MULTIPLY TRANSITIVE PERMUTATION GROUPS AND ODD PRIMES

IZUMI MIYAMOTO

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In [4] M. Hall determined all 4-fold transitive permutation groups whose stabilizer of 4 points is of odd order. In this note we give some analogous version of M. Hall's theorem for any odd prime \( p \) on \( 3p \)-fold transitive permutation groups. We note that such a version is also already obtained by E. Bannai [1] on \( (p^2+p) \)-fold transitive permutation groups.

**Theorem.** Let \( p \) be an odd prime. Let \( G \) be a \( 3p \)-fold transitive permutation group on \( \Omega = \{1, 2, \cdots, n\} \). If the order of a stabilizer of 3 points in \( G \) is prime to \( p \), then \( G = S_n (3p \leq n < 4p) \) or \( G = A_n (3p + 2 \leq n < 4p) \).

Our notation follows Nagao [6]. Let us recall some of them: For a set \( S \) of permutations on \( \Omega \) the set of the points left fixed by \( S \) will be denoted by \( I(S) \). For a permutation \( x \) let \( \alpha_i(x) \) denote the number of \( i \)-cycles. Also let \( I^*(S) = \Omega - I(S) \) and \( \alpha(x) = \alpha_i(x) \). The order of a permutation \( x \) will be denoted by \( o(x) \). \( p \mid o(x) \) will mean that \( o(x) \) is divisible by \( p \) and \( p \nmid o(x) \) will mean that \( o(x) \) is not divisible by \( p \).

1. **On \( 2p \)-fold transitive groups**

The next lemma which is indebted to Nagao [6] is essential in the present work.

**Lemma 1.1.** Let \( X \) be a \( p \)-fold transitive permutation group on a finite set \( \Omega \). Let \( P \) be a Sylow \( p \)-subgroup of \( X \). If \( P \) is semiregular on \( \Omega - I(P) \), then

(i) \( X \) has only one conjugacy class of the elements of order \( p \), and

(ii) for an element \( u \) of order \( p \), \( C_X(u) \) is transitive on \( I^*(u) \).

Proof. Since \( X \) is \( p \)-fold transitive,

\[
\frac{|X|}{p} = \sum_{x \in X} \alpha_p(x),
\]

by a result of Frobenius [1][2]. On the other hand, since \( P \) is semiregular, any element \( x \) with \( p \)-cycle is uniquely expressed as a product of an element
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Let \( u \) of order \( p \) and an element \( y \) of order prime to \( p \) which commute with each other. Then we can see easily that \( \alpha_p(x) = \frac{1}{p} \alpha^*(y) \), where \( \alpha^*(y) \) denotes the number of the fixed points of \( y \) on \( I'(u) \). Hence we have by (1)

\[
\frac{|X|}{p} = \sum_i \frac{|X|}{|C_X(u_i)|} \frac{1}{p} \sum_y \alpha^*(y),
\]

where \( \{u_1\}, \ldots, \{u_k\} \) are the conjugacy classes of \( X \) consisting of elements of order \( p \) and the second summation \( \sum_y \) ranges over all the elements of \( C_X(u_i) \) of order prime to \( p \). Now let \( t_i \) be the number of the orbits of \( C_X(u_i) \) on \( I'(u_i) \), then by [5], Theorem 16.6.13,

\[
\sum_{y \in C_X(u_i)} \alpha^*(y) = t_i \cdot |C_X(u_i)|.
\]

Since \( P \) is semiregular, \( \alpha^*(y) \) vanishes for an element \( y \) such that \( p \nmid o(y) \). Hence

\[
\sum_{y \in C_X(u_i)} \alpha^*(y) = t_i \cdot |C_X(u_i)|.
\]

Then by (2),

\[
\frac{|X|}{p} = \sum_i \frac{|X|}{|C_X(u_i)|} \frac{1}{p} \cdot t_i \cdot |C_X(u_i)|
\]

\[
= \frac{|X|}{p} \sum_i t_i.
\]

Therefore we have that \( k = 1 \) and \( t_1 = 1 \). Thus we have the assertion.

**REMARK.** The following inequality is valid whenever \( X \) is \( p \)-fold transitive.

\[
\frac{|X|}{p} \geq \sum_i \frac{|X|}{|C_X(u_i)|} \frac{1}{p} \sum_{y \in C_X(u_i)} \alpha^*(y).
\]

In this section we always assume that \( p \) is an odd prime and that \( G \) is a \( 2p \)-fold transitive permutation group on \( \Omega = \{1, \ldots, n\} \), excluding \( S_n \) and \( A_n \), where the stabilizer \( H \) of the points \( 1, \ldots, 2p \) in \( G \) is of order prime to \( p \). Then \( I(H) = \{1, \ldots, 2p\} \) by Theorem of Nagao [6]. Let \( \Delta = \{1, \ldots, 2p\} \) and let \( N = N_G(H) \), then \( N^\Delta = S_{2p} \) (cf. Wielandt [7], Theorem 9.4). Let \( P \) be a Sylow \( p \)-subgroup of \( N \) then \( P \) is an elementary abelian group of order \( p^s \). We may assume that

\[
a = (1)(2)\cdots(p)(p+1, \ldots, 2p)\cdots
\]

and

\[
b = (1, 2, \ldots, p)(p+1)\cdots(2p)\cdots
\]

generate \( P \); i.e., \( \langle a, b \rangle = P \). Since \( |H| \) is prime to \( p \), \( P \) has at most \( p+1 \) orbits of length \( p \). So we consider the following 3 cases separately.

**Case (1)** \( P \) has exactly two orbits of length \( p \); \( \{1, \ldots, p\} \) and \( \{p+1, \ldots, 2p\} \),
and $P$ is semiregular on $\Omega - I(P) - \{1, \cdots, 2p\}$.

Case (II) $P$ has $i$ orbits of length $p$ ($2 < i \leq p$) and $P$ is semiregular on $\Omega - I(P)$ - $\{1, \cdots, ip\}$ and in this case we may assume that
\[
ab = (1, 2, \cdots, p)(p+1, \cdots, 2p)(2p+1)(2p+2)\cdots(3p)\cdots.
\]

Case (III) $P$ has $p+1$ orbits of length $p$ and $P$ is semiregular on $\Omega - I(P)$ - $\{1, \cdots, (p+1)p\}$.

Let $K=G_{i-} \cdots _p$ and $L=\langle b \rangle \cdot K$. Then we have the following corollary immediately from lemma 1.1.

**Corollary 1.2.**

\[
|C_K(a)| = \sum_{y \in C_{G_K(a)}} \alpha^*(y) = \sum' \alpha^*(y),
\]
\[
|C_L(a)| = \sum_{y \in C_{G_L(a)}} \alpha^*(y) = \sum \alpha^*(y),
\]
where the summation $\sum'$ ranges over all the elements of $C_K(a)$ of order prime to $p$ and $\alpha^*(y)$ denotes the number of the fixed points of $y$ on $I^c(a)$.

**Lemma 1.3.** In cases (I) and (II) $P$ is a Sylow $p$-subgroup of $L$. In case (III) $P$ is not a Sylow $p$-subgroup of $L$.

Proof. In case (I) and (II), $n=ip+rp^s+s$ for some integers $i$ ($2 \leq i \leq p$), $r$ and $s$ ($0 \leq s < p$). $L_1=K$ and $K$ is $p$-fold transitive on $\Omega - \{1, \cdots, p\}$. Hence the first assertion holds by the assumption that $|H|$ is prime to $p$. We have the second assertion similarly.

**Lemma 1.4.** Case (I) does not hold.

Proof. Let $t$ denote the number of the orbits of $C_L(b)$ on $I^c(b)-\{1,\cdots, p\}$ and let $\alpha^*(y)$ denote the number of the fixed points of $y$ on $I^c(b)-\{1,\cdots, p\}$. By Lemma 1.3 $P$ is a Sylow $p$-subgroup of $C_L(b)$. In case (I) any element of $P$ except the identity has no fixed points on $I^c(b)-\{1,\cdots, p\}$. Therefore $\alpha^*(y)=0$ for any element $y$ of $C_L(b)$ such that $p \mid o(y)$ and
\[
t \mid C_L(b) \mid = \sum_{y \in C_L(b)} \alpha^*(y).
\]
Hence by the remark after lemma 1.1,
\[
\frac{|L|}{p} \geq \frac{|L|}{|C_L(b)|} \frac{1}{p} \sum_{y \in C_L(b)} \alpha^*(y) + \frac{|L|}{|C_L(b^{-1})|} \frac{1}{p} \sum_{y \in C_L(b^{-1})} \alpha^*(y)
\]
\[
= 2 \frac{|L|}{|C_L(b)|} \frac{1}{p} t \mid C_L(b) \mid,
\]
because there exist two elements $b$ and $b^{-1}$ of $L$ of order $p$ which are not conjugate in $L$. Hence we have $t=0$, that is, $b$ is a $p$-cycle. Then $G=S_n$ or $A_n$ (cf. Wielandt [7] §13). This is not the case.
Lemma 1.5. Case (II) does not hold.

Proof. By corollary 1.2,

$$|C_L(a)| = \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) + |C_K(a)|.$$  

Since $|C_L(a) : C_K(a)| = p$, we have

$$\frac{p-1}{p} |C_L(a)| = \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y).$$

$b, b^2, \ldots, b^{p-1}$ are not conjugate with one another in $C_L(a)$ since they are not conjugate in $L$. $b^i (i=2, 3, \ldots, p-1)$ and $ab$ are not conjugate in $C_L(a)$. We shall show that $b$ and $ab$ are not conjugate in $C_L(a)$. If $b$ and $ab$ are conjugate in $C_L(a)$ by an element $x$, i.e., $b^x = ab$. Then $x \in C_L(a) \cap N_L(P)$ and $x^p$ centralizes $b$. Hence $p \mid o(x)$, but this is a contradiction since $P$ is a Sylow $p$-subgroup of $L$. Thus we have $p$ conjugacy classes in $C_L(a) - C_K(a)$ of order $p$ represented by the elements $b, b^2, \ldots, b^{p-1}$ and $ab$, any of which has $p$ fixed points on $I^r(a)$. Since the restriction of $C_L(P)$ on the orbits of $P$ of length $p$ is self-centralizing (cf. Wielandt [5] §4), we have

$$\sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) \geq \frac{p-1}{p} |C_L(a) : C_L(P)| \cdot \{ y \in C_L(P) \mid p \mid o(y) \} | \cdot |C_L(P) : C_L(P)|.$$  

Hence

$$\sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) \geq |C_L(a)|.$$  

This contradicts the equality (3).

2. Proof of Theorem

Lemma 2.1. Let $p$ be an odd prime. Let $G$ be a $2p$-fold transitive permutation group on $\Omega = \{1, 2, \ldots, n\}$. Let $K$ be the stabilizer of the points $1, 2, \ldots, 2p$ in $G$ and let $P$ be a Sylow $p$-subgroup of $K$.

If $P$ is not identity and semiregular on $\Omega - \{1, 2, \ldots, 2p\}$, then $P$ is of order $p$.

Proof. Let $a$ be an element of order $p$ which is conjugate with some element of $P$ such that

$$a = (1)(2)\cdots(p)(p+1, p+2, \ldots, 2p)\cdots.$$  

Then $a$ normalizes $K$, hence also normalizes a Sylow $p$-subgroup $P'$ of $K$. So we find an element $b$ of $P'$ of order $p$ which commutes with $a$. Then $a$ fixes exactly $p$ points of a fixed block of $b$ and $|I(a) \cap I(b)| = p$, i.e., $|I(a, b)| = p$. Conjugating $a$ to $a'$ and $b$ to $b'$, we may assume that

$$a' = (1)(2)\cdots(p)(p+1, \ldots, 2p)\cdots.$$
Let $Q = \langle a', b' \rangle$, then any element of $Q$ has at least $p$ fixed points on $\Omega = \{1, 2, \ldots, 2p\}$. $Q$ normalizes $K$, hence also normalizes a Sylow $p$-subgroup $P''$ of $K$.

Assume $|P''| \geq p^2$. We shall find a subgroup $S$ of $P''$ of order $p^2$ which is normalized by $Q$. Since $Q$ normalizes $Z(P'')$, the center of $P''$, if $|Z(P'') \cap C_{P''}(Q)| \geq p^3$, we find such subgroup $S$ immediately. Let $R = Z(P'') \cap C_{P''}(Q)$ and we assume $|R| = p$. We can find a $Q$-invariant subgroup $\bar{S}$ of order $p$ in $P''/R$. Then the inverse image $S$ in $P''$ is $Q$-invariant and of order $p^2$. $S$ is a cyclic group of order $p^2$ or an elementary abelian group of order $p^2$. Anyhow the automorphism group of $S$ does not contain an elementary abelian group of order $p^2$. Therefore some element $c(\neq 1)$ of $Q$ centralizes $S$. Since $c$ has fixed points on $\Omega - I(S)$, $c$ has at least $p^2$ fixed points (cf. Wielandt [7] §4). Since $p$ is odd, $p^2 \geq 2p$. This contradicts the semiregularity of $P$ on $\Omega = \{1, 2, \ldots, 2p\}$. Thus we have the assertion.

Proof of Theorem. If $G$ is $3p$-fold transitive on $\Omega$, then by lemma 2.1 a Sylow $p$-subgroup of a stabilizer of $2p$ points in $G$ is of order $p$. But this contradicts lemma 1.3. Thus we have the assertion of Theorem.

Remark. A result corresponding to lemma 2.1 was also proved by E. Bannai in a little strong form. His result will be published elsewhere.

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University of Tokyo

References
