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K_G-GROUPS AND INVARIANT VECTOR FIELDS ON SPECIAL G-MANIFOLDS

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Introduction

The main purpose of this paper is to give a formula to determine the semi-group structure of G-equivalence classes of real and complex G-vector bundles over special G-manifolds, [2], [3], [5]. K. Jänich has obtained a classification theorem for regular $O(n)$-manifolds with many orbit types, and given a formula for $Vect_{O(n)}$ of these manifolds [6]. Our formula is rather simple, but it may apply just for special G-manifolds which satisfies a condition on normalizers of isotropy subgroups, (C_2) in § 2.

In § 1, we collect some known results for later use. § 2 contains a lemma which is one of our main tools. In § 3, we define an object associated with an orbit space, which we shall call a datum, and proved the formula. As an application of the formula, in § 4, we determine the complex K_G-group of Brieskorn-Hirzebruch $O(n)$-manifold $W^{2n-1}(d)$, [2]. In § 5, we shall prove the existence of an $O(n)$-invariant 1-field on $W^{2n-1}(d)$ and the non-existence of invariant 2-fields for $n\geqq 2$.

I am grateful to S. Araki and F. Uchida for their kind advices.

1. G-manifolds with one orbit type

In this section, we recall a formula due to K. Jänich and G. Segal [6], [9].

Let G be a compact Lie group and M be a compact smooth manifold. A differentiable G-action on M is a smooth map $\mu\colon G\times M\to M$ such that

$$\mu(g_1, \mu(g_2, x)) = \mu(g_1\cdot g_2, x), \text{ and } \mu(e, x) = x\,,$$

where e is the unit of G. A compact smooth manifold with a differentiable G-action is called a G-manifold. We denote by G_x the isotropy subgroup of $x\in M$, and by $G(x)$ the orbit through x. We denote by (H) the conjugate class of isotropy subgroups including H, and call it the orbit type. Let M be a G-manifold with one orbit type (H), and $P(H)$ be the set of fixed points under the action of H, i.e. $P(H)=\{x\in M;\ G_x=H\}$, then $\pi|P(H): P(H)\to\pi(M)$ is the principal $N(H)/H=\Gamma(H)$-bundle, 2.4, [2], 1.7.35, [8], where we denote by π: $M\to\pi(M)$ the orbit map, and by $N(H)$ the normalizer of H in G. The G-manifold M is G-equivariantly diffeomorphic to $G/H\times_{\Gamma(H)}P(H)$, 2.4, [2], 1.7.35, [8]. G and $P(H)$ are $N(H)$-manifolds, and H acts trivially on $P(H)$, then we have a G-equivariant diffeomorphism $G/H\times_{\Gamma(H)}P(H)=G\times_{N(H)}P(H)$.

Throughout § 1, § 2 and § 3 we denote by $\widehat{Vect}_G(M)$ the set of real or complex G-vector bundles over M, and by $Vect_G(M)$ the semi-group of G-equivalence classes of them. Let $\pi_*^{(1)}$, $(\pi_*^{(1)})^-$ be the restriction and the G-extension,

$$\pi_*^{(1)} : \widehat{Vect}_G(G\times_{N(H)}P(H)) \to \widehat{Vect}_{N(H)}(P(H))\,,$$
$$(\pi_*^{(1)})^- : \widehat{Vect}_{N(H)}(P(H)) \to \widehat{Vect}_G(G\times_{N(H)}P(H))\,,$$

then we have the isomorphism

$$(1)\qquad \pi_*^{(1)} : Vect_G(G\times_{N(H)}P(H)) \xrightarrow{\approx} Vect_{N(H)}(P(H))\,,$$

and $\pi_*^{(1)}\cdot(\pi_*^{(1)})^-$ is the identity of $\widehat{Vect}_{N(H)}(P(H))$.

Proof of (1).

Let $E\to M$ be a G-vector bundle. By the G-equivalence $M\cong G\times_{N(H)}P(H)$, we have the restriction $E_0\equiv E/P(H)\to P(H)$, which is an $N(H)$-vector bundle. Define a G-homomorphism of G-vector bundles $\alpha : G\times_{N(H)}E_0\to E$ by $\alpha(g, e_0)=g\cdot e_0$ and a homeomorphism $\beta : G\times E\to G\times E$ by $\beta(g, e_0)=(g, g^{-1}e_0)$. Let $\hat\beta$: $G\times E\to G\times_{N(H)}E$ be the composition of β with the natural projection, and p_2 : $G\times E\to E$ be the projection onto the second factor. For each $e_0\in E$, there exists $g\in G$ with $g^{-1}e_0\in E_0$, and so $\hat\beta(g, e_0)=(g, g^{-1}e_0)\in G\times_{N(H)}E_0$. For any $g'\in G$ with $g'^{-1}e_0\in E_0$, we have

$$H=G_{\pi(g^{-1}e_0)}=g^{-1}G_{\pi(e_0)}g,\ H=G_{\pi(g'^{-1}e_0)}=g'^{-1}G_{\pi(e_0)}g'\,,$$

and so $gHg^{-1}=g'Hg'^{-1}$, then $g'^{-1}g\in N(H)$ and $(g, g^{-1}e_0)=(g', g'^{-1}e_0)$ in $G\times_{N(H)}E_0$. If $g^{-1}e_0\in E_0$, then $(g_1g)^{-1}g_1e_0=g^{-1}e_0$. Thus we have a G-homomorphism

$$\tilde\beta : E \xleftarrow{p_2} \hat\beta^{-1}(G\times_{N(H)}E_0) \xrightarrow{\hat\beta} G\times_{N(H)}E_0\,.$$

By the equalities

$$\tilde\beta\alpha(g, e_0)=\tilde\beta(ge_0)=\hat\beta(g, ge_0)=(g, e_0)\,,$$
$$\alpha\tilde\beta(e_0)=\alpha\hat\beta(g, e_0)=\alpha(g, g^{-1}e_0)=e_0\,,$$

α is a G-isomorphism. Thus (1) is proved.

Now we consider the case which satisfies the condition

$$(C_1)\qquad N(H)=\Gamma(H)\times H\,.$$

For any subgroup L of G, we have $N(gLg^{-1})=gN(L)g^{-1}$, and so, if L satisfies the condition (C_1), then gLg^{-1} also does and (C_1) is satisfied for all $L_1\in(L)$.

Let $E\to P(H)$ be an $N(H)$-vector bundle over $P(H)$. By (C_1) we have an H-vector bundle $E/\Gamma(H)\to P(H)/\Gamma(H)=\pi(M)$. On the other hand, for a given H-vector bundle $E'\to P(H)/\Gamma(H)=\pi(M)$, take the vector bundle induced by the orbit map $\pi\,|\,P(H) : P(H)\to\pi(P(H))=\pi(M)$, and denote it by $P(H)\times_{\pi(M)}E'\to P(H)$. We define an $N(H)$-action on $P(H)\times_{\pi(M)}E'$ as follows : for any $(\gamma, h)\in N(H)$, and $(x, e')\in P(H)\times_{\pi(M)}E'$, $(\gamma, h)\cdot(x, e')=(\gamma x, he')$. Then the bundle $P(H)\times_{\pi(M)}E'\to P(H)$ has an $N(H)$-vector bundle structure. Let $\pi_*^{(2)}$, $(\pi_*^{(2)})^-$ be the factorization by $\Gamma(H)$ and the induced bundle construction,

$$\pi_*^{(2)} : \widehat{Vect}_{N(H)}(P(H)) \to \widehat{Vect}_H(P(H)/\Gamma(H)) ,$$
$$(\pi_*^{(2)})^- : \widehat{Vect}_H(P(H)/\Gamma(H)) \to \widehat{Vect}_{N(H)}(P(H)) ,$$

then we have the isomorphism

$$(2) \qquad Vect_{N(H)}(P(H)) \to Vect_H(P(H)/\Gamma(H)) ,$$

and$\pi_*^{(2)}\cdot(\pi_*^{(2)})^-$ is the identity of $Vect_H(P(H)/\Gamma(H))$. Denote. $\pi_*^{(2)}\cdot\pi_*^{(1)}$ by π_*, and $(\pi_*^{(1)})^-\cdot(\pi_*^{(2)})^-$ by π_*^-. By (1), (2) we have

Theorem 1. (K. Jänich, 1.4, [6], G. Segal, Proposition 2.1, [9]) *Under the condition (C_1), we have isomorphisms*

$$\pi_* : Vect_G(M)\cong Vect_H(\pi(M)),\ K_G(M)\cong K_H(\pi(M)) ,$$

and $\pi_*\cdot\pi_*^-$ is the identity of $Vect_H(P(H)/\Gamma(H))$.

2. Special G-manifolds with restricted type

For a G-manifold M, we can choose a G-invariant Riemannian metric on M. We denote by V_x the fiber over $x\in M$ of the normal bundle of the imbedding $G(x)\subset M$. A G-manifold M is called *special*, if for any $x\in M$, and for the slice representation $G_x\to GL(V_x)$, V_x is a direct sum of G_x-invariant subspaces, $V_x=W_x\oplus F_x$, such that the representation of G_x on the unit sphere in W_x is transitive, and on F_x is trivial.

In this paper we treat special G-manifolds which have the principal orbit type (H) and the singular orbit type (K). Further we assume that the orbit space $\pi(M_{(K)})$ is connected, where $M_{(K)}$ denote the set $\{x\in M;\ G_x$ is conjugate to $K\}$. $M_{(K)}$ is a closed submanifold of M. Let N be an invariant tublar neighborhood of $M_{(K)}$ of the imbedding $M_{(K)}\subset M$, and M_1 be the complement of the interior of N, i.e. $M_1=M-\text{Int } N$. Then we have a G-invariant decomposition $M=M_{(H)}\cup M_{(K)}=M_1\cup N$. Define $\rho : \partial N\times[0, 1]\to N\subset M$ by

$\rho\,|\,\partial N\times(0)=$the projection of the sphere bundle $p : \partial N\to M_{(K)}$,

$$\rho(x, t) = \mathrm{Exp}(tx) \text{ on } \partial N \times (0, 1],$$

where we identify N with a normal disc bundle, then by the speciality of M, we obtain a diffeomorphism $f: \pi(M_{(K)}) \times [0, 1] \to \pi(N)$ such that the following diagram is commutative

$$\begin{array}{ccc} \partial N \times [0, 1] & \xrightarrow{\rho} & N \\ \pi \cdot p \times \mathrm{id.} \downarrow & & \downarrow \pi \\ \pi(M_{(K)}) \times [0, 1] & \xrightarrow{f} & \pi(N), \end{array}$$ 3.0, [5], lemma p 16, [2].

Since the projection p is equivariant, it induces a smooth map $p': \pi(\partial N) \to \pi(M_{(K)})$ with $p' \cdot \pi = \pi \cdot p$. $\rho | N \times (1) =$ the identity of ∂N, then we have $p' = (f | (\partial N))^{-1}$, and it is a diffeomorphism.

For a fixed principal isotropy subgroup H and for each $y' \in \pi(M_{(K)})$, there exists $y \in \pi^{-1}(y')$ such that the slice S_y admits $x \in \partial S_y$ with $(G_y)_x = G_x = H$, $p(x) = y$. Let K be the isotropy subgroup G_y. We denote by $r^*: Vect_K(\pi(M_{(K)}) \to Vect_H(\pi(M_{(K)})$, the semigroup homomorphism induced by the inclusion $H \subset K$.

Now we cosider the case which satisfies the condition

$$(C_2)\ N(H) = H \times \Gamma(H),\ N(K) = K \times \Gamma(K), \text{ and } \Gamma(K) \subset \Gamma(H) \subset G.$$

Lemma. *The following diagram is commutative*

$$\begin{array}{ccccc} Vect_G(\partial N) & & \xleftarrow{p^*} & & Vect_G(M_{(K)}) \\ \uparrow \pi_*^- & & & & \uparrow \pi_*^- \\ Vect_H(\pi(\partial M)) & \xleftarrow{p'^*} & Vect_H(\pi(M_{(K)})) & \xleftarrow{r^*} & Vect_K(\pi(M_{(K)})). \end{array}$$

Proof of the lemma is divided into three parts.

(i) Commutativity on a fiber

The spaces $P(K) = \{y \in M_{(K)};\ G_y = K\}$ and $\partial P(H) = \{x \in \partial N;\ G_x = H\}$ are the total spaces of the principal bundles over $\pi(M_{(K)})$ and $\pi(\partial N)$ with left $\Gamma(K)$, $\Gamma(H)$-actions respectively. For a given K-vector bundle (1) $F' \to \pi(M_{(K)})$, (2) $P(K) \times_{\pi(M_{(K)})} F' \to P(K)$ is the induced bundle by the projection $\pi | P(K): P(K) \to \pi(M_{(K)})$, then $(\pi_*^- F')$ is the G-vector bundle

$$(3) \qquad G \times_{N(K)} (P(K) \times_{\pi(M_{(K)})} F') \to G \times_{N(K)} P(K) = M_{(K)},$$

and the induced bundle of (3) by p is the G-vector bundle

$$(4) \qquad [G \times_{N(H)} \partial P(H)] \times_{M_{(K)}} [G \times_{N(K)} (P(K) \times_{\pi(M_{(K)})} F')] \to \partial N.$$

The G-action in the total space of (4) is the diagonal G-action. Now we restrict the bundle (4) on $\partial P(H)$ then we have an $N(H)$-vector bundle

$$(5)\qquad \partial P(H)\times_{M_{(K)}}[G\times_{N(K)}(P(K)\times_{\pi(M_{(K)})}F')]\to\partial P(H)$$

with the diagonal $N(H)$-action. We have choosed a pair (x,y) such that $G_x=H$, $G_y=K$ and $p(x)=y$. Let $\pi(x)=b$, then $\pi(y)=\pi(p(x))=p'\pi(x)=p'(b)$. Now we restrict (5) on $\Gamma(H)x$. For $\gamma\in\Gamma(H)$, $p(\gamma x)=\gamma p(x)=\gamma y$, and so for $g\in G$, if $gy=\gamma y$ then $\gamma^{-1}g\in K$, thus $g\in\Gamma(H)\cdot K$ and $\gamma\equiv g \bmod K$. Hence the bundle (5) over $\Gamma(H)x$ is

$$(6)\qquad \Gamma(H)\{x\times K\times_K(y\times F'_{p'(b)})\}\to\Gamma(H)x.$$

On the other hand the G-vector bundle $\pi_*^-p'^*r^*F'$ is

$$(7)\qquad G\times_{N(H)}[\partial P(H)\times_{\pi(\partial N)}(p'^*r^*F')]\to G\times_{N(H)}\partial P(H)\,.$$

The restriction of (7) on $\Gamma(H)x$ is

$$(8)\qquad \Gamma(H)x\times p'^*r^*F'_{p'(b)}\to\Gamma(H)x\,.$$

(6) is H-equivariantly isomorphic to (8) by

$$\Psi(\Gamma(H)x):\gamma(x\times k\times_K(y\times f))\to(\gamma x\times kf)\,,$$

where $\gamma\in\Gamma(H)$, $k\in K$, $f\in F'_{p'(b)}$ and its inverse is given by $(\gamma x\times kf)\to\gamma(x\times e\times_K(y\times kf))=\gamma(x\times k\times_K(y\times f))$, e denotes the unit of G.

(ii) Commutativity over a neighborhood of b

Let ε be the radius of a fiber of the sphere bundle $\partial N\to M_{(K)}$, then we use the tublar neighborhood $N_1\to M_{(K)}$ with radius $\varepsilon/2$ instead of N if it is necessary. The fiber N_y over y is included in a slice and there exists $x\in\partial N_y$ such as $G_x=H$ and $p(x)=y$. For any $y_0\in S_y\cap P(K)$, $G_{y_0}=G_y=K$. Take the slice S_{y_0} at y_0 with radius ε, then $S_{y_0}\supset N_y$ and for any $x_1'\in(\overline{xy_0}-\{y_0\})$, $G_{x_1'}=G_x=H$. Thus $\{$the half line through $p(x_1')$ and $x_1'\}\cap\partial N=x_1$ has the isotropy subgroup $G_{x_1}=G_{x_1'}=H$, and $G_{p(x_1)}=K$. Hence we have local cross sections $S_y\cap P(K)\supset s_K^{(p'(b))}(p'(U(b)))$ of the bundle $P(K)\to\pi(M_{(K)})$ and $s_H^{(b)}(U(b))$ of $\partial P(H)\to\pi(\partial N)$ such that the diagram

$$\begin{tikzcd} s_H^{(b)}(U(b)) \arrow[r, "p"] & s_K^{(p'(b))}(p'(U(b))) \\ U(b) \arrow[u, "s_H^{(b)}"] \arrow[r, "p'"] & p'(U(b)) \arrow[u, "s_K^{(p'(b))}"'] \end{tikzcd}$$

is commutative. We can suppose that the bundle F' is trivial over $p'(U(b))$. By using $\Psi(\Gamma(H)x)$ in (i) and the product representation $F'|p'(U(b))=F'_{p'(b)}\times p'(U(b))$ as a K-vector bundle over $p'(U(b))$, we construct an isomorphism of $N(H)$-vector bundles over $\Gamma(H)s_H^{(b)}(U(b))$ of

(9) $\Gamma(H)\{s_H^{(b)}(U(b))\times K\times_K(s_K^{p'(b)}(p'(U(b))\times F'_{p'(b)}\times p'(U(b))\}\to\Gamma(H)s_H^{(b)}(U(b))$

onto

$$\Gamma(H)\{s_H^{(b)}(U(b))\times r^*F'_{p'(b)}\times U(b)\}\to\Gamma(H)s_H^{(b)}(U(b))\,, \tag{10}$$

which is given by

$$\Psi(\Gamma(H)s_H^{(b)}(U(b))):\gamma\{x\times k\times_K(y\times f\times p'(b_1))\}\to\gamma x\times kf\times b_1\,,$$

where $x\in s_H^{(b)}(U(b))$, $y=p(x)$, $f\in F'_{p'(b)}$, $k\in K$ and $b_1\in U(b)$.

(iii) Commutativity over ∂N

Since $\pi(M_{(K)})$ is compact connected, by the construction in (ii), we can choose an open covering of $\pi(\partial N)=\pi(M_{(K)})$, $\bigcup_{i=1}^{l} U_i=\pi(\partial N)$ which admit local cross sections $s_K^{(i)}:p'(U_i)\to P(K)$, $s_H^{(i)}:U_i\to\partial P(H)$ with $p\cdot s_H^{(i)}=s_K^{(i)}\cdot p'$. Further we can assume that $F'|p'(U_i)$ is product for each i. Now we construct isomorphisms $\Psi(\Gamma(H)s_H^{(i)})$ of $N(H)$-vector bundles as in (ii). If $b\in U_i\cap U_j$, then there exists $\gamma(b)\in\Gamma(K)$ such as $s_K^{(j)}(p'(b))=\gamma(b)s_K^{(i)}(p'(b))$. On the other hand $s_H^{(j)}(b)=\gamma'(b)s_H^{(i)}(b)$ for some $\gamma'(b)\in\Gamma(H)$, then $\gamma'(b)^{-1}\gamma(b)\in K$ and so $\gamma'(b)=\gamma(b)k$ for some $k\in K\cap\Gamma(H)$, or equivarently $\gamma(b)=\gamma'(b)k^{-1}$. Then $\Psi(\Gamma(H)s_H^{(i)})$ coincides with $\Psi(\Gamma(H)s_H^{(j)})$ over $\Gamma(H)s_H^{(i)}(U_i\cap U_j)=\Gamma(H)s_H^{(j)}(U_i\cap U_j)$ by the definition of $\Psi(\Gamma(H)s_H)$ in (ii). Since $\Gamma(H)s_H^{(i)}(U_i)$ and $\Gamma(K)s_K^{(i)}(p'(U_i))$ are open in $\partial P(H)$ and $P(K)$ respectively, we can paste the family $\Psi(\Gamma(H)s_H^{(i)})$ $i=1,\cdots,l$ to get an isomorphism of $N(H)$-vector bundles over $\partial P(H)$ of

$$\partial P(H)\times_{M_{(K)}}[G\times_{N(K)}(P(K)\times_{\pi(M_{(K)})}F')]\to\partial P(H) \tag{11}$$

onto

$$\partial P(H)\times_{\pi(\partial N)}(p'^*r^*F')\to\partial P(H)\,. \tag{12}$$

We denote the isomorphism by $\Psi(\partial P(H))$. By the first step of the proof of Theorem 1 in § 1, we have the isomorphism $1_G\times_{N(H)}\Psi(\partial P(H))$ of

$$[G\times_{N(H)}\partial P(H)]\times_{M_{(K)}}[G\times_{N(K)}(P(K)\times_{\pi(M_{(K)})}F')]\to\partial N \tag{13}$$

onto

$$G\times_{N(H)}[\partial P(H)\times_{\pi(\partial N)}p'^*r^*F']\to\partial N\,. \tag{14}$$

We denote the required isomorphism by Ψ_G.

Notational conventions. Let M be a G-manifold with one orbit type (H) and the property (C_1), and $\varphi:E\to\bar{E}$ be a G-isomorphism of G-vector bundles over M, then φ induces the H-isomorphism $\varphi':\pi_*E\to\pi_*\bar{E}$, we denote it by

$\pi_*(\varphi)$. On the other hand, for a given H-isomorphism $\varphi' : E' \to \bar{E}'$ of H-vector bundles over $\pi(M)$, the induced G-isomorphism $\pi_*^- E' \to \pi_*^- \bar{E}'$ is denoted by $\pi_*^-(\varphi')$. Suppose $f : N \to M$ to be a G-map of G-manifolds, then the above $\varphi : E \to \bar{E}$ induces the G-isomorphism $f^*E \to f^*\bar{E}$, we denote it by $f^*(\varphi)$. The G-isomorphism due to G. Segal, $E \to \pi_*^- \pi_* E$, is denoted by $\pi_*^- \pi_*$, (§ 1 of this paper, § 2, [9]).

3. A classification theorem

We consider a family $D = \{(F', E_1') \in \widehat{Vect}_K(\pi(M_{(K)})) \times \widehat{Vect}_H(\pi(M_1)), \alpha_H\}$, where we use notations in § 2 and α_H is an isomorphism of H-vector bundles $p'^*r^*F' \to E_1' \mid \partial\pi(M_1)$, say $\partial E_1'$. We call each element of D a datum.

DEFINITION 1. A datum (F', E_1', α_H) is *equivalent* to a datum $(\bar{F}', \bar{E}_1', \bar{\alpha}_H)$ if and only if there exist isomorphisms ρ_K of K-vector bundles and φ_H of H-vector bundles such that the diagram

$$\begin{array}{ccccccc}
F' & \xrightarrow{p'^*r^*} & p'^*r^*F' & \xrightarrow{\alpha_H} & \partial E_1' & \subset & E_1' \\
\downarrow{\scriptstyle \rho_K} & & \downarrow{\scriptstyle \rho_{H,K}} & & \downarrow{\scriptstyle \partial\varphi_H} & & \downarrow{\scriptstyle \varphi_H} \\
\bar{E}' & \xrightarrow{p'^*r^*} & p'^*r^*\bar{F}' & \xrightarrow{\bar{\alpha}_H} & \partial\bar{E}_1' & \subset & \bar{E}_1'
\end{array}$$

is commutative, where $\rho_{H,K}$ is the isomorphism ρ_K as an H-vector bundle isomorphism.

The relation in the definition is an equivalence relation.

Proposition 1. *For two data* (F', E_1', α_H), $(F', E_1', \bar{\alpha}_H)$ if α_H is homotopic *to* $\bar{\alpha}_H{}'$ *by a homotopy* $\{h_t\,; 0 \leqq t \leqq 1\}$ *such that* h_t *is an* H*-isomorphism for each* t*, then the data are equivalent each other.*

Proof. We choose a coloring $\partial\pi(M_1) \times I \subset \pi(M_1)$. Since $h_0 \cdot h_0^{-1} =$ the identity of $\partial E_1'$, the homotopy $h_{1-t} \cdot h_0^{-1} : E_1' \mid \partial\pi(M_1) \times I \to E_1' \mid \partial\pi(M_1) \times I$ can be extended to an H-automorphism $\varphi_H : E_1' \to E_1'$ such that the diagram

$$\begin{array}{ccccc}
 & \overset{\alpha_H}{\nearrow} \partial E_1' & \subset & E_1' \\
p'^*r^*F' & & \downarrow{\scriptstyle \bar{\alpha}_H \cdot \alpha_H^{-1}} & & \downarrow{\scriptstyle \varphi_H} \\
 & \underset{\bar{\alpha}_H}{\searrow} \partial E_1' & \subset & E_1'
\end{array}$$

is commutative.

REMARK. The isomorphism α_H of a datum (F', E_1', α_H) determines a canonical G-isomorphism between G-vector bundles $p^*\pi_*^- F'$ and $\pi_*^- \partial E_1'$. In fact, by the lemma in § 2, we have the G-isomorphism $\Psi_G : p^*\pi_*^- F' \to \pi_*^- p'^*r^*F'$. Using α_H, we have a G-isomorphism $1_G \times_{N(H)} (1_{\partial P(H)} \times_{\pi(\partial N)} \alpha_H) : \pi_*^- p'^*r^*F' \to$

$\pi_*^-(\partial E_1')$, i.e. $\pi_*^-(\alpha_H)$. Let $\Phi_G(\alpha_H)$ be the composition $\pi_*^-(\alpha_H)\cdot\Psi_G$, which we call the canonical G-isomorphism.

Using the deformation along geodesics which are perpendicular to $M_{(K)}$, we have the equivariat deformation retract $\tilde{p}: N\to M_{(K)}$ with $\tilde{p}|\partial N=p$. Precisely $\tilde{p}$ is defined to be $\tilde{p}\cdot\rho(x, t)=p(x)$ over $\rho(\partial N\times(0.1])$ and $\tilde{p}(x)=x$ for $x\in M_{(K)}$, where ρ has been used in § 2.

Proposition 2. *If a datum (F', E_1',α_H) is equivalent to a datum $(\bar{F}', \bar{E}_1', \bar{\alpha}_H)$, then $\tilde{p}^*\pi_*^-F'\cup_{\Phi_G(\alpha_H)}\pi_*^-E_1'$ is G-isomorphic to $\tilde{p}^*\pi_*^-\bar{F}'\cup_{\Phi_G(\bar{\alpha}_H)}\pi_*^-\bar{E}_1'$, where we denote by $\cup_{\Phi_G}$ the clutching construction.*

Proof. From the equivalence

$$\begin{array}{ccccccc}
F' & \longrightarrow & p'^*r^*F' & \xrightarrow{\alpha_H} & \partial E_1' & \subset & E_1' \\
\downarrow{\scriptstyle \rho_K} & & \downarrow{\scriptstyle \rho_{H,K}} & & \downarrow{\scriptstyle \partial\varphi_H} & & \downarrow{\scriptstyle \varphi_H} \\
\bar{F}' & \longrightarrow & p'^*r^*\bar{F}' & \xrightarrow{\bar{\alpha}_H} & \partial\bar{E}_1' & \subset & \bar{E}_1',
\end{array}$$

we have the commutative diagram

$$\begin{array}{ccccccccc}
\tilde{p}^*\pi_*^-F' & \supset & p^*\pi_*^-F' & \xrightarrow{\Psi_G} & \pi_*^-(p'^*r^*F' & \xrightarrow{\pi_*^-(\alpha_H)} & \pi_*^-\partial E_1' & \subset & \pi_*^-E_1' \\
\downarrow{\scriptstyle 1_N\times\pi_*^-(\rho_K)} & & \downarrow{\scriptstyle 1_{\partial N}\times\pi_*^-(\rho_K)} & & \downarrow{\scriptstyle \pi_*^-(\rho_{H,K})} & & \downarrow{\scriptstyle \pi_*^-(\partial\varphi_H)} & & \downarrow{\scriptstyle \pi_*^-(\varphi_H)} \\
\tilde{p}^*\pi_*^-\bar{F}' & \supset & p^*\pi_*^-\bar{F}' & \xrightarrow{\Psi_G} & \pi_*^-(p'^*r^*\bar{F}') & \xrightarrow{\pi_*^-(\bar{\alpha}_H)} & \pi_*^-\partial\bar{E}_1' & \subset & \pi_*^-\bar{E}_1',
\end{array}$$

for the second square from the left, its commutativity is obtained from the commutative diagram

$$\begin{array}{ccc}
x\times k\times_K(y\times f\times p'(b_1)) & \xrightarrow{\Psi(\Gamma(H)s_H^{(b)}(U(b)))} & x\times kf\times b_1 \\
\downarrow{\scriptstyle 1_{P(H)}\times\pi_*^-(\rho_K)} & & \downarrow{\scriptstyle 1_{\partial P(H)}\times\pi_*^-(\rho_{H,K})} \\
x\times k\times_K(y\times\rho_K(f)\times p'(b_1)) & \xrightarrow{\Psi(\Gamma(H)s_H^{(b)}(U(b)))} & x\times\rho_{H,K}(k\cdot f)\times b_1,
\end{array}$$

c.f. (ii), the proof of the lemma, § 2. For other squares, the commutativities are resulted by the definition of π_*^-. Since each arrow is a G-isomorphism, we have the proposition.

For each G-vector bundle E over M, we use the notations, $E|M_1=E_1$, $\pi_*E_1=E_1'$, $E|\partial N=\partial E_1$, $E|M_{(K)}=F$, $\pi_*F=F'$.

Since N is a compact differentiable manifold, using $\tilde{p}$ and the covering homotopy theorem, we have a G-equivalence $p_G^*:\tilde{p}^*F\to E|N$, and we get a G-isomorphism $p_G^*\cup 1_{E_1}:\tilde{p}^*F\cup_{\partial p_G^*}E_1\to E$. Let $\bar{p}_G^*:\tilde{p}^*F\to E|N$ be another G-isomorphism. By the commutative diagram

$$\begin{array}{ccc}
\tilde{p}^*F\cup_{\partial p_G^*}E_1 & \xrightarrow{p_G^*\cup 1_{E_1}} & E \\
\Big\downarrow (\bar{p}_G^*)^{-1}\cdot p_G^*\cup 1_{E_1} & & \| \\
\tilde{p}^*F\cup_{\partial \bar{p}_G^*}E_1 & \xrightarrow{\bar{p}_G^*\cup 1_{E_1}} & E,
\end{array}$$

$\tilde{p}^*F\cup_{\partial p_G^*}E_1$ is G-isomorphic to $\tilde{p}^*F\cup_{\partial\bar{p}_G^*}E_1$.

G-isomorphisms ∂p_G^*, Ψ_G and $\pi_*^-\pi_*$ induce H-isomorphisms $\partial p_H^*=\pi_*(\partial p_G^*):\pi_*(p^*F)\to\partial E_1'$, $\Psi_H=\pi_*(\Psi_G):\pi_*(p^*\pi_*^-F')\to p'^*r^*F'$ and $q=\pi_*(p^*([\pi_*^-\pi_*]^{-1})):\pi_*(p^*\pi_*^-F')\to\pi_*(p^*F)$ respectively. To the bundle $p^*F\cup_{\partial p_G^*}E_1$, we make to correspond a datum $(F', E_1', \partial p_H^*\cdot q\cdot\Psi_H^{-1})$. By the next proposition the correspondence is independent of the choice of p_G^*.

Proposition 3. *If a G-vector bundle E is G-isomorphic to a G-vector bundle $\bar{E}$, then the resulting data are equivalent.*

Proof. Let $\varphi_G:E\to\bar{E}$ be a G-isomorphism. Choose representations $\tilde{p}^*F\cup_{\partial p_G^*}E_1\to E$, $\tilde{p}^*\bar{F}\cup_{\partial\bar{p}_G^*}\bar{E}_1\to\bar{E}$. Let $\tilde{\varphi}_G$ be $(\bar{p}_G^*)^{-1}(\varphi_G|N)(p_G^*):\tilde{p}^*F\to\tilde{p}^*\bar{F}$. Since $\tilde{\varphi}_G$ is resulted from $\tilde{\varphi}_G|M_{(K)}:F\to\bar{F}$, we have commutative diagrams

$$\begin{array}{ccccccccc}
\pi_*^-(p'^*r^*F') & \xleftarrow{\Psi_G} & p^*\pi_*^-F' & \xrightarrow{p^*[(\pi_*^-\cdot\pi_*)^{-1}]} & p^*F & \xrightarrow{\partial p_G^*} & \partial E_1 & \subset & E_1 \\
\Big\downarrow \pi_*^-(p'^*r^*\pi_*(\tilde{\varphi}_G|\partial N)) & & \Big\downarrow p^*\pi_*^-\pi_*(\tilde{\varphi}_G|\partial N) & & \Big\downarrow \tilde{\varphi}_G|\partial N & & \Big\downarrow \varphi_G|\partial N & & \Big\downarrow \varphi_G|M_1 \\
\pi_*^-(p'^*r^*\bar{F}') & \xleftarrow{\Psi_G} & p^*\pi_*^-\bar{F}' & \xrightarrow{p^*[(\pi_*^-\pi_*)^{-1}]} & p^*\bar{F} & \xrightarrow{\partial\bar{p}_G^*} & \partial\bar{E}_1 & \subset & \bar{E}_1
\end{array}$$

and

$$\begin{array}{ccccccc}
F' & \longrightarrow & p'^*r^*F' & \xrightarrow{\partial p_H^*\cdot q\cdot\Psi_H^{-1}} & \partial E_1' & \subset & E_1' \\
\Big\downarrow \rho_K=\pi_*(\tilde{\varphi}_G|\partial N)) & & \Big\downarrow \rho_{H,K} & & \Big\downarrow \pi_*(\varphi_G|\partial N) & & \Big\downarrow \pi_*(\varphi_G|M_1) \\
\bar{F}' & \longrightarrow & p'^*r^*\bar{F}' & \xrightarrow{\partial p_H^*\cdot q\cdot\Psi_H^{-1}} & \partial\bar{E}_1' & \subset & \bar{E}_1'.
\end{array}$$

The equivalence classes of elements of D has a semi group structure by the Whitney sum, we denote it by $D_{H,K}(M)$. By Proposition 3 we get a homomorphism $S:Vect_G(M)\to D_{H,K}(M)$ which is defined to be $S(E)=(F', E_1', \partial p_H^*\cdot q\cdot\Psi_H^{-1})$ for a representation $\tilde{p}^*F\cup_{\partial p_G^*}E_1$ up to G-isomorphisms. We define $\hat{T}:D\to\widehat{Vect}_G(M)$ by $\hat{T}(F', E_1', \alpha_H)=\tilde{p}^*\pi_*^-F'\cup_{\Phi_G(\alpha_H)}\pi_*^-E_1'$, then by Proposition 2, it induces a homomorphism $T:D_{H,K}(M)\to Vect_G(M)$. Now we are in a position to prove our main theorem.

Theorem 2. *The homomorphism $S:Vect_G(M)\to D_{H,K}(M)$ is an isomorphism of semi groups and T is its inverse.*

Proof. For $E \in \widehat{Vect_G}(M)$ we choose a representation $\tilde{p}^*F \cup_{\partial p_G^*} E_1 = E$ and take the datum $(F', E_1', \partial p_H^* \cdot q \cdot \Psi_H^{-1})$. We consider the following diagram,

$$
\begin{tikzcd}
\pi_*^{-}\pi_*(p^*F) \arrow[rr, "\pi_*^{-}\pi_*(p^*(\pi_*^{-}\pi_*))"] & & \pi_*^{-}\pi_*(p^*\pi_*^{-}F') \arrow[r, "\pi_*^{-}(q)"] \arrow[d, "\pi_*^{-}(\Psi_H)"] & \pi_*^{-}\pi_*(p^*F) \arrow[d, "\pi_*^{-}(\partial p_H^*)"] \\
& p^*\pi_*^{-}F' \arrow[r, "\Psi_G"] \arrow[ur, "\pi_*^{-}\pi_*"] & \pi_*^{-}(p'^*r^*F') \arrow[r, "\pi_*^{-}(\partial p_H^* \cdot q \cdot \Psi_H^{-1})"] & \pi_*^{-}(\partial E_1') \\
& p^*F \arrow[u, "p^*(\pi_*^{-}\pi_*)"] \arrow[uul, "\pi_*^{-}\pi_*"] \arrow[rr, "\partial p_G^*"] & & \partial E_1 . \arrow[u, "\pi_*^{-}\pi_*"']
\end{tikzcd}
$$

In order to get the commutativity of the lower square, we use commutativities of other parts, and we have

$$
\begin{aligned}
&\pi_*^{-}(\partial p_H^* \cdot q \cdot \Psi_H^{-1}) \cdot \Psi_G \cdot p^*(\pi_*^{-}\pi_*) = \pi_*^{-}(\partial p_H^*) \cdot \pi_*^{-}(q) \cdot \pi_*^{-}(\Psi_H^{-1}) \cdot \Psi_G \cdot p^*(\pi_*^{-}\pi_*) \\
&= \pi_*^{-}(\partial p_H^*) \cdot \pi_*^{-}(q) \cdot \pi_*^{-}\pi_* \cdot p^*(\pi_*^{-}\pi_*) \\
&= \pi_*^{-}\pi_*(\partial p_G^*) \cdot \pi_*^{-}\pi_*(p^*[(\pi_*^{-}\pi_*)^{-1}]) \cdot \pi_*^{-}\pi_*(p^*(\pi_*^{-}\pi_*)) \cdot \pi_*^{-}\pi_* \\
&= \pi_*^{-}\pi_*(\partial p_G^*) \cdot \pi_*^{-}\pi_* = \pi_*^{-}\pi_* \cdot \partial p_G^* ,
\end{aligned}
$$

and G-isomorphisms,

$$E \cong p^*F \cup_{\partial p_G^*} E_1 \cong p^*\pi_*^{-}F' \cup_{\Phi(\partial p_H^* \cdot q \cdot \Psi_H^{-1})} \pi_*^{-}E_1' ,$$

where $\Phi(\partial p_H^* \cdot q \cdot \Psi_H^{-1}) = \pi_*^{-}(\partial p_H^* \cdot q \cdot \Psi_H^{-1}) \cdot \Psi_G$, (Remark after Proposition 1).

Let $[E]$ be the equivalence class which contains E, then we have $T \cdot S([E]) = [E]$ by the above equalities and Propositions 2,3. Let (F', E_1', α_H) be a datum, then $T(F', E_1', \alpha_H) = p^*\pi_*^{-}F' \cup_{\Phi(\alpha_H)} \pi_*^{-}E_1'$. Since $\Phi(\alpha_H)\Psi_G^{-1} = \pi_*^{-}(\alpha_H)$ and $\pi_*\pi_*^{-} =$ the identity, (F', E_1', α_H) is a datum of this representation. Thus we have proved that $S \cdot T =$ the identity of $D_{H,K}(M)$.

4. $K_{O(n)}(W^{2n-1}(d))$, $n \geqq 2$.

Brieskorn-Hirzebruch $O(n)$-manifold $W^{2n-1}(d)$ is the loci of equations $z_0^d + z_1^2 + \cdots + z_n^2 = 0$, $|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 2$. By 4.5 of [2], the manifold is a special $O(n)$-manifold with the orbit type $(O(n-2), O(n-1))$, and the orbit space is D^2, the 2-disc, and $\partial D^2 = S^1 = \pi(W^{2n-1}(d)_{(O(n-1))})$.

In this section, we consider complex vector bundles, then any vector bundle is orientable. Since the boundary S^1 is a trivial $O(n-1)$-manifold, any vector bundle over S^1 is equivalent to a product $O(n-1)$-vector bundle, and we have an isomorphism $Vect_{O(n-1)}(S^1) \cong \widehat{O}(n-1)$, where $\widehat{O}(n-1)$ is the semi group of isomorphism classes of complex $O(n-1)$-modules, (Prop. 2.2, [9]).

Let K be the Grothendieck functor, then by Theorem 2 in § 3, we have

$$K(D_{O(n-2),O(n-1)}) \cong K(Vect_{O(n)}(W^{2n-1}(d))) = K_{O(n)}(W^{2n-1}(d)),$$
$$K_{O(n-1)}(S^1) = K(Vect_{O(n-1)}(S^1)) \cong K(O(n-1)) = R(O(n-1)),$$

where $R(G)$ is the complex representation ring of G.

Using notations in § 3, we define a homomorphism of semi groups $j^*: D_{H,K} \to Vect_K(\pi(M_{(K)}))$ by $j^*(F', E_1', \alpha_H) = F'$.

In the case of $M = W^{2n-1}(d)$, $\pi(M_{1,(O(n-2))}) = D_\varepsilon^2$, where ε is the radius of the disc $\pi(M_{1,(O(n-2))})$. We define a homomorphism $k^*: Vect_{O(n-1)}(S^1) \to D_{O(n-2),O(n-1)}$ by $k^*(S^1 \times V) = (S^1 \times V, D_\varepsilon^2 \times r^*V, \alpha_{O(n-2)} = p' \times 1_{r^*V})$ for each $O(n-1)$-module V. Then we have $j^* \cdot k^* =$ the identity of $Vect_{O(n-1)}(S^1)$, and $K_{O(n-1)}(S^1) \cong R(O(n-1))$ is a direct summand of $K_{O(n)}(W^{2n-1}(d))$.

Now we prove our main result in this section.

Theorem 3.

$$K_{O(n)}(W^{2n-1}(d)) \cong R(O(n-1)).$$

Proof. At first we seek a linear form of a clutching function $\alpha_{O(n-2)}$. We can do this quite parallely to the proof of the Bott periodicity due to Atiyah-Bott, [4]. For any datum $(S^1 \times V, D_\varepsilon^2 \times r^*V, \alpha_{O(n-2)})$, the clutching function $\alpha_{O(n-2)}$ is equivariantly homotopic to a Laurent polynomial clutching function $\beta_{O(n-2)} = \sum_{|k| \leq l} a_k z^k$, 2.5, Proposition p. 130 [4], then $(S^1 \times V, D_\varepsilon^2 \times r^*V, \alpha_{O(n-2)})$ is equivalent to $(S^1 \times V, D_\varepsilon^2 \times r^*V, \beta_{O(n-2)})$ by the proposition 1, § 3. There exists a polynomial clutching function $p(z) = b_0 + b_1 z + \cdots + b_s z^s$ with $\beta_{O(n-2)} = p(z) z^{-s}$. By the diagram

$$\begin{tikzcd}
S^1 \times V \arrow[r, "p'^* r^*"] \arrow[d, "1_{S^1} \times z^{-s}"] & S_\varepsilon^1 \times r^*V \arrow[r, "p(z)z^{-s}"] \arrow[d, "1_{S^1} \times z^{-s}"] & S_\varepsilon^1 \times r^*V \arrow[d, equal] & \subset & D_\varepsilon^2 \times r^*V \arrow[d, equal] \\
S \times V \arrow[r, "p'^* r^*"] & S_\varepsilon^1 \times r^*V \arrow[r, "p(z)"] & S_\varepsilon^1 \times r^*V & \subset & D_\varepsilon^2 \times r^*V,
\end{tikzcd}$$

$(S^1 \times V, D_\varepsilon^2 \times r^*V, p(z)z^{-s})$ is equivalent to $(S^1 \times V, D_\varepsilon^2 \times r^*V, p(z))$. $p(z) + 1_{(1)} + \cdots + 1_{(s)}$ is equivariantly homotopic to a linear clutching function $az+b$, further to $a_+ z \oplus b_-$, and $(S^1 \times (s+l)V, D_\varepsilon^2 \times r^*(s+l)V, p(z) + 1_{(1)} + \cdots + 1_{(s)})$ is equivalent to $(S^1 \times (s+1)V, D_\varepsilon^2 \times \{(r^*(s+1)V)_+^0 \oplus (r^*(s+1)V)_-^0\}, a_+ z \oplus b_-)$, where $(r^*(s+1)V)_+^0$ and $(r^*(s+1)V)_-^0$ are $O(n-2)$-modules and a_+, b_- are $O(n-2)$-automorphisms, Proof of 3,2. p. 132, 4.6 p. 135, [4], (Since $az+b$ is $O(n-2)$-equivariant, then p_0, p_∞ are $O(n-2)$-equivariant and the decomposition $\operatorname{im} p_0 \oplus \ker p_0$ is $O(n-2)$-invariant).

By the corollary 2 (i) [7], $r^*: R(O(n-1)) \to R(O(n-2))$ is epimorphic, then for any $O(n-2)$-module L there exist $O(n-1)$-modules L_1, L_2 with $L = r^*L_2 - r^*L_1$, and so $L + r^*L_1 = r^*L_2$ in $R(O(n-1))$. Thus $L + r^*L_1 + L_3 = r^*L_2 + L_3$, where L_3 is a trivial $O(n-2)$-module and it can be considered as a trivial $O(n-1)$-module. Then we can choose $O(n-1)$-modules V_+, V_- with $(r^*(s+1)V)^0_\pm \oplus r^*V \in \operatorname{im} r^*$. Since $[a_+z\oplus z]\oplus[b_-\oplus 1_{V_-}]=\{[a_+\oplus 1_{V_+}]\oplus[b_-\oplus 1_{V_-}]\}\cdot\{[z]\oplus[1]\}$, adding the datum $(S^1\times(V_+\oplus V_-),\ D^2_\varepsilon\times(r^*V_+\oplus r^*V_-),\ z\oplus 1)\in \operatorname{im} k^*$ to the last one, the datum

$$(1)\quad (S^1\times\{(s+1)V\oplus V_+\oplus V_-\},\ D^2_\varepsilon\times\{[(r^*(s+1)V)^0_+\oplus r^*V_+]\oplus[r^*(s+1)V)^0_- \oplus r^*V_-]\},\ [a_+z\oplus z]\oplus[b_-\otimes 1_{V_-}])$$

is equivalent to

$$(2)\quad (S^1\times\{(s+1)V\oplus V_+\oplus V_-\},\ D^2_\varepsilon\times\{[(r^*(s+1)V)^0_+\oplus r^*V_+]\oplus[r^*(s+1)V)^0_- \oplus r^*V_-]\},\ [a_+\oplus 1_{V_+}]\oplus[b_-\oplus 1_{V_-}])\,.$$

The $O(n-2)$-automorphism $[a_+\oplus 1_{V_+}]\oplus[b_-\oplus 1_{V_-}]$ has the extension to an $O(n-2)$-automorphism of $D^2_\varepsilon\times\{[(r^*(s+1)V)^0_+\oplus r^*V_+]\oplus[(r^*(s+1)V)^0_-\oplus r^*V_-]\}$, thus the datum (2) is equivalent to

$$(3)\quad (S^1\times\{(s+1)V\oplus V_+\oplus V_-\},\ D^2_\varepsilon\times\{[(r^*(s+1)V)^0_+\oplus r^*V_+]\oplus[r^*(s+1)V)^0_- \oplus r^*V_-]\},\ \text{the identity}),$$

which belongs to im k^*. By the remark before the theorem 3, we have proved the theorem.

REMARK. S. Araki has obtained the theorem by using a Fáry type spectral sequence.

5. Invariant vector field on $W^{2n-1}(d)$, $n\geqq 2$

5.1 A Killing vector field on $W^{2n-1}(d)$

The manifold $W^{2n-1}(d)$ is an $SO(2)\times O(n)$-manifold. In fact for $A\in O(n)$, the action is defined by

$$A(z_0, z_1, \cdots, z_n) = (z_0, A(z_1, \cdots, z_n))\,.$$

On the other hand the 1-parameter group $\{\mathrm{Diag}\,(e^{2it}, e^{dit}, \cdots, e^{dit});\ 0\leqq t\leqq 2\pi\}\cong SO(2)$ acts by

$$\mathrm{Diag}\,(e^{2it}, e^{dit}, \cdots, e^{dit})(z_0, \cdots, z_n)=(e^{2it}z_0, e^{dit}z_1, \cdots, e^{dit}z_n)\,,$$

and the action is free for sufficiently small $|t|$. The actions of $SO(2)$ and $O(n)$ are commutative.

Choosing an $SO(2)\times O(n)$-invariant Riemannian metric on $W^{2n-1}(d)$, we have $SO(2)\times O(n)\subset I(W^{2n-1}(d))$, the group of isometries of $W^{2n-1}(d)$, and $SO(2)$ is an 1-parameter group of transformations. Define a vector field on $W^{2n-1}(d)$ by

$$(1) \qquad X_p f = \left.\frac{df(\varphi_t(p))}{dt}\right|_{t=0} \quad \text{for any } f \in C^\infty(U(p), R)\,,$$

where $U(p)$ is a neighborhood of a point p in $W^{2n-1}(d)$, and $\varphi_t = \mathrm{Diag}\,(e^{2it}, e^{dit}, \cdots, e^{dit})$, then by definition, X is a complete vector field on $W^{2n-1}(d)$.

The next proposition is well known in differential geometry.

Proposition 5. *Let X be a complete vector field on a Riemannian manifold M, then X is a Killing vector field if and only if Exp tX is an isometry of M for each $t \in R$.*

Thus the vector field X defined by (1) is a Killing vector field. Since φ_t acts freely for sufficiently small $|t|$, the vector field X has no singularity.

DEFINITION. Let G be a compact Lie group. A vector field X on a G-manifold M is called *G-invariant* if it satisfies the equality

$$(2) \qquad (dg)_p X_p = X_{gp} \qquad \text{for all } p \in M \text{ and } g \in G\,.$$

Let $\{\varphi_t : t \in R\}$ be an 1-parameter group of transformations of a G-manifold M, and suppose to be $g\varphi_t = \varphi_t g$ for all $g \in G$ and $t \in R$, then for any $f \in C^\infty(U(gp), R)$,

$$\{(dg)_p \times X_p\}(f) = X_p(f \cdot g) = \left.\frac{d(f \cdot g)(\varphi_t(p))}{dt}\right|_{t=0} = \left.\frac{df(\varphi_t(gp))}{dt}\right|_{t=0} = X_{gp} f\,,$$

then the condition (2) is satisfied, and the vector field X is G-invariant.
By these discussions, we have proved

Theorem 4. *There exists an $O(n)$-invariant Killing vector field without singularity on $W^{2n-1}(d)$.*

The next proposition is well known in the case without G-action.

Proposition 5. *A G-manifold M admits a G-invariant vector field without singularity if and only if the tangent bundle $T(M)$ of M has a G-invariant decomposition $T(M) = E \oplus \theta^1$, where E is a G-vector bundle and θ^1 is the product G-line bundle over M, and the decomposition is smooth.*

We can prove the proposition quite similarly to the case without G-action.

REMARK (1). Suppose n to be a positive odd integer and $n \geqq 3$, then $W^{2n-1}(2k+1)$ is diffeomorphic to S^{2n-1}, the standard sphere if $2k+1 \equiv +1 \bmod 8$, and to $\sum^{2n-1}$, the Kervaire sphere if $2k+1 \equiv +3 \bmod 8$, and $\sum^{2n-1}$ is not diffeomorphic to S^{2n-1} if $2k+1 \equiv +3 \bmod 8$ and $n+1$ is not a power of 2. (11.3, [2]).

REMARK (2) S^{4l+1}admits 1-field but not 2-field (27.11, [11]). Here we quote a theorem in [10]. Let $f: S^n \to \sum^n$ be an orientation preserving homotopy equivalence of the standard n-sphere S^n onto a homotopy sphere $\sum^n$, then we have an equivalence $f^*T(\sum^n) \approx T(S^n)$. Thus $\sum^{4l+1}$ admits 1-field but not 2-field.

5.2 Non existence of invariant 2-fields
Now we proved the following

Theorem 5. *For $n \geqq 2$, the $O(n)$-manifold $W^{2n-1}(d)$ admits an $O(n)$-invariant 1-field, but not $O(n)$-invariant 2-fields.*

Proof. The orbit map $\pi: W^{2n-1}(d) \to D^2$ is the projection $(z_0, z_1, \cdots, z_n) \to z_0$. Since $|z_0| \leqq 1$, $|z_0|^2 = |z_1|^2 = 1$ for $(z_0, z_1) \in P(O(n-1))$ and $z_0 = e^{2\pi it}$, $z_1 = \pm ie^{d\pi it}$ for $0 \leqq t \leqq 1$. Then

$$
\begin{aligned}
ie^{d\pi i(t+1)} &= -ie^{d\pi it} \text{ if } d \text{ is odd}\,, \\
&= ie^{d\pi it} \text{ if } d \text{ is even}\,.
\end{aligned}
$$

Thus

(3) $P(O(n-1)) = S^1$ if d is odd and the orbit map $P(O(n-1)) \to S^1$ is the non trivial covering,
$= S^1 \cup S^1$, the disjoint sum, if d is even and the orbit map is the trivial covering.

Let X be a vector field on a G-manifold M and generate the 1-parameter group of transformations $\{\varphi_t\}$. The next proposition is well known.

Proposition 6. *X is G-invariant if and only if $g\varphi_t = \varphi_t g$ for each $t \in R$ and $g \in G$.*

Proof. The if part has been proved in 5.1. Suppose X to be G-invariant. For any $f \in C^\infty(U(gp))$, $f \cdot g \in C^\infty(U(p))$, 5.1 for notations. By the equalities

$$
\begin{aligned}
(dgX_p)f &= X_p(f \cdot g) \\
&= \lim_{t \to 0} \frac{(f \cdot g \cdot \varphi_t - f \cdot g) g^{-1}(gp)}{t} \\
&= \lim_{t \to 0} \frac{(f \cdot g \cdot \varphi_t \cdot g^{-1} - f)(gp)}{t}
\end{aligned}
$$

dgX generates $g \cdot \varphi_t \cdot g^{-1}$. Since $dgX = X$, we have $g \cdot \varphi_t \cdot g^{-1} = \varphi_t$ by the uniqueness of the solution of ordinary differential equations.

Proposition 7. *Suppose M to be a G-manifold with non empty fixed point* *set F, and admits an invariant vector field X without singularity. Then the restriction $X|F$ is a tangent vector field on F.*

Proof. Suppose X to be a vector field on M and the restriction $X|F$ to be a non trivial, non tangential vector field on F, then X can not be G-invariant. For, if X is G-invariant and generates the 1-parameter group of transformations $\{\varphi_t\}$, then there exist $p_0 \in F$ and $t_0 \in R$ with $\varphi_{t_0} p_0 \notin F$ since X is not tangential to F. By Proposition 6 $g \cdot \varphi_{t_0} \cdot p_0 = \varphi_{t_0} \cdot g p_0 = \varphi_{t_0} p_0$ for any $g \in G$, then $\varphi_{t_0} p_0 \in F$ which is a contradiction.

Now we return to the proof of the theorem. If X is an $O(n)$-invariant, then it is $O(n-1)$-invariant. By (11) and Proposition 7, $W^{2n-1}(d)$ can not admit $O(n-1)$-invariant 2-fields. Thus we proved the theorem.

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