G-manifold is a G-ANR.*

Remark. It is known that separable smooth G-manifolds having finite number of orbit types are G-ANR's for normal G-spaces and hence G-ANR's in our sense. C.f., [17], 1.6.6.

Theorem 8.9. *Let X be a metrizable (resp. completely regular) G-space. Then X is a G-ANR (resp. G-ANE) iff every point x of X has a G_x-nbd which is a G_x-ANR (resp. G_x-ANE).*

Proof. Let $x \in X$ and V be a G_x-nbd of x which is a G_x-ANR (G_x-ANE). There are a slice S_x at x and an open G_x-nbd U of G_x in G such that G_x is a G_x-retract of U and $US_x \subset V$. Then US_x is an open G_x-subspace of V and S_x is a G_x-retract of US_x. By Proposition 6.7 (6.1) S_x is a G_x-ANR (G_x-ANE). So GS_x is a G-ANR (G-ANE) by Proposition 8.6. Thus X is a local G-ANR (local G-ANE) and hence a G-ANR (G-ANE) by Theorem 7.2.

The converse follows from Propositions 6.7 (6.1) and 8.1 (8.3). q.e.d.

Let p: $E \rightarrow X$ be a locally trivial bundle with fibre F. Then p is called a *G-bundle* iff E and X are G-spaces and p is a G-map. A G-bundle $p: E \to X$ with fibre F is called to be *G-locally trivial* iff there is a covering $\{GV_\alpha\}$ of X by tubes such that $p| : p^{-1}(GV_\alpha) \to GV_\alpha$ is G-equivalent to

$$G \times_{H_\alpha} (V_\alpha \times F_\alpha) \to G \times_{H_\alpha} V_\alpha \quad (\cong GV_\alpha),$$

where V_α is a slice at $x_\alpha \in X$, $H_\alpha = G_{x_\alpha}$, and $F_\alpha = p^{-1}(x_\alpha)$ (F_α has an H_α-action). Then $\{(V_\alpha, H_\alpha)\}$ is called a *G-atlas* of p.

Theorem 8.10. *Let $p: E \to X$ be a G-locally trivial G-bundle with a G-atlas $\{(V_\alpha, H_\alpha)\}$. If X is a G-ANR and each F_α is an H_α-ANR, then E is a G-ANR.*

The proof follows from Propositions 6.8, 8.6 and Theorem 7.2.

9. Small G-homotopies and G-homotopy extension property

Let $\mathcal{U}$ be a given covering of a G-space X, and Y a G-space. Two G-maps $f, f': Y \to X$ are said to be *$\mathcal{U}$-near* iff, for each $y \in Y$, there is a set $U \in \mathcal{U}$ such that $f(y) \in U$ and $f'(y) \in U$.

A G-homotopy $h_t: Y \to X$, $t \in I$, is called a *$\mathcal{U}$-G-homotopy* iff, for each $y \in Y$, there is a set $U \in \mathcal{U}$ such that $h_t(y) \in U$ for every $t \in I$.

Proposition 9.1. *If X is a G-ANR and $\mathcal{U}$ a given open (G-)covering of X, then there exists an open G-covering $\mathcal{V}$, which is a refinement of $\mathcal{U}$, such that, for any metrizable G-pair (Y, B), any $\mathcal{V}$-near G-maps $f, f': Y \to X$, and any $\mathcal{V}$-G-homotopy $h_t: B \to X$, $t \in I$, with $h_0 = f|_B$ and $h_1 = f'|_B$, there exists a $\mathcal{U}$-G-homotopy $H_t: Y \to X$, $t \in I$, with $H_0 = f$, $H_1 = f'$, and $H_t|_B = h_t$ for every $t \in I$.*

This is proved in a parallel way to [5], Theorem 4.1, [7], (IV, 1.2) by using Proposition 6.6, Lemmas 1.5 and 6.12.

Theorem 9.2. *A metrizable G-space X is a G-ANR iff there exists an open G-covering $\mathcal{U}$ of X such that, for any metrizable G-pair (Y, B), any two $\mathcal{U}$-near G-maps $f, f': Y \to X$, and any $\mathcal{U}$-G-homotopy $h_t: B \to X$, $t \in I$, with $h_0 = f|_B$ and $h_1 = f'|_B$, there exists a G-homotopy $H_t: Y \to X$ with $H_0 = f$, $H_1 = f'$, and $H_t|_B = h_t$ for every $t \in I$.*

Proof. The necessity follows from Proposition 9.1.

To prove the sufficiency, let $x \in X$ and choose $U \in \mathcal{U}$ with $x \in U$. There is an open tube segment S generated by a *nbd* of x in Gx with a G-retraction $r: GS \to Gx$ such that $S \subset U$. Define two G-maps $f, f': GS \to X$ and a G-homotopy $h_t: Gx \to X$ by

$$f(y) = r(y) \qquad \text{for } y \in GS$$

$$f'(y) = y \qquad \text{for } y \in GS$$
$$h_t(y) = y \qquad \text{for } y \in Gx \text{ and } t \in I\,.$$

Obviously f and f' are $\mathcal{U}$-near, and h_t is a $\mathcal{U}$-G-homotopy. So we have a G-extension H_t: $GS \to X$ of h_t. By the compactness of I and Gx, and by Lemma 1.1 there is a G-nbd V of Gx such that $H(V\times I)\subset GS$ and $H_t|_V$: $V\to X$ is a $\mathcal{U}$-G-homotopy ($V \subset GS$).

We prove that V is a G-ANR: Let (Y, B) be a metrizable G-pair and k: $B\to V$ a G-map. Then there is a G-nbd extension $\bar{k}$: $W\to Gx$ of $r\circ k$, $B\subset W\subset Y$. Define F_0, F_1: $W\to X$ and h'_t: $B\to X$ by

$$F_0(y) = F_1(y) = \bar{k}(y) \qquad \text{for } y \in W\,,$$
$$h'_t(y) = \begin{cases} H_{2t}(k(y)) & \text{for } y\in B,\ 0\leqq t\leqq \frac{1}{2} \\ H_{2-2t}(k(y)) & \text{for } y\in B,\ \frac{1}{2}\leqq t\leqq 1\,. \end{cases}$$

Clearly F_0 and F_1 are $\mathcal{U}$-near, and h'_t is a $\mathcal{U}$-G-homotopy. Hence there is a G-extension H'_t: $W\to X$ of h'_t.
Set $W'=H'^{-1}_{1/2}(V)$ and define F: $W'\to V$ by

$$F(y) = H'_{1/2}(y) \qquad \text{for } y\in W'\,.$$

Then F is a G-nbd extension of k, which shows that V is a G-ANR. Thus X is a local G-ANR, and hence a G-ANR by Theorem 7.2. q.e.d.

Proposition 9.3. *If X is a G-ANR, then every metrizable G-pair (Y, B) has the G-homotopy extension property with respect to X, i.e., every G-map h: $Y\times \{0\} \cup B\times I\to X$ has a G-extension H: $Y\times I\to X$.*

The proof is similar to [7], (IV, 2.2).

Proposition 9.4. *Let X be a G-space and every metrizable G-pair has the G-homotopy extension property with respect to X. Then every metrizable H-pair (Y, B) has the H-homotopy extension property with respect to X, where H is a closed subgroup of G.*

Proof. Let $Z=Y\times \{0\} \cup B\times I$ and h: $Z\to X$ be an H-map. Consider $\phi\circ G\times_H h$: $G\times_H Z\to G\times_H X\to X$, where $\phi(g, x)=gx$. Since $G\times_H Z=(G\times_H Y)\times \{0\} \cup (G\times_H B)\times I\subset (G\times_H Y)\times I$, $\phi\circ G\times_H h$ has a G-extension $\bar{h}$: $(G\times_H Y)\times I\to X$. Then $\bar{h}|_{Y\times I}$: $Y\times I\to X$ is the required H-extension of h.

Parallelly to [5], Theorem 5.1, [7], (IV, 2.3), by Propositions 6.11, 9.3, 9.4, and Theorem 8.9, we obtain the following

Theorem 9.5. *For a given metrizable G-space X the following four statements are equivalent:*

(1) *X is a G-ANR.*
(2) *X is locally equivariantly contractible, and every metrizable G-pair has the G-homotopy extension property with respect to X.*
(3) *Every point $x \in X$ has a G_x-nbd V such that, for any metrizable G_x-pair (Y, B), any G_x-map f: $B \to V$ has a G_x-extension $\bar{f}$: $Y \to X$.*
(4) *Every point $x \in X$ has a G_x-nbd which is a G_x-ANR.*

Let (X, A) be a G-pair.

A G-homotopy h_t: $X \to X$, $t \in I$, is called a *G-nbd deformation retraction to A in X* iff $h_t(x)=x$ for $(x, t) \in X \times \{0\} \cup A \times I$ and there exists a G-nbd U of A such that $h_1(U)=A$.

A G-pair (X, A) is called a *G-NDR pair* iff there exist an invariant function l: $X \to I$ such that $A=l^{-1}(0)$, and a G-nbd deformation retraction h_t to A in X such that $h_1(i^{-1}[0, 1))=A$. Then the inclusion $A \to X$ is a G-cofibration. If a metrizable G-pair (X, A) has a G-nbd deformation retraction h_t to A in X, then (X, A) is a G-NDR pair.

A G-pair (X, A) is called a *G-ANR pair* iff both X and A are G-ANR's.

Proposition 9.6. *If (X, A) is a G-ANR pair, then for any open G-covering $\mathcal{U}$ of X there exists a G-nbd deformation retraction to A in X which is a $\mathcal{U}$-G-homotopy. In particular, (X, A) is a G-NDR pair.*

The proof is parallel to [7], (IV, 3.4).

Corollary 9.7. *If X is a G-ANR, then for each orbit Gx in X, any G-nbd V of Gx, and any open G-covering $\mathcal{U}$ of V there exist a tube (T, r) about Gx in V and a $\mathcal{U}$-G-homotopy h_t: $T \to V$, $t \in I$, joining r with the inclusion $T \hookrightarrow V$.*

10. G-domination

A G-space X is called to be *G-dominated* by a G-space Y iff there exist two G-maps f: $X \to Y$ and f': $Y \to X$ such that $f' \circ f$: $X \to X$ is G-homotopic to Id_X ($f' \circ f \underset{G}{\simeq} Id_X$). Then X is called to be *$\mathcal{V}$-G-dominated* by Y for a (G-)covering $\mathcal{V}$ of X iff $f' \circ f$ is $\mathcal{V}$-G-homotopic to Id_X.

Proposition 10.1. *Let X be a G-ANR. Then, for any open (G-)covering $\mathcal{V}$ of X, there exists a TN G-covering $\mathcal{S}$ of X such that X is $\mathcal{V}$-G-dominated by the G-nerve $K(\mathcal{S})$ of $\mathcal{S}$.*

Proof. By Proposition 6.6 X is regarded as a G-nbd retract of a convex G-subset C of a Banach G-space B with a G-nbd retraction r: $U \to X$. By Lemma 1.5 we may regard $\mathcal{V}$ as a G-covering. Since C is locally convex and U is a G-nbd of X in C, there is a G-covering $\mathcal{W}=\{W_\beta\}$ of X by convex open sets in U which is a refinement of the G-covering $\{r^{-1}(V)\}_{V \in \mathcal{V}}$. Put

$U_\beta = W_\beta \cap X$ and $\mathcal{U} = \{U_\beta\}$. Then $\mathcal{U}$ is an open G-covering of X. By Proposition 2.3 there is a TN G-covering $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ with $\{p_\alpha\}_{\in \Lambda/G}$ which is a star-refinement of $\mathcal{U}$.

Let $K = K(\mathcal{S})$, the G-nerve of $\mathcal{S}$, and $P: X \to K$ be the G-map given by Proposition 2.4. Define $q': K \to U$ $(\subset C)$ by letting $y = |x_{\alpha_0}, \cdots, x_{\alpha_n}; t_0, \cdots, t_n| \in K$ to

$$q'(y) = \sum_{i=0}^{n} t_i x_{\alpha_i} .$$

Clearly q' is a G-map to C. We show that $q'(K) \subset U$: By definition there is $\{S_{\lambda_0}, \cdots, S_{\lambda_n}\} \subset S$ such that $\lambda_i \in \alpha_i$, $x_{\alpha_i} \in O_{\lambda_i} \subset S_{\lambda_i}$, $i = 0, \cdots, n$, and $\bigcap_{i=0}^{n} S_{\lambda_i} \neq \emptyset$. Choose $x \in \bigcap_{i=0}^{n} S_{\lambda_i}$. Then $x \in \bigcup_{i=0}^{n} S_{\lambda_i} \subset St(x, \mathcal{S})$. Since $\mathcal{S}$ is a star-refinement of $\mathcal{U}$, there is a $U_\beta \in \mathcal{U}$ such that

$$\{x_{\alpha_0}, \cdots, x_{\alpha_n}\} \subset St(x, \mathcal{S}) \subset U_\beta \subset W_\beta .$$

For W_β is convex, we see

$$q'(y) = \sum_{i=0}^{n} t_i x_{\alpha_i} \in W_\beta \subset U .$$

Hence $q'(K) \subset U$ and q' is well-defined. Put

$$q = r \circ q' : K \to X .$$

Define a $\mathcal{V}$-G-homotopy h_t: $q \circ P \underset{G}{\simeq} Id_X$, $t \in I$, by

$$h_t(x) = r(tx + (1-t) q' \circ P(x))$$

for $x \in X$ and $t \in I$. Let $x \in X$ and $\{\alpha_0, \cdots, \alpha_n\} = \{\alpha \in \Lambda/G \mid p_\alpha(x) > 0\}$. There is $\{S_{\lambda_0}, \cdots, S_{\lambda_n}\} \subset \mathcal{S}$ such that $\lambda_i \in \alpha_i$, $i = 0, \cdots, n$, and $x \in \bigcap_{i=0}^{n} S_{\lambda_i}$. There is a $U_\beta \in \mathcal{U}$ such that

$$\{x, r_{\alpha_0}(x), \cdots, r_{\alpha_n}(x)\} \subset \bigcup_{i=0}^{n} S_{\lambda_i} \subset St(x, \mathcal{S}) \subset U_\beta \subset W_\beta .$$

Since W_β is convex, we see

$$tx + (1-t) q' \circ P(x) = tx + (1-t) \sum_{i=0}^{n} p_{\alpha_i}(x) r_{\alpha_i}(x) \in W_\beta \subset U .$$

Thus h_t is well-defined. As there is a $V \in \mathcal{V}$ such that $W_\beta \subset r^{-1}(V)$, we see that $h_t(x) \in r(W_\beta) \subset V$ for every $t \in I$, which complete the proof. q.e.d.

In the preceding proof, if X is separable, we may choose $\mathcal{S}$ such that $\tilde{\mathcal{S}}$ is countable. Hence we have

Corollary 10.2. *If X is a separable G-ANR, then there exists a TN G-*

covering $\mathcal{S}$ of X such that X is G-dominated by $K(\mathcal{S})$ and that the nerve $N(\tilde{\mathcal{S}})$ is countable.

11. Mapping spaces

For G-spaces X and Y the mapping space $\mathrm{Map}(X, Y)$ in the *compact-open topology* is a G-space with the following G-action

$$(gf)(x) = gf(g^{-1}x)$$

for $f \in \mathrm{Map}(X, Y)$, $g \in G$, and $x \in X$.

Theorem 11.1. *Let Y be a compact G-space and X a G-ANR. Then the mapping space $Map(Y, X)$ is a G-ANR.*

Proof. By Proposition 6.6, X is a G-nbd retract of a convex G-subset C of a Banach G-space B with a G-nbd retraction $r: U \to X$. Let $|| \ ||$ be the norm of B. Then $\mathrm{Map}(Y, B)$ is a Banach G-space with the norm $|| \ ||^*$ defined by $||f||^* = \sup_{y \in Y} ||f(y)||$. Thus $\mathrm{Map}(Y, X)$ is a G-nbd retract of the convex G-subset $\mathrm{Map}(Y, C)$ of the Banach G-space $\mathrm{Map}(Y, B)$ with the G-nbd retraction r_*: $\mathrm{Map}(Y, U) \to \mathrm{Map}(Y, X)$, $r_*(f)(y) = r(f(y))$. Again by Proposition 6.6 we complete the proof. q.e.d.

Theorem 11.2. *Let (X, A) be a G-ANR pair and (Y, B) a compact metrizable G-pair. Then the relative mapping space $Map(Y, B; X, A)$ is a closed G-ANR subspace of a G-ANR $Map(Y, X)$. If X is separable, then so are both $Map(Y, X)$ and $Map(Y, B; X, A)$.*

The proof is parallel to [7], (VI, 3.1), (VI, 2.2)

A pointed G-space X has a base point $*$ in X^G. For a Euclidian G-space V the one-point compactification $V \cup \{\infty\}$ is denoted by V^c. For a pointed G-space X, $\mathrm{Map}(V^c, \{\infty\}; X, \{*\})$ is denoted by $\Omega^V X$ and called the *V-th loop space* of X. The path space $\mathrm{Map}(I, \{0\}; X, \{*\})$ is denoted by PX. The one-point set is clearly a G-ANR.

Corollary 11.3. *Let X be a pointed G-ANR and V a Euclidian G-space. Then both the path space PX and the V-th loop space $\Omega^V X$ of X are G-ANR's. If X is separable, then so are PX and $\Omega^V X$.*

12. Small G-deformation and adjunction spaces

By a *G-deformation* of a G-space X, we mean a G-homotopy $h_t: X \to X$, $t \in I$, such that $h_0 = Id_X$. Then h_t is said a *$\mathcal{U}$-G-deformation* whenever h_t is a $\mathcal{U}$-G-homotopy for a (G-)covering $\mathcal{U}$ of X.

When X is a metric G-space, a G-deformation $h_t: X \to X$ is called an ε-*deformation* iff, for each $x \in X$, the set $\{h_t(x) \mid t \in I\}$ is of diameter less than ε.

A sequence of G-deformations

$$\{h_t^n: X \to X, t \in I, \quad n=1, 2, \cdots\},$$

of a G-space X is called to *converge to the identity map* Id_X iff, for each $x \in X$ and any G-nbd V of x in X, there exist a G-nbd W of x in X and an integer k such that

$$h_t^n(W) \subset V$$

for every $n \geqq k$ and every $t \in I$.

Theorem 12.1. *For any metrizable G-space X embedded as a closed G-subspace of a convex G-subset C in a Banach G-space, the following four statements are equivalent:*
(1) *X is a G-ANR.*
(2) *For each open (G-) covering $\mathcal{U}$ of X, there exists a $\mathcal{U}$-G-deformation $h_t: X \to X$, $0 \leqq t \leqq 1$, of X such that h_1 has a G-extension $\bar{h}_1: U \to X$ to a G-nbd U of X in C.*
(3) *For some metric d of X, there exists for each $\varepsilon > 0$ an ε-deformation $h_t: X \to X$, $0 \leqq t \leqq 1$, such that h_1 has a G-extension $\bar{h}_1: U \to X$ to a G-nbd U of X in C.*
(4) *There exists a sequence of G-deformations*

$$\{h_t^n: X \to X, 0 \leqq t \leqq 1, \quad n = 1, 2, \cdots\}$$

of X converging to the identity map Id_X such that each h_1^n has a G-extension $\bar{h}_t^n: U_n \to X$ to a G-nbd U_n of X in C.

The proof is a routine translation of [5], Theorem 7.1, [7], (IV, 5.3) into the terminology of Top^G.

Similarly to [5], Theorem 8.2, [7], (VI, 5.3) we obtain

Theorem 12.2. *Let (X, A) be a G-ANR pair, Y a G-ANR, and $f: A \to Y$ a G-map. If the adjunction space $Y \underset{f}{\cup} X$ is metrizable, then $Y \underset{f}{\cup} X$ is a G-ANR.*

Combining Theorem 12.2 with Proposition 3.1, we have

Corollary 12.3. *Let A be a compact G-ANR subspace of a G-ANR X, Y a G-ANR, and $f: A \to Y$ a G-map. Then the adjunction space $Y \underset{f}{\cup} X$ is a G-ANR.*

By induction on the number of G-cells of a finite G-CW complex we get

Corollary 12.4. *Every finite G-CW complex is a G-ANR.*

By Theorem 7.3 and Proposition 3.4 we obtain

Theorem 12.5.
(1) *Every G-CW complex is a G-ANE.*
(2) *Every locally finite G-CW complex is a G-ANR.*

Lemma 12.6. *Every join of a finite collection of G-orbits is a G-ANR.*

The proof is obtained by Corollary 12.3 and by induction.

Let $\mathcal{S}$ be a *TN* G-covering of a G-space.
Then clearly the G-nerve $K(\tilde{\mathcal{S}})$ of the saturation $\tilde{\mathcal{S}}$ of $\mathcal{S}$ has the weak topology with respect to the closed covering $\{\pi^{-1}(|\sigma|)\}_{\sigma\in N(\tilde{\mathcal{S}})}$, $\pi=\pi_{K(\tilde{\mathcal{S}})}: K(\tilde{\mathcal{S}})\to |N(\tilde{\mathcal{S}})|$, and each $\pi^{-1}(|\sigma|)$ is the join of G-orbits. Thus, by Proposition 6.1, Theorem 7.3, and Lemma 12.6 we have the following

Proposition 12.7. *If $\mathcal{S}$ is a TN G-covering of a G-space, then $K(\tilde{\mathcal{S}})$ is a G-ANE and $K(\mathcal{S})$ is an open G-ANE subspace of $K(\tilde{\mathcal{S}})$.*

Corresponding to [5], Theorem 7.2, [7], (IV, 6.3), by Proposition 10.1 and Theorem 12.1 we obtain the following

Theorem 12.8. *For any metrizable G-space X, the following three statements are equivalent:*
(1) *X is a G-ANR.*
(2) *For each open (G-)covering $\mathcal{U}$ of X, there exists a G-ANE Y such that X is $\mathcal{U}$-G-dominated by Y.*
(3) *There exists a sequence of G-ANE's $\{Y_n\}$ such that X is G-dominated by each Y_n with G-maps f_n: $X\to Y_n$, f'_n: $Y_n\to X$, and the corresponding sequence of G-homotopies $\{h^n_t: Id_X \underset{G}{\simeq} f'_n\circ f_n\}$ is a sequence of G-deformation converging to Id_X.*

13. *G*-homotopy types of *G*-ANR's

In this section we shall show that G-ANR's have the G-homotopy types of G-CW complexes.

Proposition 13.1. *If a G-space X is G-dominated by a G-CW complex (resp. countable G-CW complex), then X has the G-homotopy type of a G-CW complex (resp. countable G-CW complex).*

Proof. Let X be G-dominated by a G-CW complex Y. Then there are two G-maps f: $X\to Y$ and f': $Y\to X$ such that $f'\circ f\underset{G}{\simeq} Id_X$. By [11] there are a functor K from Top^G to the category of G-CW complexes (which is regarded as a subcategory of Top^G) and a natural transformation ρ: $K\to Id_{\mathrm{Top}^G}$ such that ρ_X: $K_X\to X$ is a weak G-homotopy equivalence for every G-space X, where

Id_{Top^G} denotes the identity functor. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} K_X & \xrightarrow{K_f} & K_Y & \xrightarrow{K_{f'}} & K_X \\ \downarrow{\rho_X} & & \downarrow{\rho_Y} & & \downarrow{\rho_X} \\ X & \xrightarrow{f} & Y & \xrightarrow{f'} & X. \end{array}$$

Note that $K_{f'}\circ K_f \underset{G}{\simeq} Id_{K_X}$. Since Y and K_Y are G-CW complexes and ρ_Y is a weak G-homotopy equivalence, ρ_Y is a G-homotopy equivalence by [10]. Let j be a G-homotopy inverse to ρ_Y. Then $K_{f'}\circ j\circ f$: $X\to K_X$ is a G-homotopy inverse to ρ_X.

The proof for the countable case is the same as [16], § 4. q.e.d.

Proposition 13.2. *Let K_* be a simplicial G-space without degeneracy such that each n-th space K_n has the G-homotopy type of a G-CW complex (resp. countable G-CW complex) for $n=0, 1, \cdots$. Then the geometric realization $|K_*|$ has the G-homotopy type of a G-CW complex (resp. countable G-CW complex).*

Proof. (C.f., [19], Proposition 7.2)
Let K^n denote the n-skeleton of $|K_*|$. Since K_* has no degeneracy and $1\times\iota_n$ is a G-cofibration, the diagram

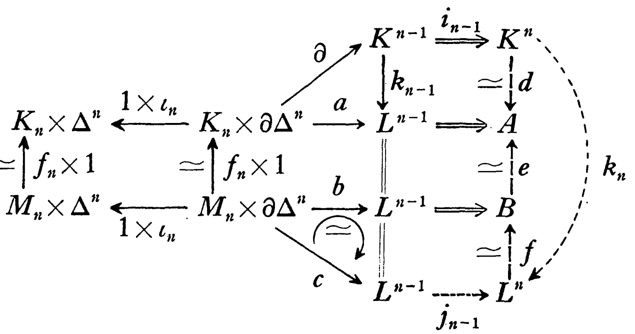
$$\begin{array}{ccc} K_n\times\partial\Delta^n & \xrightarrow{1\times\iota_n} & K_n\times\Delta^n \\ \downarrow & & \downarrow \\ K^{n-1} & \xrightarrow{i_{n-1}} & K^n \end{array}$$

is pushout and the inclusion i_{n-1} is a G-cofibration. By an equivariant version of Milnor's theorem (see [19], Theorem 1.2) it suffices to construct a G-homotopy commutative diagram:

$$\begin{array}{ccccc} K^0 & \xrightarrow{i_0} & K^1 & \xrightarrow{i_1} & \cdots \\ \downarrow{k_0} & & \downarrow{k_1} & & \\ L^0 & \xrightarrow[j_0]{} & L^1 & \xrightarrow[j_1]{} & \cdots \end{array}$$

in which each L^n is a G-CW complex, each j_n is a G-cellular inclusion, and each k_n is a G-homotopy equivalence. Then $|K_*|$ is G-homotopy equivalent to the G-CW complex $L=\mathrm{colim}\, L^n$.

By assumption, for each K_n, there exists a G-CW complex M_n and a G-homotopy equivalence f_n: $M_n\to K_n$. Inductively, let $L^0=M_0$, let k_0: $K^0=K_0\to L^0$ be a G-homotopy inverse of f_0, and suppose that L^{n-1}, k_{n-1}, and j_{n-2} have been defined with the required properties. Consider the diagram which is commutative except the lower triangle:



Here, $a=k_{n-1}\circ\partial$, $b=a\circ(f_n\times 1)$, c is a G-cellular approximation of b, A (resp. B) is the pushout of a and $1\times\iota_n$ (resp. b and $1\times\iota_n$). By an equivariant glueing theorem ([19], Theorem 1.1), the induced maps d and e are G-homotopy equivalences. Let L^n be the double mapping cylinder of c and $1\times\iota_n$, and j_{n-1} be the inclusion. Then L^n has the structure of a G-CW complex and there is a G-homotopy equivalence f which makes the diagram commutative. Define k_n by the composite of d and G-homotopy inverses of e and f. Then $k_n\circ i_{n-1}\underset{G}{\simeq} j_{n-1}\circ k_{n-1}$. Therefore these L^n, k_n, and j_{n-1} are the required ones.

Moreover, when every M_n is countable, inductively each L^n becomes countable, and so is $L=\operatorname{colim} L^n$. q.e.d.

Theorem 13.3. *Every G-ANR has the G-homotopy type of a G-CW complex and every separable G-ANR has the G-homotopy type of a countable G-CW complex.*

Proof. We use [17], 1.8.1 Metatheorem, and assume that the theorem holds for actions of all proper closed subgroup of G. By Proposition 10.1 there is a TN G-covering $\mathcal{S}$ such that X is G-dominated by the G-nerve $K=K(\mathcal{S})$. Moreover, if X is separable, we may choose $\mathcal{S}$ such that $N=N(\tilde{\mathcal{S}})$ is countable by Corollary 10.2. Now $K_n=K_n(\mathcal{S})=\sum_{\sigma\in N_n} K_\sigma$. Since K_σ is an open G-subspace of the product $O_{\alpha_0}\times\cdots\times O_{\alpha_n}$ of orbits, K_σ is an open G-submanifold of $O_{\alpha_0}\times\cdots\times O_{\alpha_n}$ and hence separable. We will show that K_σ has the G-homotopy type of a countable G-CW complex for every $\sigma=\{\alpha_0,\cdots,\alpha_n\}\in N(\tilde{\mathcal{S}})$. If all O_{α_i}'s are of type G/G, then K_σ is a point and hence a G-CW complex. We assume that there is an orbit O_{α_i} of type G/H such that H is a proper closed subgroup of G. Let π_σ be the composite:

$$O_{\alpha_0}\times\cdots\times O_{\alpha_n}\xrightarrow{\text{proj.}} O_{\alpha_i}\underset{G}{\approx} G/H\,.$$

Let $L_\sigma=(\pi_\sigma|_{K_\sigma})^{-1}([H])$. Since $K_\sigma \underset{G}{\approx} G \underset{H}{\times} L_\sigma$ and L_σ is an open H-submanifold of $O_{\alpha_0}\times\cdots\times O_{\alpha_{i-1}}\times O_{\alpha_{i+1}}\times\cdots\times O_{\alpha_n}$, L_σ has the H-homotopy type of a countable H-CW complex M_σ by the assumption of the metatheorem. Thus K_σ has the G-homotopy type of the countable G-CW complex $G\underset{H}{\times} M_\sigma$. Therefore each $K_n=\sum_{\sigma\in N_n} K_\sigma$ has the G-homotopy type of a G-CW complex M_n, $n\geqq 0$. Moreover, if X is separable, then N_n is countable, and so is M_n, $n\geqq 0$. By Propositions 13.1 and 13.2 we complete the proof. q.e.d.

Corollary 13.4. *If a G-space X has the G-homotopy type of a G-ANR, then X has the G-homotopy type of a G-CW complex.*

By [10], Theorem 5.3 (an equivariant J.H.C. Whitehead theorem) we have

Corollary 13.5. *If f: $X\rightarrow Y$ is a weak G-homotopy equivalence between G-ANR's X and Y, then f is a G-homotopy equivalence between X and Y.*

This is a generalization of [22], Theorem (1.1).

14. *G*-homotopy types of countable *G-CW* complexes

Proposition 14.1. *Every countable G-CW complex has the G-homotopy type of a locally finite countable G-CW complex.*

The proof is similar to [20], Theorem 13.

Theorem 14.2. *The following restrictions on a G-space X are equivalent:*
(1) *X has the G-homotopy type of a countable G-CW complex.*
(2) *X has the G-homotopy type of a separable G-ANR.*

Proof. The implication (1)⇒(2) follows from Theorem 12.5, (2) and Proposition 14.1.
The converse is the result of Theorem 13.3. q.e.d.

By the above theorem, Corollary 11.3, and Proposition 9.6 we have the following

Corollary 14.3. *Let V be a Euclidian G-space. If a G-space X has the G-homotopy type of a pointed countable G-CW complex, then so is the V-th loop space $\Omega^V X$ of X.*

Added in Proof. The same result as Theorem 13.3 is announced in "S. Kwasik: *On the equivariant homotopy type of G-ANR's*, Proc. Amer. Math. Soc. **83** (1981), 193–194".

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