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This note gives some supplementary results of [6]. The first one shows an application of the idea in the proof of [6], Lemma 7 and gives a characterization of artinian rings. The second one gives a refinement of [6], Corollary 2 to Theorem A.2 and the final one is a special type of the exchange property.

Throughout we shall assume that $R$ is a ring with identity and modules are unitary right $R$-modules. First, we shall recall definitions in [6].

Let $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ be two infinite sets of $R$-modules. We take a countable set $\{M_i\}_I$ such that $M_{2i-1}=P_{\alpha(2i-1)}\in\{P_\alpha\}_I$, and $M_{2j}=Q_{\beta(2j)}\in\{Q_\beta\}_J$. Further we take a set of non-isomorphisms $f_i: M_i\to M_{i+1}$. If for any element $m$ in $M_i$ there exists $n$ such that $f_if_n\cdots f_i(m)=0$, we say $\{f_i\}_I$ is locally T-nilpotent. If for any countable sets $\{M_i\}_I$ above such that $\alpha(2i-1)=\alpha(2i'-1)$ and $\beta(2j)=\beta(2j')$ if $i\neq i'$ ($j\neq j'$) any sets $\{f_i\}$ of non-isomorphisms are always locally T-nilpotent, then we say $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ are relatively and locally semi-T-nilpotent. If we omit the assumptions $\alpha(2i-1)=\alpha(2i')$ and $\beta(2j)=\beta(2j')$ in the above, we say $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ are relatively and locally T-nilpotent. If $\{P_\alpha\}_I=\{Q_\beta\}_J$, we say $\{P_\alpha\}_I$ is locally semi-T-nilpotent or T-nilpotent, corresponding to the above cases. We shall assume that the definition of relatively semi-T-nilpotency contains a case of either $I$ or $J$ being finite. If $K=\sum_{\alpha}P_\alpha=\sum_{\beta}Q_\beta$ and $\{P_\alpha\}_I$, $\{Q_\beta\}_J$ are locally and relatively T-nilpotent, then we say $\sum_{\alpha}P_\alpha$ and $\sum_{\beta}Q_\beta$ are relatively T-nilpotent decompositions of $K$.

Finally, let $M=N\oplus P$ be $R$-modules and $\kappa$ a cardinal number. If for any decomposition $M=\sum L_\alpha$ with $\kappa$-components there exist submodules $L_\alpha'$ of $L_\alpha$ such that $M=N\oplus\sum L_\alpha'$, then we say $N$ has the $\kappa$-exchange property in $M$. In case $\kappa$ is any cardinal, we say $N$ has the exchange property in $M$.

1. T-nilpotent decompositions

First, we study a property of relative T-nilpotency. If the endomorphism ring of a module $M$ is a local ring, then we call $M$ completely indecomposable.

Lemma 1. Let $M$ be an $R$-module and $f,g$ in $\text{End}_R(M)$. If $fg$ is isomorphic, $M=\text{Im }g\oplus\text{Ker }f$. 
Lemma 2. Let $P$ be an $R$-module. If $P$ is itself locally $T$-nilpotent, $P$ is a completely indecomposable module.

Proof. Put $S = \text{End}_R(P)$. If $e \in S$ and $e^2 = e$, then $e = 1$ or $0$ by the assumption. Let $x, y$ be elements in $S$ with $x$ non-unit. Then neither $xy$ nor $yx$ is unit in $S$ from the above and Lemma 1. Furthermore, consider a sequence $\{x^n\}^\infty_1$ of non-units in $S$. For any element $p$ in $P$, there exists $n = n(p)$ such that $x^{n(p)}(p) = 0$ from the assumption. Therefore, $X = 1 + \sum_1^\infty x^i$ is an element in $S$ and $((1-x)X)(p) = (1-x)(1+x+\cdots+x^{n(p)-1})(p) = (1-x^{n(p)})(p) = p$. Hence, $1-x$ is unit in $S$ from Lemma 1. Let $x, y$ be non-unit in $S$. We assume that $x+y$ is unit in $S$. Then we may assume $x+y=1$, which is a contradiction to the above. Therefore, $S$ is a local ring.

Theorem 1. Let $M$ be an $R$-module and $M = \sum\oplus P_\alpha = \sum\oplus Q_\beta$ two relatively $T$-nilpotent decompositions of $M$. Then all $P_\alpha$ and $Q_\beta$ are completely indecomposable modules and hence, those decompositions are unique up to isomorphism and every direct summand of $M$ has the exchange property in $M$.

Proof. We put $I = \{\alpha \in I | P_\alpha \cong Q_\beta$ for some $\beta \in J\}$ and $J = \{\beta \in J | Q_\beta \cong P_\alpha$ for some $\alpha \in I\}$. We first show $I \neq \emptyset$ and so $J \neq \emptyset$. We assume the contrary. Let $p_\alpha, q_\beta$ be projections of $M$ to $P_\alpha$ and $Q_\beta$, respectively. Let $x_\alpha \neq 0 \in P_\alpha$. Then there exists $\beta_\alpha \in J$ such that $q_\beta(x_\alpha) = x_\alpha \neq 0$. Again there exists $\alpha_\beta \in I$ such that $p_\alpha(x_\beta) = x_\beta \neq 0$. Repeating those arguments, we obtain a contradiction to the $T$-nilpotency, since $I = J = \emptyset$ (cf. [6], Lemma 7). Hence, $I \neq \emptyset$ and so $J \neq \emptyset$. Furthermore, $\{P_\alpha\}_I$, $\{Q_\beta\}_J$ are sets of completely indecomposable modules and locally $T$-nilpotent by Lemma 2 and the assumption. We put $M = \sum\oplus P_\alpha \oplus \sum\oplus P_\alpha' = \sum\oplus Q_\beta \oplus \sum\oplus Q_\beta'$, where $I' = I - I$, and $J' = J - J$. Let $\{P_\alpha\}_I$ be any finite subset of $\{P_\alpha\}_I$. Since $\sum\oplus P_\alpha$ has the exchange property by [2], Lemma 3.11 and [9], Proposition 1, $M = \sum\oplus P_\alpha + \sum\oplus Q_\beta + \sum\oplus Q_\beta' + \cdots$ (*),

where $Q_\beta = Q_\beta' \oplus Q_\beta''$. Then $\sum\oplus Q_\beta''$ is isomorphic to a direct summand of $\sum\oplus P_\alpha$. If $Q_\beta'' \neq (0)$, $Q_\beta''$ contains a direct summand isomorphic to some $P_\alpha$ by Krull-Remak-Schmidt's theorem, say $Q_\beta'' = X \oplus Y$; $X \cong P_\alpha$. Since $\beta' \in J'$, $Y \neq (0)$. Let $p$ be a projection of $Q_\beta''$ to $X$ and $i$ the inclusion of $X$ to $Q_\beta''$, then $p$ and $i$ are not isomorphic. However, $\varphi pi \varphi^{-1} = 1_{P_\alpha}$, and neither $\varphi p$ nor $i \varphi^{-1}$ is isomorphic. Which contradicts the relative $T$-nilpotency. Accordingly, $Q_\beta'' = (0)$ and $(\sum\oplus P_\alpha) \cap (\sum\oplus Q_\beta') = (0)$. Therefore, $(\sum\oplus P_\alpha) \cap
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(∑_{β}Q_{β})=(0). Let \( M = M/ (∑_{β}Q_{β}) \) and \( ψ \) the natural epimorphism of \( M \) to \( \tilde{M} \). Then \( \tilde{M} = ∑_{β}ψ(Q_{β}) \), \( ∑_{β}ψ(P_{α}) \) (note \( ψ(Q_{β}) \approx Q_{β} \) and \( ψ(P_{α}) \approx P_{α} \)).

On the other hand, \( ∑_{β}ψ(P_{α}) \) is locally direct summand of \( \tilde{M} \) from \( (*) \) and \( \{ψ(P_{α})\}_{β} \), is locally T-nilpotent. Hence, \( ∑_{β}ψ(P_{α}) \) is a direct summand of \( \tilde{M} \) by [3], Theorem 9 and [7], Corollary 2 to Lemma 2 and Lemma 3, since \( \{ψ(Q_{β})\}_{β} \) is a set of completely indecomposable modules. Furthermore, \( ∑_{β}ψ(P_{α}) \) has the exchange property in \( \tilde{M} \) by [4], Theorem 4 and so \( \tilde{M} = ∑_{β}ψ(P_{α}) \) where \( J_{i}'' \subseteq J_{i} \). Therefore, \( M = ∑_{β}P_{α} \) has \( ∑_{β}ψ(Q_{β}) \) with \( ∑_{β}ψ(Q_{β}) \approx Q_{β} \). Hence, \( ∑_{β}ψ(Q_{β}) \approx ∑_{β}ψ(P_{α}) \). However, \( \{ψ(P_{α})\}_{β} \) are locally and relatively T-nilpotent and so some \( P_{α} \) is isomorphic to some \( Q_{β} \) by the first part, provided \( P_{α} = (0) \). Therefore, \( J_{i}'' = J_{i} = P_{α} \). The remaining parts are clear from [1] and [4], Theorem 4.

REMARK. Theorem 1 does not remain valid if we replace the T-nilpotency by the semi-T-nilpotency in the assumption.

**Corollary 1.** Let \( P \) be \( R \)-projective. Then \( P \) has two relatively T-nilpotent decompositions if and only if \( P \) is a perfect module.

Proof. It is clear from Theorem 1 and [4], Theorem 6.

**Corollary 2.** \( R \) is right artinian if and only if every projective modules and every injective modules have relatively T-nilpotent decompositions.

Proof. It is clear from Corollary 1, [3], Corollary 1 to Proposition 1 and [8].

2. Exchange property

Let \( \{M_{α}\} \) be a set completely indecomposable modules and \( M = ∑_{α}M_{α} \).

We consider a relation between the concept of the exchange property in \( M \) and that of the 2-exchange property in \( M \) for a direct summand \( N \) of \( M \). If \( N \) is also a direct sum of indecomposable modules, those concepts are equivalent by [6], Theorem A.2. We do not know whether this fact is true without any assumptions.

**Lemma 3.** Let \( A = B \oplus C \oplus D \) be \( R \)-modules. If \( B \oplus C \) has the 2-exchange property in \( A \), then \( C \) has the same property in \( C \oplus D \).

Proof. We assume \( C \oplus D = K \oplus L \). Then \( A = B \oplus C \oplus D = B \oplus K \oplus L \). Hence, \( A = B \oplus C \oplus K' \oplus L' \) for some \( K' \subseteq K \) and \( L' \subseteq L \) from [6], Lemma A.4. Since \( C \oplus D \supseteq C \oplus K' \oplus L' \), \( C \oplus D = C \oplus K' \oplus L' \).
Let \( M \) be as above and \( M = S_1 \oplus S_2 \). Let \( \{M_a\}_J \) be the isomorphic representative classes of \( \{M_a\}_J \) and we shall denote it by \([M]\). We put \( J' = \{\alpha \in J \mid M_\alpha \text{ is isomorphic to a direct summand of both } S_1 \text{ and } S_2 \} \) and \( J'' = \{\alpha \in J' \mid \text{directsums of any finite copies of } M_\alpha \text{ are isomorphic to direct summands of both } S_1 \text{ and } S_2 \} \).

**Theorem 2.** Let \( M, S_1, \) etc. be as above. Then \( S_1 \) has the exchange property in \( M \) if and only if \( S_1 \) has the 2-exchange property in \( M, \{M_a\}_J \) is locally semi-T-nilpotent and \( \{M_a\}_J'' \) is locally T-nilpotent.

**Proof.** Let \( N_i \) be a dense submodule of \( S_i \) and \( N_i = \sum_{j_1} \sum_{\beta \in \Gamma_{i\beta}} \oplus N_{i\beta j} \), where \( i = 1, 2 \) and \( N_{i\beta j} \)'s are isomorphic to some \( M_\alpha \) in \( \{M_a\}_J \) and \( N_{i\beta j} \approx N_{i'\beta j} \approx N_{i''\beta j} \) if \( j \neq j' \) (see [4]). We put

\[
\begin{align*}
J_1 &= \{j \in J \mid \text{I}_{i\beta j} \text{ and } I_{i\beta j} \text{ are infinite}\} \\
J_2 &= \{j \in J \mid I_{i\beta j} = \phi \text{ and } I_{i\beta j} \text{ is finite}\} \\
J_3 &= \{j \in J \mid \text{I}_{i\beta j} \text{ is finite and } I_{i\beta j} \text{ is infinite}\} \\
J_4 &= \{j \in J \mid I_{i\beta j} = \phi \} \\
J_5 &= \{j \in J \mid I_{i\beta j} = \phi \}.
\end{align*}
\]

Then \( J_i = J'' \) and \( J_1 \cup J_2 \cup J_3 = J' \). Furthermore, we put \( N_i(k) = \sum_{j \in J} \sum_{\beta \in \Gamma_{i\beta}} \oplus N_{i\beta j} \). Then \( N_1 = \sum_{i=1}^{N_1(3)} \oplus N_i(k) \) and \( N_2 = \sum_{i=1}^{N_2(3)} \oplus N_i(k) \). We shall show "if" part. \( N_1(1), N_1(3) \) and \( N_2(2) \) are direct summands of \( M \) from the assumption and [4], Proposition 2. Since \( N_1(1) \oplus N_1(3) \subseteq S_1 \) and \( N_2(2) \subseteq S_2 \), \( M = N_1(1) \oplus N_1(3) \oplus S_1' \oplus N_2(2) \oplus S_2' \), where \( S_1' \subseteq S_1 \). Furthermore, a dense submodule of \( S_1' \) (resp. \( S_1' \)) is isomorphic to \( N_2(2) \oplus N_1(5) \) (resp. \( N_2(1) \oplus N_1(3) \oplus N_1(4) \)). Since \( S_1 = N_1(1) \oplus N_1(3) \oplus S_1', S_2 = N_2(2) \oplus S_2' \) and \( S_1 \) has the 2-exchange property in \( M, S_1' \) has the same property in \( S_1 \oplus S_1' \oplus S_1' \) (\( = C \)) from Lemma 3. We assume \( S_1' \oplus S_2' = A \oplus B' \). Then \( C = A \oplus B \oplus N_1(2) \). Hence, \( C = S_1' \oplus A' \oplus B' \oplus N_1(2) \) from [6], Lemma A.4, where \( A' \subseteq A \), etc. Therefore, \( S_1' \oplus A' \oplus B' = S_1' \oplus S_1' \), which means \( S_1' \) has the 2-exchange property in \( S_1' \oplus S_1' \). Accordingly, \( S_1' \) has the exchange property in \( S_1' \oplus S_1' \) by [6], Corollary 2 to Theorem A.2, since \( N_1(1) \oplus N_1(3) \oplus N_1(2) \) has the exchange property in \( M \) by [5], Theorem 2 and so \( S_1' \oplus S_1' \) is a directsum of completely indecomposable modules. Therefore, \( S_1' \) is also a directsum of completely indecomposable modules and hence, \( S_1 \) has the exchange property in \( M \) by [6], Theorem A.2. The converse is clear from [6], Theorem A.2.

Finally, we shall study some special properties concerning with the exchange property in \( M \) of a direct summand of \( M \). We are interested in a relation between the exchange property in \( M \) and the relative semi-T-nilpotency. We assume \( M = N_1 \oplus N_2 \) and \( N_i = \sum \oplus N_i(k) \) as in the proof of Theorem 2.
We know from the proof above that $N_i$ has the exchange property in $M$ if and only if $N_i(2) \oplus N_i(5)$ has the same property in $N_i(2) \oplus N_i(5) \oplus N_i(3) \oplus N_i(4)$ and $\{N_{i1}\}, \{N_{i2}\}$ are locally and relatively semi-T-nilpotent (cf. [6], the proof of Corollary 1 to Theorem). Hence, we may restrict ourselves to a case of $[N_i] \cap [N_i] = \phi$.

In the following we shall use the category $\mathcal{A}$ induced from a set of completely indecomposable modules and its factor category $\mathcal{A}/\mathcal{A}'$ studied in [3]. We refer to [3] for the notations and results on $\mathcal{A}$.

**Lemma 4.** Let $M$ be in $\mathcal{A}$ and $A, B$ two locally direct summands of $M$. If $[A] \cap [B] = \phi$, $A \cap B = (0)$ and $A \oplus B$ is a locally direct summand of $M$.

Proof. Let $i_A, i_B$ be the inclusions of $A$ and $B$ into $M$, respectively. Since $[A] \cap [B] = \phi$, $\text{Im} \ i_A \cap \text{Im} \ i_B = (0)$ from [3], Theorem 7 and [7], Lemma 3. Let $A \oplus B$ be the external direct sum and $i = (i_A, i_B): A \oplus B \to M$. Then it is clear from the above that $i$ is monomorphic in $\mathcal{A}$. Hence, $\text{Im} \ i = A + B$ is a locally direct summand of $M$ and $A \cap B = (0)$.

**Lemma 5.** Let $M=S \oplus T$ and $M$ in $\mathcal{A}$. For any element $x$ in $S$ there exists a finite set of indecomposable modules $S_i$ such that $S = \sum \oplus S_i \oplus S'$, $x \in \sum \oplus S_i$ and $S_i$'s are isomorphic to some in $[M]$.

Proof. See [4], the proof of Proposition 3.

Let $M=N_1 \oplus N_2 = \sum \oplus S_i$ and $N_i \in \mathcal{A}$ with $[N_i] \cap [N_i] = \phi$. We put $S_i(t) = \sum S_{a}$, where $S_{a}$ runs through all indecomposable direct summands of $S_i$ which are isomorphic to some in $[N_i]$. By $[S_{a}(t)]$ we denote the representative classes of such $S_{a}$'s. Then $S_{a}(t)$ is also the union of all locally direct summands $A$ of $S_i$ with $[A] \subseteq [N_i]$. It is clear, from Lemma 5, $S_i = S_i(1) + S_i(2)$. If $N_i$ or $N_i$ has the exchange property in $M$, $S_i = S_i \oplus S_i$ where $S_i \subseteq S_i(t)$ and every indecomposable direct summand of $S_i$ is isomorphic to some in $[N_i]$. In the following, we shall study a case of $S_i = S_i(1)$.

The following lemma is a slight generalization of [6], Lemma 7.

**Lemma 6.** Let $M=N_1 \oplus N_2 = \sum \oplus S_i$ and $N_i = \sum \oplus N_{i a}$; $N_{i a}$'s are completely indecomposable modules. We assume $\{N_{i a}\}_{i a}$ and $\{N_{i a}\}_{i a}$ are locally and relatively semi-T-nilpotent and $[N_i] \cap [N_i] = \phi$. Then $M = \sum \oplus S_i(1) + N_2 = N_i + \sum \oplus S_i(2)$.

Proof. We shall give a sketch of the proof (cf. [6], Lemma 7). We assume $I_i$ and $I_{i a}$ are infinite. Put $M^* = \sum \oplus S_i(1) + N_2$. We assume $M \neq M^*$. Then there exists $N_{i a}$ not contained in $M^*$. Let $x_i \in N_{i a} - M^*$ and $x_i = \sum x_i, x_i \in S_i$. 


We may assume $x_{\alpha} \in M^*$. Then from Lemma 5 we have $x_{\alpha} = \sum y_{\beta}$; $y_{\beta} \in A_\beta \oplus S_{\alpha}$, and $A_\beta$'s are indecomposable. Since $x_{\alpha} \in M^*$, there exists $y_{\beta} \in A_\beta - M^*$ and so $[A_\beta] \in [N_\alpha]$. Now, we can find $S_{\alpha' \nu}$ which contains a direct summand $S_{\nu' \nu}$ isomorphic to $N_{\nu}$. Since $S_{\alpha' \nu}$ has the exchange property by [9], Proposition 1,

$$M = S_{\alpha' \nu} \oplus N_{\nu} \oplus N_{\nu}'; \ N_{\nu}' \subseteq N_1,$$

since $[N_1] \cap [N_\alpha] = \phi$. Let $y_{\beta} = a + b + c; \ a \in S_{\alpha' \nu}$, $b \in N_{\nu}'$ and $c \in N_\alpha$. Then $b \in N_{\nu}' - M^*$. Hence, we can find an indecomposable direct summand $N_{\nu}$ of $N_{\nu}'$ such that $b = d + \sum y_{\beta}$; $d \in N_{2\nu}, f_{\beta} \in S_{\nu}'$. Again there exists $f_{\beta} \in S_{\nu'} - M^*$. Similarly to the above, we can find a direct summand $A_{\nu}$ of $S_{\nu'}$ such that $[A_{\nu}] \in [N_\nu], f_{\beta} = \sum y_{\beta}'$ and $y_{\beta'} \in A_{\nu} - M^*$. Furthermore, since $N_{\nu}$ is a direct summand of $N_{1\nu}$, there exists $S_{\nu'}$ which contains a direct summand $S_{\nu'}$ isomorphic to $N_{\nu}$. Let $x_{1\nu} = d + \sum f_{\beta}; \ d \in N_{2\nu}, f_{\beta} \in S_{\nu}'$. Then $x_{1\nu} \in N_{1\nu}' - M^*$. Hence, we obtain $x_{1\nu}' \subseteq N_{1\nu}'$ and $S_{\nu}' \approx S_{\nu'}$. Repeating those arguments we obtain a sequence $\{x_{\nu}, y_{\beta}, x_{\nu}, y_{\beta}, \ldots\}$, which contradicts the assumption of relative semi-T-nilpotency.

**Theorem 3.** Let $M$ be a direct sum of completely indecomposable modules and $M = N_1 \oplus N_2 = \sum \oplus S_\gamma$. We assume that $M_{1\alpha}$ is a direct sum of indecomposable modules $\{N_{1\alpha}\}_{\gamma}$ such that $N_{1\alpha} \approx N_{2\beta}$ for any $\alpha \in I_1, \beta \in I_2$, and $\{N_{1\alpha}\}_{\gamma}$ is locally and relatively semi-T-nilpotent. Then the following conditions are equivalent.

1. $M = \sum \oplus S_\gamma(1) \oplus N_2$
2. $\text{Hom}_R([S_\gamma(1)], [S_\gamma(2)]) = (0)$ for every $\gamma$
3. $S_\gamma(1)$ is a direct summand of $S_\gamma$ and $[A] \in [N_\alpha]$ for every indecomposable direct summand $A$ of $S_\gamma(1)$, where $\text{Hom}_R([S_\gamma(1)], [S_\gamma(2)]) = (0)$ means $\text{Hom}_R(B, C) = (0)$ for all $B \in [S_\gamma(1)]$ and all $C \in [S_\gamma(2)]$.

Proof. 1) $\rightarrow$ 2) and 3). 1) implies $S_{\gamma} = S_{\gamma}(1) \oplus S_{\gamma}'$ and $\sum \oplus_{\gamma} S_{\gamma}(1) \approx N_1, \sum \oplus_{\gamma} S_{\gamma}' \approx N_2$. Hence, we obtain 3). Furthermore, every $C \in [S_\gamma(2)]$ is isomorphic to a direct summand $C'$ of $S_{\gamma}'$. Let $B$ be in $[S_{\gamma}(1)]$ and $f' \in \text{Hom}_R(B, C) \cong \text{Hom}_R(B, C')$. Then $f = \varphi(f')$ is not isomorphic from the assumption. Put $B' = \{x + f(x) \mid x \in B\}$. Then $B' \approx B$ and $B'$ is a direct summand of $S_{\gamma}$ from [7],
Lemma 3. Hence, \( B' \subseteq S_{\gamma}(1) \). Therefore, \( \text{Im } f \subseteq (B + B') \cap S_{\gamma}' \subseteq S_{\gamma}(1) \cap S_{\gamma}' = (0) \).

2)\( \rightarrow 1) \). Let \( x \) be in \( S_{\gamma}(1) \) and \( x = \sum x_i ; x_i \in A_i \ll S_{\gamma} \) with \( [A_i] \subseteq [N_i] \). On the other hand, there exists, from Lemma 5, a direct summand \( B = \sum B_j \) of \( S_{\gamma} \) such that \( x \in B \) and \( [B_j] \subseteq [M] \). Let \( p_j \) be the projection of \( S_{\gamma} \) onto \( B_j \). If \( 0 \neq p_j(x) = \sum p_j(x_i)[B_j] \subseteq [N_i] \) from 2). Hence, \( x \in \sum B_j \) with \( [B_j] \subseteq [N_i] \).

Now, let \( y \) be in \((\sum \oplus S_{\gamma}(1)) \cap N_z \) and \( y = \sum y_i ; y_i \in S_{\gamma}(1) \). Then there exists a direct summand \( \sum \oplus B_j \) containing \( y \) as above. Hence, \( y \in (\sum \oplus B_j) \cap N_z = (0) \) by Lemma 4. Therefore, \( M = \sum \oplus S_{\gamma}(1) \oplus N_z \) from Lemma 6.

3)\( \rightarrow 1) \). Let \( S_{\gamma}' = S_{\gamma}(1) \oplus S_{\gamma}' \) and \( C \) an indecomposable direct summand of \( S_{\gamma}' \). Then \( C \subseteq [N_z] \) otherwise \( C \subseteq S_{\gamma}(1) \). Hence, for dense submodules \( B_{\eta_1} \), \( B_{\eta_2} \) in \( S_{\gamma}(1) \) and \( S_{\gamma}' \), respectively, we have \([B_{\eta_1}] \subseteq [N_i] \). Since \( \sum \oplus B_{\eta_1} \) and \( \sum \oplus B_{\eta_2} \) are dense submodules of \( \sum \oplus S_{\gamma}(1) \) and \( \sum \oplus S_{\gamma}' \), respectively by [4], Theorem 1, \( M = \sum \oplus S_{\gamma}(1) \oplus N_z \) by [6], Lemma 7.

**Corollary 1.** Let \( M \) be as above and \( M = N_1 \oplus N_2 \). If either \( \text{Hom}_R(N_1, N_2) = (0) \) or \( \text{Hom}_R(N_2, N_1) = (0) \), then \( N_1 \) and \( N_2 \) have the exchange property in \( M \), (cf. [6], Corollary 5 to Theorem).

Proof. \( N_i \) is in A by [6], Corollary 5 to Theorem. The condition 2) in the theorem is satisfied for any decompositions \( M = \sum \oplus S_{\gamma} \). Hence, \( N_1 \) and \( N_2 \) have the exchange property in \( M \) from Theorem 3 and [6], Theorem A.2.

**Corollary 2.** Let \( M, N_i \) and \( S_{\gamma} \) be as in Theorem 3. Then the following conditions are equivalent.

1) \( M = (\sum \oplus S_{\gamma}(1)) \oplus N_z = N_1 \oplus (\sum \oplus S_{\gamma}(2)) \)

2) \( \text{Hom}_R([S_{\gamma}(1)], [S_{\gamma}(2)]) = (0) = \text{Hom}_R([S_{\gamma}(2)], [S_{\gamma}(1)]) \)

3) \( S_{\gamma} = S_{\gamma}(1) \oplus S_{\gamma}(2) \).

Proof. 1)\( \rightarrow 3) \). 1) implies \( S_{\gamma} = S_{\gamma}(1) \oplus S_{\gamma}' = S_{\gamma}(2) \oplus S_{\gamma}'' \). Hence, \( S_{\gamma}(2) = S_{\gamma}' \oplus S_{\gamma}(1) \cap S_{\gamma}(2) \), since \( S_{\gamma}' \subseteq S_{\gamma}(2) \). If \( S_{\gamma}(1) \cap S_{\gamma}(2) = T \neq (0) \), \( T \) contains an indecomposable direct summand \( A \) of \( S_{\gamma} \) from Lemma 5. Then \([A] \subseteq [N_i] \cap [N_z] \) from the first decompositions and 1). Therefore, \( S_{\gamma}(1) \cap S_{\gamma}(2) = (0) \) and \( S_{\gamma} = S_{\gamma}(1) \oplus S_{\gamma}(2) \). Other implications are clear from Theorem 3.

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References


