Miyamoto, I.
Osaka J. Math.
11 (1974), 9-13

MULTIPLY TRANSITIVE PERMUTATION GROUPS AND ODD PRIMES

IZUMI MIYAMOTO

(Received June 11, 1973)

In [4] M. Hall determined all 4-fold transtitive permutation groups whose stabilizer of 4 points is of odd order. In this note we give some analogous version of M. Hall's theorem for any odd prime p on $3p$-fold transtive permutation groups. We note that such a version is also already obtained by E. Bannai [1] on (p^2+p)-fold transitive permutation groups.

Theorem. *Let p be an odd prime. Let G be a $3p$-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. If the order of a stabilizer of $3p$ points in G is prime to p, then $G=S_n$ $(3p\leqslant n<4p)$ or $G=A_n$ $(3p+2\leqslant n<4p)$.*

Our notation follows Nagao [6]. Let us recall some of them: For a set S of permutations on Ω the set of the points left fixed by S will be denoted by $I(S)$. For a permutation x let $\alpha_i(x)$ denote the number of i-cycles. Also let $I^c(S)=\Omega-I(S)$ and $\alpha(x)=\alpha_1(x)$. The order of a permutation x will be denoted by $o(x)$. $p|o(x)$ will mean that $o(x)$ is divisible by p and $p\not| o(x)$ will mean that $o(x)$ is not divisible by p.

1. On $2p$-fold transitive groups

The next lemma which is indebted to Nagao [6] is essential in the present work.

Lemma 1.1. *Let X be a p-fold transitive permutation group on a finite set Ω. Let P be a Sylow p-subgroup of X. If P is semiregular on Ω-$I(P)$, then*

(i) *X has only one conjugacy class of the elements of order p, and*

(ii) *for an element u of order p, $C_X(u)$ is transitive on $I^c(u)$.*

Proof. Since X is p-fold transitive,

$$(1) \qquad \frac{|X|}{p}=\sum_{x\in X}\alpha_p(x),$$

by a result of Frobenius [1][2]. On the other hand, since P is semiregular, any element x with p-cycle is uniquely expressed as a product of an element

u of order p and an element y of order prime to p which commute with each other. Then we can see easily that $\alpha_p(x)=\frac{1}{p}\alpha^*(y)$, where $\alpha^*(y)$ denotes the number of the fixed points of y on $I^c(u)$. Hence we have by (1)

$$(2) \qquad \frac{|X|}{p}=\sum_i \frac{|X|}{|C_X(u_i)|}\frac{1}{p}\sum_y{}' \alpha^*(y),$$

where $\{u_1\}, \cdots, \{u_k\}$ are the conjugacy classes of X consisting of elements of order p and the second summuation $\sum_y{}'$ ranges over all the elements of $C_X(u_i)$ of order prime to p. Now let t_i be the number of the orbits of $C_X(u_i)$ on $I^c(u_i)$, then by [5], Theorem 16.6.13,

$$\sum_{y\in C_X(u_i)} \alpha^*(y)=t_i\cdot|C_X(u_i)|\,.$$

Since P is semiregular, $\alpha^*(y)$ vanishes for an element y such that $p\,|\,o(y)$. Hence

$$\sum_{y\in C_X(u_i)}{}' \alpha^*(y)=t_i|C_X(u_i)|\,.$$

Then by (2),
$$\frac{|X|}{p}=\sum_i \frac{|X|}{|C_X(u_i)|}\frac{1}{p}\cdot t_i\cdot|C_X(u_i)|$$
$$=\frac{|X|}{p}\sum_i t_i\,.$$

Therefore we have that $k=1$ and $t_1=1$. Thus we have the assertion.

REMARK. The following inequality is valid whenever X is p-fold transitive.

$$\frac{|X|}{p}\geqslant\sum_i \frac{|X|}{|C_X(u_i)|}\frac{1}{p}\sum_{y\in C_X(u_i)}{}' \alpha^*(y).$$

In this section we always assume that p is an odd prime and that G is a $2p$-fold transitive permutation group on $\Omega=\{1, \cdots, n\}$, excluding S_n and A_n, where the stabilizer H of the points $1, \cdots, 2p$ in G is of order prime to p. Then $I(H)=\{1, \cdots, 2p\}$ by Theorem of Nagao [6]. Let $\Delta=\{1, \cdots, 2p\}$ and let $N=N_G(H)$, then $N^\Delta=S_{2p}$ (cf. Wielandt [7], Theorem 9.4). Let P be a Sylow p-subgroup of N then P is an elementary abelian group of order p^2. We may assume that

$$a=(1)(2)\cdots(p)(p+1, \cdots, 2p)\cdots\cdots$$
and
$$b=(1, 2, \cdots, p)(p+1)\cdots(2p)\cdots\cdots$$

generate P; i.e., $\langle a, b\rangle=P$. Since $|H|$ is prime to p, P has at most $p+1$ orbits of length p. So we consider the following 3 cases separately.

Case (I) *P has exactly two orbits of length* p; $\{1, \cdots, p\}$ *and* $\{p+1, \cdots, 2p\}$, *and P is semiregular on $\Omega-I(P)-\{1, \cdots, 2p\}$.*

Case (II) *P has i ordbits of length p $(2<i\leqslant p)$ and P is semiregular on $\Omega-I(P)-\{1, \cdots, ip\}$ and in this case we may assume that*

$$ab=(1, 2, \cdots, p)(p+1, \cdots, 2p)(2p+1)(2p+2)\cdots(3p)\cdots\cdots$$

Case (III) *P has $p+1$ orbits of length p and P is semiregular on $\Omega-I(P)-\{1, \cdots, (p+1)p\}$.*

Let $K=G_{1,\cdots,p}$ and $L=\langle b\rangle\cdot K$. Then we have the following corollary immediately from lemma 1.1.

Corollary 1.2.
$$|C_K(a)|=\sum_{y\in C_K(a)}\alpha^*(y)=\sum_{y\in C_K(a)}{}'\alpha^*(y),$$
$$|C_L(a)|=\sum_{y\in C_K(a)}\alpha^*(y)=\sum_{y\in C_L(a)-C_K(a)}\alpha^*(y),$$
where the summuation $\sum'$ ranges over all the elements of $C_K(a)$ of order prime to p and $\alpha^*(y)$ denotes the number of the fixed points of y on $I^c(a)$.

Lemma 1.3. *In cases* (I) *and* (II) *P is a Sylow p-subgroup of L. In case* (III) *P is not a Sylow p-subgroup of L.*

Proof. In case (I) and (II), $n=ip+rp^2+s$ for some integers $i\,(2\leqslant i\leqslant p)$, r and $s\,(0\leqslant s<p)$. $L_1=K$ and K is p-fold transitive on $\Omega-\{1, \cdots, p\}$. Hence the first assertion holds by the assumption that $|H|$ is prime to p. We have the second assertion similarly.

Lemma 1.4. *Case* (I) *does not hold.*

Proof. Let t denote the number of the orbits of $C_L(b)$ on $I^c(b)-\{1, \cdots, p\}$ and let $\alpha^*(y)$ denote the number of the fixed points of y on $I^c(b)-\{1, \cdots, p\}$. By Lemma 1.3 P is a Sylow p-subgroup of $C_L(b)$. In case (I) any element of P except the identity has no fixed points on $I^c(b)-\{1, \cdots, p\}$. Therefore $\alpha^*(y)=0$ for any element y of $C_L(b)$ such that $p\,|\,o(y)$ and

$$t|C_L(b)|=\sum_{y\in C_L(b)}{}'\alpha^*(y).$$

Hence by the remark after lemma 1.1,

$$\frac{|L|}{p}\geqslant\frac{|L|}{|C_L(b)|}\frac{1}{p}\sum_{y\in C_L(b)}{}'\alpha^*(y)+\frac{|L|}{|C_L(b^{-1})|}\frac{1}{p}\sum_{y\in C_L(b^{-1})}{}'\alpha^*(y)$$
$$=2\frac{|L|}{|C_L(b)|}\frac{1}{p}t|C_L(b)|,$$

because there exist two elements b and b^{-1} of L of order p which are not conjugate in L. Hence we have $t=0$, that is, b is a p-cycle. Then $G=S_n$ or A_n(cf. Wielandt [7] §13). This is not the case.

Lemma 1.5. *Case* (II) *does not hold.*

Proof. By corollary 1.2,

$$|C_L(a)| = \sum_{y \in C_L(a)-C_K(a)} \alpha^*(y) + |C_K(a)| .$$

Since $|C_L(a): C_K(a)| = p$, we have

$$(3) \qquad \frac{p-1}{p}|C_L(a)| = \sum_{y \in C_L(a)-C_K(a)} \alpha^*(y) .$$

$b, b^2, \cdots, b^{p-1}$ are not conjugate with one another in $C_L(a)$ since they are not conjugate in L. $b^i (i=2, 3, \cdots, p-1)$ and ab are not conjugate in $C_L(a)$. We shall show that b and ab are not conjugate in $C_L(a)$. If b and ab are conjugate in $C_L(a)$ by an element x, *i.e.*, $b^x = ab$. Then $x \in C_L(a) \cap N_L(P)$ and x^p centralizes b. Hence $p | o(x)$, but this is a contradiction since P is a Sylow p-subgroup of L. Thus we have p conjugacy classes in $C_L(a) - C_K(a)$ of order p represented by the elements $b, b^2, \cdots, b^{p-1}$ and ab, any of which has p fixed points on $I^c(a)$. Since the restriction of $C_L(P)$ on the orbits of P of length p is self-centralizing (cf. Wielandt [5] §4), we have

$$\begin{aligned} \sum_{y \in C_L(a)-C_K(a)} \alpha^*(y) &\geqslant p \cdot p |C_L(a): C_L(P)| \cdot |\{y \in C_L(P) | p \nmid o(y)\}| \\ &= p^2 |C_L(a): C_L(P)| \cdot |C_L(P): P| . \end{aligned}$$

Hence

$$\sum_{y \in C_L(a)-C_K(a)} \alpha^*(y) \geqslant |C_L(a)| .$$

This contradicts the equality (3).

2. Proof of Theorem

Lemma 2.1. *Let p be an odd prime. Let G be a $2p$-fold transitive permutation group on $\Omega = \{1, 2, \cdots, n\}$. Let K be the stabilizer of the points* $1, 2, \cdots, 2p$ *in G and let P be a Sylow p-subgroup of K.*

If P is not identity and semiregular on $\Omega - \{1, 2, \cdots, 2p\}$, then P is of order p.

Proof. Let a be an element of order p which is conjugate with some element of P such that

$$a = (1)(2)\cdots(p)(p+1, p+2, \cdots, 2p)\cdots\cdots .$$

Then a normalizes K, hence also normalizes a Sylow p-subgroup P' of K. So we find an element b of P' of order p which commutes with a. Then a fixes exactly p points of a fixed block of b and $|I(a) \cap I(b)| = p$, *i.e.*, $|I(\langle a, b \rangle)| = p$. Conjugating a to a' and b to b', we may assume that

$$a' = (1)(2)\cdots(p)(p+1, \cdots, 2p)\cdots\cdots ,$$

$$b' = (1, 2, \cdots, p)(p+1) \cdots (2p) \cdots\cdots .$$

Let $Q = \langle a', b' \rangle$, then any element of Q has at least p fixed points on $\Omega - \{1, 2, \cdots, 2p\}$. Q normalizes K, hence also normalizes a Sylow p-subgroup P'' of K.

Assume $|P''| \geqslant p^2$. We shall find a subgroup S of P'' of order p^2 which is normalized by Q. Since Q normalizes $Z(P'')$, the center of P'', if $|Z(P'') \cap C_{P''}(Q)| \geqslant p^2$, we find such subgroup S immediately. Let $R = Z(P'') \cap C_{P''}(Q)$ and we assume $|R| = p$. We can find a Q-invariant subgroup $\bar{S}$ of order p in P''/R. Then the inverse image S in P'' is Q-invariant and of order p^2. S is a cyclic group of order p^2 or an elementary abelian group of order p^2. Anyhow the automorphism group of S does not contain an elementary abelian group of order p^2. Therefore some element $c(\neq 1)$ of Q centralizes S. Since c has fixed points on $\Omega - I(S)$, c has at least p^2 fixed points (cf. Wielandt [7] §4). Since p is odd, $p^2 > 2p$. This contradicts the semiregularity of P on $\Omega - \{1, 2, \cdots, 2p\}$. Thus we have the assertion.

Proof of Theorem. If G is $3p$-fold transitive on Ω, then by lemma 2.1 a Sylow p-subgroup of a stabilizer of $2p$ points in G is of order p. But this contradicts lemma 1.3. Thus we have the assertion of Theorem.

REMARK. A result corresponding to lemma 2.1 was also proved by E. Bannai in a little strong form. His result will be published elsewhere.

At the end the author thanks Dr. E. Bannai and Mr. H. Enomoto for their helpful advice and kind encouragement.

UNIVERSITY OF TOKYO

References

[1] E. Bannai: *A note on multiply transitive permutation groups*, J. Algebra **26** (1973), 383–384.

[2] G. Frobenius: *Über die Charaktere der symmetrischen Gruppen*, S.–B. Preuss. Akad. Wiss. Berlin (1900), 516–534.

[3] G. Frobenius: *Über die Charaketere mehrfach transitiven Gruppen*, S.–B. Preuss. Akad. Wiss. Berlin (1904), 528–571.

[4] M. Hall: *On a theorem of Jordan*, Pacific J. Math. **4** (1954), 219–226.

[5] M. Hall: The Theory of Groups, Macmillan, New York, 1959.

[6] H. Nagao: *On multiply transitive groups* IV, Osaka J. Math. **2** (1965), 327–341.

[7] H. Wielandt: Finite Permutation Groups, Academic Press, New York, 1964.