A NOTE ON RELATIVE T-NILPOTENCY

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This note gives some supplementary results of [6]. The first one shows an application of the idea in the proof of [6], Lemma 7 and gives a characterization of artinian rings. The second one gives a refinement of [6], Corollary 2 to Theorem A.2 and the final one is a special type of the exchange property.

Throughout we shall assume that $R$ is a ring with identity and modules are unitary right $R$-modules. First, we shall recall definitions in [6].

Let $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ be two infinite sets of $R$-modules. We take a countable set $\{M_\gamma\}_\Gamma$ such that $M_{\gamma+1}=P_{\alpha(2\gamma+1)}\subseteq\{P_\alpha\}_I$, and $M_{\beta(j)}=Q_{\beta(j)}\subseteq\{Q_\beta\}_J$. Further we take a set of non-isomorphisms $f_i: M_i\rightarrow M_{i+1}$. If for any element $m$ in $M_i$ there exists $n$ such that $f_{m}f_{m-1}\cdots f_{i}(m)=0$, we say $\{f_{i}\}_I$ is locally $T$-nilpotent. If for any countable sets $\{M_\gamma\}_\Gamma$ above such that $\alpha(2\gamma+1)\neq\alpha(2\gamma'+1)$, we say $\{f_{i}\}_I$ are relatively and locally semi-$T$-nilpotent. If we omit the assumptions $\alpha(2\gamma+1)\neq\alpha(2\gamma'+1)$ in the above, we say $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ are relatively and locally $T$-nilpotent.

Finally, let $M=N\oplus P$ be $R$-modules and $\kappa$ a cardinal number. If for any decomposition $M=\sum\oplus L_\alpha$ with $\kappa$-components there exist submodules $L_\alpha'$ of $L_\alpha$ such that $M=N\oplus\sum\oplus L_\alpha'$, then we say $N$ has the $\kappa$-exchange property in $M$. In case $\kappa$ is any cardinal, we say $N$ has the exchange property in $M$.

1. T-nilpotent decompositions

First, we study a property of relative $T$-nilpotency. If the endomorphism ring of a module $M$ is a local ring, then we call $M$ completely indecomposable.

Lemma 1. Let $M$ be an $R$-module and $f, g$ in $\text{End}_R(M)$. If $fg$ is isomorphic, $M=\text{Im} \ g \oplus \text{Ker} \ f$. 

Lemma 2. Let $P$ be an $R$-module. If $P$ is itself locally $T$-nilpotent, $P$ is a completely indecomposable module.

Proof. Put $S = \text{End}_R(P)$. If $e^S$ and $e^2 = e$, then $e = 1$ or $0$ by the assumption. Let $x, y$ be elements in $S$ with $x$ non-unit. Then neither $xy$ nor $yx$ is unit in $S$ from the above and Lemma 1. Furthermore, consider a sequence $\{x^n\}$ of non-units in $S$. For any element $p$ in $P$, there exists $n = n(p)$ such that $x^{n(p)}(p) = 0$ from the assumption. Therefore, $X = 1 + \sum x^n$ is an element in $S$ and $((1-x)x)(p) = (1-x)(1+x + \cdots + x^{n(p)-1})(p) = (1-x^{n(p)})(p) = p$. Hence, $1-x$ is unit in $S$ from Lemma 1. Let $x, y$ be non-unit in $S$. We assume that $x+y$ is unit in $S$. Then we may assume $x+y = 1$, which is a contradiction to the above. Therefore, $S$ is a local ring.

Theorem 1. Let $M$ be an $R$-module and $M = \sum_{\ell}^\beta P_{\beta} = \sum_{\ell}^\gamma Q_{\beta}$ two relatively $T$-nilpotent decompositions of $M$. Then all $P_{\beta}$ and $Q_{\beta}$ are completely indecomposable modules and hence, those decompositions are unique up to isomorphism and every direct summand of $M$ has the exchange property in $M$.

Proof. We put $I_1 = \{\alpha \in I | P_{\alpha} \approx Q_{\beta}$ for some $\beta \in J\}$ and $J_1 = \{\beta \in J | Q_{\beta} \approx P_{\alpha}$ for some $\alpha \in I\}$. We first show $I_1 \neq \emptyset$ and so $J_1 \neq \emptyset$. We assume the contrary. Let $p_{\alpha}, q_{\beta}$ be projections of $M$ to $P_{\alpha}$ and $Q_{\beta}$, respectively. Let $x_i \neq 0 \in P_{\alpha_i}$. Then there exists $\beta_i \in J$ such that $q_{\beta_i} (x_i) = x_i \neq 0$. Again there exists $\alpha_i \in I$ such that $p_{\alpha_i} (x_i) = x_i \neq 0$. Repeating those arguments, we obtain a contradiction to the $T$-nilpotency, since $I_1 = J_1 = \emptyset$ (cf. [6], Lemma 7). Hence, $I_1 \neq \emptyset$ and so $J_1 \neq \emptyset$. Furthermore, $\{P_{\alpha}\}_{I_1}, \{Q_{\beta}\}_{J_1}$ are sets of completely indecomposable modules and locally $T$-nilpotent by Lemma 2 and the assumption. We put $M = \sum_{I_1}^\beta P_{\alpha} \oplus \sum_{J_1}^{\gamma} Q_{\beta}' \oplus \sum_{J_1}^{\gamma} Q_{\beta}''$, where $I_1' = I - I_1$ and $J_1' = J - J_1$. Let $\{P_{\alpha}\}_{I_1}$ be any finite subset of $\{P_{\alpha}\}_{I_1}$. Since $\sum_{I_1}^\beta P_{\alpha}$ has the exchange property by [2], Lemma 3.11 and [9], Proposition 1, $M = \sum_{I_1}^\beta P_{\alpha} \oplus \sum_{J_1}^{\gamma} Q_{\beta}' \oplus \sum_{J_1}^{\gamma} Q_{\beta}'' \cdots (**)$, where $Q_{\beta} = Q_{\beta}' \oplus Q_{\beta}''$. Then $\sum_{J_1}^{\gamma} Q_{\beta}''$ is isomorphic to a direct summand of $\sum_{I_1}^\beta P_{\alpha}$. If $Q_{\beta}''' \neq (0)$, $Q_{\beta}'''$ contains a direct summand isomorphic to some $P_{\alpha_j}$ by Krull-Remak-Schmidt's theorem, say $P_{\alpha_j}' = X \oplus Y$. $X \approx P_{\alpha_j}$. Since $\beta_j \in J_1', Y \neq 0$, let $\phi$ be a projection of $Q_{\beta}'''$ to $X$ and $i$ the inclusion of $X$ to $Q_{\beta}'''$, then $\phi$ and $i$ are not isomorphic. However, $\phi p \phi^{-1} = 1_{P_{\alpha_j}}$ and neither $\phi \phi^{-1}$ is isomorphic. Which contradicts the relative $T$-nilpotency. Accordingly, $Q_{\beta}''' = (0)$ and $(\sum_{I_1}^\beta P_{\alpha}) \cap (\sum_{J_1}^{\gamma} Q_{\beta}') = (0)$. Therefore, $(\sum_{I_1}^\beta P_{\alpha}) \cap
Let \( \sum_{i} Q_{i} = (0) \). Let \( \tilde{M} = M / (\sum_{i} Q_{i}) \) and \( \psi \) the natural epimorphism of \( M \) to \( \tilde{M} \). Then \( \tilde{M} = \sum_{i} \psi(Q_{i}) \sum_{i} \psi(P_{i}) \) (note \( \psi(Q_{i}) \approx Q_{i} \) and \( \psi(P_{i}) \approx P_{i} \)). On the other hand, \( \sum_{i} \psi(P_{i}) \) is locally direct summand of \( \tilde{M} \) from (\( \ast \)) and \( \{ \psi(P_{i}) \}_{i} \) is locally T-nilpotent. Hence, \( \sum_{i} \psi(P_{i}) \) is a direct summand of \( \tilde{M} \) by \( [3] \), Theorem 9 and \( [7] \), Corollary 2 to Lemma 2 and Lemma 3, since \( \{ \psi(Q_{i}) \}_{i} \) is a set of completely indecomposable modules. Furthermore, \( \sum_{i} \psi(P_{i}) \) has the exchange property in \( \tilde{M} \) by \( [4] \), Theorem 4 and so \( \tilde{M} = \sum_{i} \psi(P_{i}) \sum_{i} \psi(Q_{i}) \), where \( J_{i}'' \subseteq J_{i} \). Therefore, \( M = \sum_{i} \psi(P_{i}) \sum_{i} \psi(Q_{i}) \). Hence, \( \sum_{i} Q_{i}'' = \sum_{i} \psi(Q_{i}) \sum_{i} \psi(P_{i})'' \). However, \( \{ P_{i} \}_{i} \) and \( \{ Q_{i} \}_{i} \) are locally and relatively T-nilpotent and so some \( P_{i}'' \) is isomorphic to some \( Q_{i}'' \) by the first part, provided \( P_{i}'' \neq (0) \). Therefore, \( I_{i}'' = J_{i}'' = J_{i}'' = \phi \). The remaining parts are clear from \( [1] \) and \( [4] \), Theorem 4.

**Remark.** Theorem 1 does not remain valid if we replace the T-nilpotency by the semi-T-nilpotency in the assumption.

**Corollary 1.** Let \( P \) be \( R \)-projective. Then \( P \) has two relatively T-nilpotent decompositions if and only if \( P \) is a perfect module.

**Proof.** It is clear from Theorem 1 and \( [4] \), Theorem 6.

**Corollary 2.** \( R \) is right artinian if and only if every projective modules and every injective modules have relatively T-nilpotent decompositions.

**Proof.** It is clear from Corollary 1, \( [3] \), Corollary 1 to Proposition 1 and \( [8] \).

### 2. Exchange property

Let \( \{ M_{a} \} \) be a set completely indecomposable modules and \( M = \sum_{i} M_{i} \). We consider a relation between the concept of the exchange property in \( M \) and that of the 2-exchange property in \( M \) for a direct summand \( N \) of \( M \). If \( N \) is also a direct sum of indecomposable modules, those concepts are equivalent by \( [6] \), Theorem A.2. We do not know whether this fact is true without any assumptions.

**Lemma 3.** Let \( A = B \oplus C \oplus D \) be \( R \)-modules. If \( B \oplus C \) has the 2-exchange property in \( A \), then \( C \) has the same property in \( C \oplus D \).

**Proof.** We assume \( C \oplus D = K \oplus L \). Then \( A = B \oplus C \oplus D = B \oplus K \oplus L \). Hence, \( A = B \oplus C \oplus K' \oplus L' \) for some \( K' \subseteq K \) and \( L' \subseteq L \) from \( [6] \), Lemma A.4. Since \( C \oplus D \geq C \oplus K' \oplus L' \), \( C \oplus D = C \oplus K' \oplus L' \).
Let $M$ be as above and $M = S_1 \oplus S_2$. Let $\{M_\alpha\}_J$ be the isomorphic representative classes of $\{M_\alpha\}_J$ and we shall denote it by $[M]$. We put $J' = \{ \alpha \in J \mid M_\alpha \text{ is isomorphic to a direct summand of both } S_1 \text{ and } S_2 \}$ and $J'' = \{ \alpha \in J' \mid \text{directsums of any finite copies of } M_\alpha \text{ are isomorphic to direct summands of both } S_1 \text{ and } S_2 \}$.

**Theorem 2.** Let $M, S_1, \ldots$ be as above. Then $S_1$ has the exchange property in $M$ if and only if $S_1$ has the $2$-exchange property in $M$, $\{M_\alpha\}_J$ is locally semi-$T$-nilpotent and $\{M_\alpha\}_J$ is locally $T$-nilpotent.

Proof. Let $N_i$ be a dense submodule of $S_i$ and $N_i = \sum_{j \in I} \sum_{\beta \in I_j} \oplus N_{ij \beta}$, where $i=1,2$ and $N_{ij \beta}$'s are isomorphic to some $M_\alpha$ in $\{M_\alpha\}_I$ and $N_{ij \beta} \approx N_{i' j' \beta} \approx N_{i' j' \beta}$ if $J \neq J'$ (see [4]). We put

\[
J_1 = \{ j \in J \mid I_{ij} \text{ and } I_{ij} \text{ are infinite} \} \\
J_2 = \{ j \in J \mid I_{ij} \neq \emptyset \text{ and } I_{ij} \text{ is finite} \} \\
J_3 = \{ j \in J \mid I_{ij} \text{ is finite and } I_{ij} \text{ is infinite} \} \\
J_4 = \{ j \in J \mid I_{ij} = \emptyset \} \\
J_5 = \{ j \in J \mid I_{ij} = \emptyset \} .
\]

Then $J_i = J''$ and $J_1 \cup J_2 \cup J_3 = J'$. Furthermore, we put $N_i(k) = \sum_{j \in I} \sum_{\beta \in I_j} \oplus N_{ij \beta}$. Then $N_1 = \sum_{i=1}^4 \oplus N_i(k)$ and $N_2 = \sum_{i=1}^4 \oplus N_i(k)$. We shall show "if" part. $N_i(1)$, $N_i(3)$ and $N_i(2)$ are direct summands of $M$ from the assumption and [4], Proposition 2. Since $N_i(1) \oplus N_i(3) \subseteq S_i$ and $N_i(2) \subseteq S_i$, $M = N_i(1) \oplus N_i(3) \oplus S_i' \oplus N_i(2) \oplus S_i'$, where $S_i' \subseteq S_i$. Furthermore, a dense submodule of $S_i'$ (resp. $S_i$) is isomorphic to $N_i(2) \oplus N_i(5)$ (resp. $N_i(1) \oplus N_i(3) \oplus N_i(4)$). Since $S_i = N_i(1) \oplus N_i(3) \oplus S_i'$, $S_i = N_i(2) \oplus N_i(2) \oplus S_i'$ and $S_i$ has the 2-exchange property in $M$, $S_i'$ has the same property in $S_i(2) \oplus S_i'$ (resp. $S_i'$). Therefore, $S_i'$ is a directsum of completely indecomposable modules. Hence, $S_i$ has the exchange property in $M$ by [5], Theorem 2 and so $S_i'$ has the exchange property in $S_i' \oplus S_i'$. Accordingly, $S_i'$ has the exchange property in $S_i' \oplus S_i'$ by [6], Corollary 2 to Theorem A.2, since $N_i(1) \oplus N_i(3) \oplus N_i(2)$ has the exchange property in $M$ by [5], Theorem 2 and so $S_i' \oplus S_i'$ is a directsum of completely indecomposable modules. Therefore, $S_i'$ is also a directsum of completely indecomposable modules and hence, $S_i$ has the exchange property in $M$ by [6], Theorem A.2.

The converse is clear from [6], Theorem A.2.

Finally, we shall study some special properties concerning with the exchange property in $M$ of a direct summand of $M$. We are interested in a relation between the exchange property in $M$ and the relative semi-$T$-nilpotency. We assume $M = N_1 \oplus N_2$ and $N_i = \sum_{i=1}^4 \oplus N_i(k)$ as in the proof of Theorem 2.
We know from the proof above that $N_i$ has the exchange property in $M$ if and only if $N_i(2) \oplus N_i(5)$ has the same property in $N_i(2) \oplus N_i(5) \oplus N_i(3) \oplus N_i(4)$ and \{\{N_i(2)\}, \{N_i(5)\}\} are locally and relatively semi-T-nilpotent (cf. [6], the proof of Corollary 1 to Theorem). Hence, we may restrict ourselves to a case of $[N_i] \cap [N_j] = \phi$.

In the following we shall use the category $A$ induced from a set of completely indecomposable modules and its factor category $\tilde{A} = A/J'$ studied in [3]. We refer to [3] for the notations and results on $A$.

**Lemma 4.** Let $M$ be in $A$ and $A, B$ two locally direct summands of $M$. If $[A] \cap [B] = \phi$, $A \cap B = (0)$ and $A \oplus B$ is a locally direct summand of $M$.

**Proof.** Let $i_A, i_B$ be the inclusions of $A$ and $B$ into $M$, respectively. Since $[A] \cap [B] = \phi$, $\operatorname{Im} i_A \cap \operatorname{Im} i_B = (0)$ from [3], Theorem 7 and [7], Lemma 3. Let $A \oplus B$ be the external direct sum and $i = (i_A, i_B): A \oplus B \to M$. Then it is clear from the above that $i$ is monomorphic in $\tilde{A}$. Hence, $\operatorname{Im} i = A + B$ is a locally direct summand of $M$ and $A \cap B = (0)$.

**Lemma 5.** Let $M = S \oplus T$ and $M$ in $A$. For any element $x$ in $S$ there exists a finite set of indecomposable modules $S_i$ such that $S = \sum_i S_i \oplus S', x \in \sum_i S_i$ and $S_i$'s are isomorphic to some in $[M]$.

**Proof.** See [4], the proof of Proposition 3.

Let $M = N_1 \oplus N_2 = \sum \oplus S_\gamma$ and $N_i \in A$ with $[N_1] \cap [N_2] = \phi$. We put $S_\gamma(i) = \sum S_\alpha$, where $S_\alpha$ runs through all indecomposable direct summands of $S_\gamma$ which are isomorphic to some in $[N_i]$. By $[S_\gamma(i)]$ we denote the representative classes of such $S_\alpha$'s. Then $S_\gamma(i)$ is also the union of all locally direct summands $A$ of $S_\gamma$ with $[A] \subseteq [N_i]$. It is clear, from Lemma 5, $S_\gamma = S_\gamma(1) + S_\gamma(2)$. If $N_i$ has the exchange property in $M$, $S_\gamma = S_\gamma(1) \oplus S_\gamma(2)$ where $S_\gamma \subseteq S_\gamma(i)$ and every indecomposable direct summand of $S_\gamma$ is isomorphic to some in $[N_i]$. In the following, we shall study a case of $S_\gamma = S_\gamma(1)$.

The following lemma is a slight generalization of [6], Lemma 7.

**Lemma 6.** Let $M = N_1 \oplus N_2 = \sum \oplus S_\gamma$ and $N_i = \sum_{\alpha} N_i \alpha$; $N_i \alpha$'s are completely indecomposable modules. We assume \{\{N_1 \alpha\}\}_{I_1}$ and \{\{N_2 \alpha\}\}_{I_2}$ are locally and relatively semi-T-nilpotent and $[N_1] \cap [N_2] = \phi$. Then $M = \sum \oplus S_\gamma(1) + N_2 = N_1 + \sum \oplus S_\gamma(2)$.

**Proof.** We shall give a sketch of the proof (cf. [6], Lemma 7). We assume $I_1$ and $I_2$ are infinite. Put $M^* = \sum \oplus S_\gamma(1) + N_\alpha$. We assume $M \neq M^*$. Then there exists $N_i \alpha$ not contained in $M^*$. Let $x_i \in N_i \alpha - M^*$ and $x_i = \sum x_i \beta; x_i \beta \in S_\gamma$. 
We may assume $x_{\gamma i} \in M^\ast$. Then from Lemma 5 we have $x_{\gamma i} = \sum_{\beta} y_{i\beta}$; $y_{i\beta} \in A_{\beta} \oplus S_{\gamma i}$ and $A_{\beta}$'s are indecomposable. Since $x_{\gamma i} \in M^\ast$, there exists $y_{i\beta} \in A_{\beta} - M^\ast$ and so $[A_{\beta}] \in [N_\beta]$. Now, we can find $S_{\gamma i}'$ which contains a direct summand $S_{\gamma i}'$ isomorphic to $N_{i\alpha}$. Since $S_{\gamma i}'$ has the exchange property by [9], Proposition 1,

$$M = S_{\gamma i}' + N_1' \oplus N_2' ; N_1' \subseteq N_1,$$

since $[N_1] \cap [N_2] = \phi$. Let $y_{i\beta} = a + b + c$; $a \in S_{\gamma i}'$, $b \in N_1'$ and $c \in N_2$. Then $b \in N_1' - M^\ast$. Hence, we can find an indecomposable direct summand $N_{i\beta}'$ of $N_1'$ such that $b = x_{i\beta} + \cdots$, $x_{i\beta} \in N_{i\beta} - M^\ast$. On the other hand, since $[A_{\beta}] \in [N_\beta]$, there exists $N_2\beta$ isomorphic to $A_{\beta}$ and

$$M = N_2 \oplus \sum_{k} \oplus S_{\gamma k} ; S_{\gamma k} \subseteq S_{\gamma}.$$

Let $x_{i\beta} = d + \sum f_\gamma$; $d \in N_2\beta$ and $y_{i\beta} \in S_{\gamma}$. Again there exists $f_\gamma \in S_{\gamma} - M^\ast$. Similarly to the above, we can find a direct summand $A_{\gamma}$ of $S_{\gamma}$ such that $[A_{\gamma}] \in [N_\gamma]$, $f_\gamma = \sum y_{i\beta}'$ and $y_{i\beta} \in A_{\gamma} - M^\ast$. Furthermore, since $N_{i\beta}$ is a direct summand of $N_1'$, there exists $S_{\gamma}$ which contains a direct summand $S_{\gamma}$ isomorphic to $N_{i\beta}$ such that

$$M = S_{\gamma} + S_{\gamma} \oplus N_1' \oplus N_2' ; N_1' \subseteq N_1,$$

and $S_{\gamma} \approx S_{\gamma}$. Repeating those arguments we obtain a sequence $\{x_{i\alpha}, y_{i\beta}, x_{i\beta}, y_{2\beta}, \cdots\}$, which contradicts the assumption of relative semi-T-nilpotency.  

**Theorem 3.** Let $M$ be a direct sum of completely indecomposable modules and $M = N_1 \oplus N_2 = \sum_{k} S_{\gamma}$, $N_1$ is a direct sum of indecomposable modules $\{N_{i\alpha}\}_{I_1}$ such that $N_{i\alpha} \approx N_{i\beta}$ for any $\alpha \in I_1$, $\beta \in I_2$ and $\{N_{i\alpha}\}_{I_1}$, $\{N_{i\beta}\}_{I_2}$ are locally and relatively semi-T-nilpotent. Then the following conditions are equivalent.

1) $M = \sum_{k} S(1) \oplus N_2$

2) $\text{Hom}_R([S(1)], [S_\gamma(2)]) = (0)$ for every $\gamma$

3) $S(1)$ is a direct summand of $S_\gamma$ and $[A] \in [N_\gamma]$ for every indecomposable direct summand $A$ of $S(1)$, where $\text{Hom}_R([S(1)], [S_\gamma(2)]) = (0)$ means $\text{Hom}_R(B, C) = (0)$ for all $B \in [S(1)]$ and all $C \in [S_\gamma(2)]$.

Proof. 1) $\rightarrow$ 2) and 3). 1) implies $S_\gamma = S(1) \oplus S_{\gamma}'$ and $\sum_{k} \oplus S(1) \approx N_1$, $\sum_{k} \oplus S_{\gamma}' \approx N_2$. Hence, we obtain 3). Furthermore, every $C$ in $[S(2)]$ is isomorphic to a direct summand $C'$ of $S_{\gamma}'$. Let $B$ be in $[S(1)]$ and $f' \in \text{Hom}_R(B, C) \cong \text{Hom}_R(B, C')$. Then $f' = \varphi(f')$ is not isomorphic from the assumption. Put $B' = \{x + f(x) | x \in B\}$. Then $B' \approx B$ and $B'$ is a direct summand of $S_{\gamma}$ from [7],

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Lemma 3. Hence, \( B' \subseteq S_\gamma(1) \). Therefore, \( \text{Im} f \subseteq (B + B') \cap S_\gamma' \subseteq S_\gamma(1) \cap S_\gamma' = (0) \).

2)\( \rightarrow \)1). Let \( x \) be in \( S_\gamma(1) \) and \( x = \sum x_i; x_i \in A_i \triangleleft S_\gamma \) with \( [A_i] \subseteq [N_i] \). On the other hand, there exists, from Lemma 5, a direct summand \( B = \sum_{j=1}^I B_j \) of \( S_\gamma \) such that \( x \in B \) and \( [B_j] \subseteq [M] \). Let \( p_j \) be the projection of \( S_\gamma \) onto \( B_j \). If \( 0 \neq p_j(x) = \sum p_j(x_i) [B_j] \subseteq [N_i] \) from \( 2) \). Hence, \( x = \sum_{j=1}^I B_j \) with \( [B_j] \subseteq [N_i] \).

Now, let \( y \) be in \( (\sum \oplus S_\gamma(1)) \cap N_2 \) and \( y = \sum y_i; y_i \in S_\gamma(1) \). Then there exists a direct summand \( \sum_{j=1}^I \oplus B_j \), containing \( y \) as above. Hence, \( y \in (\sum \oplus B_j) \cap N_2 = (0) \) by Lemma 4. Therefore, \( M = \sum \oplus S_\gamma(1) \oplus N_2 \) from Lemma 6.

3)\( \rightarrow \)1). Let \( S'_\gamma = S_\gamma(1) \oplus S'_\gamma \) and \( C \) an indecomposable direct summand of \( S_\gamma' \). Then \( C \subseteq [N_i] \) otherwise \( C \subseteq S_\gamma(1) \). Hence, for dense submodules \( B_\gamma, B_z \) in \( S_\gamma(1) \) and \( S'_\gamma \), respectively, we have \( [B_\gamma] \subseteq [N_i] \). Since \( \sum_{k} \oplus B_k \) and \( \sum_{k} \oplus B_k \) are dense submodules of \( \sum_{k} \oplus S_\gamma(1) \) and \( \sum_{k} \oplus S' \), respectively by [4], Theorem 1, \( M = \sum \oplus S_\gamma(1) \oplus N_2 \) by [6], Lemma 7.

**Corollary 1.** Let \( M \) be as above and \( M = N_1 \oplus N_2 \). If either \( \text{Hom}_R(N_1, N_2) = (0) \) or \( \text{Hom}_R(N_1, N_2) = (0) \), then \( N_1 \) and \( N_2 \) have the exchange property in \( M \), (cf. [6], Corollary 5 to Theorem).

Proof. \( N_i \) is in \( A \) by [6], Corollary 5 to Theorem. The condition 2) in the theorem is satisfied for any decompositions \( M = \sum \oplus S_\gamma \). Hence, \( N_1 \) and \( N_2 \) have the exchange property in \( M \) from Theorem 3 and [6], Theorem A.2.

**Corollary 2.** Let \( M, N_1 \) and \( S_\gamma \) be as in Theorem 3. Then the following conditions are equivalent.

1) \( M = (\sum \oplus S_\gamma(1)) \oplus N_\gamma = N_1 \oplus (\sum \oplus S_\gamma(2)) \)

2) \( \text{Hom}_R([S_\gamma(1)], [S_\gamma(2)]) = (0) = \text{Hom}_R([S_\gamma(2)], [S_\gamma(1)]) \)

3) \( S_\gamma = S_\gamma(1) \oplus S_\gamma(2) \).

Proof. 1)\( \rightarrow \)3). 1) implies \( S_\gamma = S_\gamma(1) \oplus S' \). Hence, \( S_\gamma(2) = S'_\gamma \oplus S_\gamma(1) \cap S_\gamma(2) \), since \( S'_\gamma \subseteq S_\gamma(2) \). If \( S_\gamma(1) \cap S_\gamma(2) = T \neq (0) \), \( T \) contains an indecomposable direct summand \( A \) of \( S_\gamma \) from Lemma 5. Then \( [A] \subseteq [N_1] \cap [N_2] \) from the first decompositions and 1). Therefore, \( S_\gamma(1) \cap S_\gamma(2) = (0) \) and \( S_\gamma = S_\gamma(1) \oplus S_\gamma(2) \). Other implications are clear from Theorem 3.

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References


