Convolution for the transform induced by Fourier integral transform and its inverse

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Abstract

The convolutions for $T = 3F + F^{-1}$ transforms are formulated, its properties and applications to solving integral equations are considered.

1 Introduction

The convolution for integral transforms were studied in the 19^{th} century, at first the convolutions for Fourier transform, for the Laplace transform, for the Mellin transform and after that for Hilbert transform [13], Hankel transform [12], Kontorovich- Lebedev transform and Stieltjes transform.
The convolution for the Fourier integral transform [10]

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x-y)g(y)dy, \tag{1}$$

for which satisfies the factorization equality

$$F(f * g)(y) = (Ff)(y)(Fg)(y), \qquad y \in \mathbb{R}$$

in which F is the Fourier transform [4]and is defined as follows

$$\tilde{f}(y) \equiv (Ff)(y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ixy} f(x)dx.$$

$$f(x) \equiv (F^{-1}\tilde{f})(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{ixy} \tilde{f}(y) dy.$$

In 1941, Churchill R. V. introduced the convolution of two functions f and g for the Fourier cosine transforms [3]

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} f(y)[g(|x-y|) + g(x+y)]dy, \quad x \in \mathbb{R}_+$$

for which satisfies the factorization equality

$$F_c(f * g)(y) = (F_c f)(y)(F_c g)(y), \qquad \forall y \in \mathbb{R}_+,$$

here F_c is the Fourier cosine transform.

Afterwards, in 1967, V. A. Kakichev proposed a constructive method for defining the convolution with a weight- function which is more general than the convolution (1). And as by- products, convolutions of many integral transforms such as the Meijer, Hankel, Fourier- sine were found. For instance, the convolution with the weight- function $\gamma(y) = \sin y$ of the functions f and g for the Fourier- sine integral transform F_s was studied in [1], [5]

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{0}^{+\infty} f(t)[\operatorname{sign}(x-t+1)g(|x-t+1|)$$
$$-\operatorname{sign}(x-t-1)g(|x-t-1|) - g(x+t+1) + \operatorname{sign}(x+t-1)g(|x+t-1|)]dt. \tag{2}$$

for which the factorization property holds

$$F_s(f \overset{\gamma}{*} g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$

In 1998, Kakichev V. A and Nguyen xuan Thao proposed a construction method for defining the generalized convolutions of three arbitrary integral transforms [5]. In recents years, several generalized convolutions of integral transforms were published [6]-[8].

In this talk, we define the convolution with the weight- function for the $T = 3F + F^{-1}$ transform, study some its properties and apply them to solving integral equation.

2 Convolution

We consider T transform

$$\begin{aligned}(Tf)(y) =&3(Ff)(y) + (F^{-1}f)(y)\\ =&\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}[4\cos(yx) - 2i\sin(yx)]f(x)dx, \quad y \in \mathbb{R}.\end{aligned}$$

Definition 1. *The convolution with the weight- function $\gamma(x) = e^{\frac{-x^2}{2}}$ of two function f, g for the T transform is defined as follows*

$$\begin{aligned}(f \overset{\gamma}{*} g)(x) = \frac{1}{8\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} f(u)g(v)\big[13e^{\frac{-(x-u-v)^2}{2}} + 3e^{\frac{-(x+u-v)^2}{2}}\\ - 3e^{\frac{-(x+u+v)^2}{2}} + 3e^{\frac{-(x-u+v)^2}{2}}\big]dudv. \quad (3)\end{aligned}$$

Theorem 1. *Let f, g be function in $L(\mathbb{R})$. Then the convolution with the weight- function $\gamma(y) = e^{\frac{-y^2}{2}}$ of them for the T transform belongs to $L(\mathbb{R})$ and the fractorization property holds*

$$T(f \overset{\gamma}{*} g)(y) = \gamma(y)(Tf)(y)(Tg)(y).$$

Proof. We prove $(f \overset{\gamma}{*} g)(x) \in L(\mathbb{R})$.
We have

$$\begin{aligned}\int\limits_{-\infty}^{+\infty} |(f \overset{\gamma}{*} g)|(x)dx \leqslant&\frac{1}{8\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |f(u)||g(v)|\Big|13e^{\frac{-(x-u-v)^2}{2}}\\ &+ 3e^{\frac{-(x+u-v)^2}{2}} - 3e^{\frac{-(x+u+v)^2}{2}} + 3e^{\frac{-(x-u+v)^2}{2}}\Big|dudvdx\\ \leqslant&N_1 + N_2 + N_3 + N_4,\end{aligned}$$

where

$$N_1 = \frac{13}{8\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |f(u)||g(v)|e^{\frac{-(x-u-v)^2}{2}}dudvdx.$$

$$N_2 = \frac{3}{8\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(u)||g(v)|e^{\frac{-(x+u-v)^2}{2}}dudvdx.$$

$$N_3 = \frac{3}{8\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(u)||g(v)|e^{\frac{-(x+u+v)^2}{2}}dudvdx.$$

$$N_4 = \frac{3}{8\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(u)||g(v)|e^{\frac{-(x-u+v)^2}{2}}dudvdx.$$

By using the formula $\int\limits_{-\infty}^{+\infty} e^{-u^2/2}du = \sqrt{2\pi}$ and since $f, g \in L(\mathbb{R})$ then $N_i < +\infty$, $i = 1 \ldots 4$.
So

$$\int_{-\infty}^{+\infty} |(f \overset{\gamma}{*} g)|(x)dx < +\infty.$$

We prove the fractorization property

$$T(f \overset{\gamma}{*} g)(y) = \gamma(y)(Tf)(y)(Tg)(y).$$

We have

$$\begin{aligned}
&\gamma(y)(Tf)(y)(Tg)(y) = \\
=&\gamma(y)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)(3e^{-iyu}+e^{iyu})du\int_{-\infty}^{+\infty} g(v)(3e^{-iyv}+e^{iyv})dv \\
=&\frac{1}{8\pi\sqrt{2\pi}}4\sqrt{2\pi}e^{-\frac{y^2}{2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u)g(v)(3e^{-iyu}+e^{iyu})(3e^{-iyv}+e^{iyv})dudv.
\end{aligned}$$

Put

$$B_1 = \sqrt{2\pi}e^{\frac{-y^2}{2}}\left[39e^{-iyu}e^{-iyv} + 13e^{iyu}e^{iyv}\right].$$

$$B_2 = \sqrt{2\pi}e^{\frac{-y^2}{2}}\left[9e^{iyu}e^{-iyv} + 3e^{-iyu}e^{iyv}\right].$$

$$B_3 = -\sqrt{2\pi}e^{\frac{-y^2}{2}}\left[9e^{iyu}e^{iyv} + 3e^{-iyu}e^{-iyv}\right].$$

$$B_4 = \sqrt{2\pi}e^{\frac{-y^2}{2}}\left[9e^{-iyu}e^{iyv} + 3e^{iyu}e^{-iyv}\right].$$

We consider

$$\begin{aligned}
B_1 =& 39\sqrt{2\pi}e^{-\frac{y^2}{2}}e^{-iyu}e^{-iyv} + 13\sqrt{2\pi}e^{iyu}e^{iyv}e^{\frac{-y^2}{2}} \\
=& 39\int\limits_{-\infty}^{+\infty} e^{-i(x-u-s)y}e^{\frac{-(x-u-v)^2}{2}}e^{-iyu}e^{-iyv}dx + 13\int\limits_{-\infty}^{+\infty} e^{i(x-u-v)y}e^{\frac{-(x-u-v)^2}{2}}e^{iyu}e^{iyv}dx \\
=& 13\int\limits_{-\infty}^{+\infty} 3e^{-iyx}e^{\frac{-(x-u-v)^2}{2}}dx + 13\int\limits_{-\infty}^{+\infty} e^{iyx}e^{\frac{-(x-u-v)^2}{2}}dx \\
=& 13\int\limits_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx})e^{\frac{-(x-u-v)^2}{2}}dx. \qquad (4)
\end{aligned}$$

Similarly, we have

$$B_2 = 3\int\limits_{-\infty}^{+\infty}(3e^{-iyx} + e^{iyx})e^{\frac{-(x+u-v)^2}{2}}dx. \qquad (5)$$

$$B_3 = -3\int\limits_{-\infty}^{+\infty}(3e^{-iyx} + e^{iyx})e^{\frac{-(x+u+v)^2}{2}}dx. \qquad (6)$$

$$B_4 = 3\int\limits_{-\infty}^{+\infty}(3e^{-iyx} + e^{iyx})e^{\frac{-(x-u+v)^2}{2}}dx. \qquad (7)$$

From (4), (5), (9), (7) we obtain

$$\begin{aligned}
4\sqrt{2\pi}e^{\frac{-t^2}{2}}(3e^{-iyu} + e^{iyu})(3e^{-its} + e^{its}) = \int\limits_{-\infty}^{+\infty}(3e^{-iyx} + e^{iyx})\big[13e^{\frac{-(x-u-v)^2}{2}} \\
+ 3e^{\frac{-(x+u-v)^2}{2}} - 3e^{\frac{-(x+u+v)^2}{2}} + 3e^{\frac{-(x-u+v)^2}{2}}\big]dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\gamma(y)(Tf)(y)(Tg)(y) = \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{8\pi} \int\limits_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} f(u)g(v) \big[13e^{\frac{-(x-u-v)^2}{2}} \\
&\quad + 3e^{\frac{-(x+u-v)^2}{2}} - 3e^{\frac{-(x+u+v)^2}{2}} + 3e^{\frac{-(x-u+v)^2}{2}} \big] dudvdx \\
&= T(f \overset{\gamma}{*} g)(y).
\end{aligned}$$

□

Remark 1. In the $L(\mathbb{R})$ space, the convolution (3) is comutative, associative and distributive.

Theorem 2. *In the space $L(\mathbb{R})$, there does not exist the unit element for the operation of the convolution with a weight function for the T transform.*

Proof. Suppose that exists e, the unit element of the operation of convolution in the space $L(\mathbb{R})$: $e \overset{\gamma}{*} g = g \overset{\gamma}{*} e = g$ for any function g belonging to $L(\mathbb{R})$. Then we have

$$T(e \overset{\gamma}{*} g)(y) = (Tg)(y), \qquad \forall y \in \mathbb{R}.$$

Hence

$$e^{-y^2/2}(Te)(y)(Tg)(y) = (Tg)(y), \quad \forall y \in \mathbb{R}.$$

The last is eqivalent to the equality

$$(Tg)(y)[e^{-y^2/2}(Te)(y) - 1] = 0, \qquad \forall y \in \mathbb{R}.$$

Choosing g so that $(Tg)(y) \neq 0, \forall y \in \mathbb{R}$, we see that $e^{-y^2/2}(Te)(y) - 1 = 0$ or $(Te)(y) = e^{y^2/2}$. Since assumption that $e \in L_1(\mathbb{R})$ then

$$(Te)(y) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} (3e^{-ixy} + e^{ixy}) e(x)dx.$$

$$|(Te)(y)| \leqslant \frac{4}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} |e(x)|dx < +\infty, \forall y.$$

So $(Te)(y)$ is bounded function and $e^{y^2/2}$ is not bounded function. This is a contradiction. Hence there does not exists unit element for the operation of convolution with a weight function for the T tramsform in the space $L(\mathbb{R})$. $\square$

Definition 2. *The norm in the space $L(\mathbb{R})$ is defined by*

$$\|f\| = \frac{11}{2\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} |f(x)|dx.$$

Theorem 3. *If f, g are functions in to $L(\mathbb{R})$, then the following inequality holds*

$$\|f \overset{\gamma}{*} g\| \leqslant \|f\|.\|g\|.$$

Put

$$L(e^{|x|}, \mathbb{R}) = \{h, \int\limits_{-\infty}^{+\infty} e^{|x|}|h(x)|dx < \infty, \quad h \in L(\mathbb{R})\}.$$

Theorem 4 (A Titchmarch theorem). *Let $f, g \in L(e^{|x|}, \mathbb{R})$. If $(f \overset{\gamma}{*} g)(x) = 0, \forall x \in \mathbb{R}$ then either $f(x) = 0$ or $g(x) = 0, \forall x \in \mathbb{R}$.*

Proof. Under the hypothesis $(f \overset{\gamma}{*} g)(x) = 0, \quad \forall x \in \mathbb{R}$, it follows that

$$T(f \overset{\gamma}{*} g)(x) = \gamma(x)(Tf)(x)(Tg)(x), \forall x \in \mathbb{R}.$$

$$\gamma(x)(Tf)(x)(Tg)(x) = 0, \quad x \in \mathbb{R}.$$

Consider

$$(Tf)(y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} (3e^{-ixy} + e^{ixy})f(x)dx, \quad \forall x \in \mathbb{R}.$$

Since

$$\begin{aligned} \left| \frac{d^n}{dy^n}[3e^{-ixy} + e^{ixy}]f(x) \right| &= |(ix)^n[(-1)^n 3e^{-ixy} + e^{ixy}]f(x)| \\ &\leqslant |4x^n f(x)| = |4x^n e^{-|x|}.h(x)| \\ &\leqslant c|h(x)|. \end{aligned}$$

Due to Weierstrass' criterion, the integral $\int\limits_{-\infty}^{+\infty} \frac{d^n}{dy^n}[3e^{-ixy} + e^{ixy}]f(x)dx$ uniformly converges on $\mathbb{R}$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(Tf)(y)$ is analytic for $y \in \mathbb{R}$.
Similarly, $(Tg)(y)$ is analytic for $y \in \mathbb{R}$.
So we have $(Tf)(y) = 0$ or$(Tg)(y) = 0$. It follows that either $f(x) = 0$ or $g(x) = 0$. □

Theorem 5. *If f and g are functions in $L(\mathbb{R})$, then the following equality holds*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{8\pi}\Big[13[f \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](x) + 3[f_1 \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](x) \\ - 3[f \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](-x) + 3[f_1 \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](-x)\Big]$$

where $f(-x) = f_1(x)$.

Proof. We have

$$\frac{13}{8\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x-u-v)}{2}}dudv = \frac{13}{8\pi}\int\limits_{-\infty}^{+\infty} f(u)\int\limits_{-\infty}^{+\infty} e^{\frac{-(x-u-v)}{2}}g(v)dv$$

$$=\frac{13}{8\pi}\int\limits_{-\infty}^{+\infty} f(u)(e^{-s^2/2} \underset{F}{*} g(v))(x-u)du$$

$$=\frac{13}{8\pi}[f \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](x). \tag{8}$$

Similarly, we obtain

$$\frac{3}{8\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x+u-v)}{2}}dudv = \frac{3}{8\pi}\int\limits_{-\infty}^{+\infty} f(u)(e^{-v^2/2} \underset{F}{*} g(v))(x+u)du$$

$$= \frac{3}{8\pi}[f_1 \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](x). \tag{9}$$

$$-\frac{3}{8\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x+u+v)}{2}}dudv = -\frac{3}{8\pi}\int\limits_{-\infty}^{+\infty} f(u)(e^{-v^2/2} \underset{F}{*} g(v))(-x-u)du$$

$$= -\frac{3}{8\pi}[f \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](-x). \tag{10}$$

$$\frac{3}{8\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x-u+v)}{2}}dudv = \frac{3}{8\pi}\int_{-\infty}^{+\infty} f(u)(e^{-v^2/2} \underset{F}{*} g(v))(-x+u)du$$
$$= \frac{3}{8\pi}[f_1 \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](-x) \qquad (11)$$

in which $f(-x) = f_1(x)$.
From (8), (9), (10), (11) we obtain

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{8\pi}\Big[13 f \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](x) + 3[f_1 \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](x)$$
$$- 3[f \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](-x) + 3[f_1 \underset{F}{*} (e^{-v^2/2} \underset{F}{*} g(v))](-x)\Big],$$

here $f(-x) = f_1(x)$. □

3 Application to solving integral equations

Consider the integral equation

$$f(x) + \frac{\lambda}{8\pi}\int_{-\infty}^{+\infty} f(u)\psi(x,u)du = f(x). \qquad (12)$$

Here

$$\psi(x,u) = \int_{-\infty}^{+\infty} g(v)[13e^{\frac{-(x-u-v)^2}{2}} + 3e^{\frac{-(x+u-v)^2}{2}} - 3e^{\frac{-(x+u+v)^2}{2}} + 3e^{\frac{-(x-u+v)^2}{2}}]dv$$

$\lambda \in \mathbb{C}, g, h$ are functions in $\in L(\mathbb{R})$, f is unknown function . To solving the integral equation we introduce the following definition

Definition 3. *The generalized convolution of two function f, g for the T, F transforms with the weight- function $\gamma(x) = e^{-x^2/2}$ is defined as follows*

$$(f \underset{T}{*} g)(x) = \frac{1}{16\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u)g(v)\big[9e^{\frac{-(x-u-v)^2}{2}} + 3e^{\frac{-(x+u-v)^2}{2}}$$
$$- 3e^{\frac{-(x+u+v)^2}{2}} - e^{\frac{-(x-u+v)^2}{2}}\big]dudv. \qquad (13)$$

Lema 1. *Let f, g be function in $L(\mathbb{R})$. Then the generalized convolution* (13) *belongs to $L(\mathbb{R})$ and the fractorization property holds*

$$T(f \underset{T}{*} g)(y) = \gamma(y)(Tf)(y)(Fg)(y).$$

Theorem 6. *With the condition $1+\lambda e^{-y^2/2}(Tg)(y) \neq 0, \forall y \in \mathbb{R}$, there exists a solution in $L(\mathbb{R})$ of* (12) *which is defined by*

$$f = h - \lambda(h \underset{T}{*} l),$$

here $l = l_1 + l_2$, $l_1(x), l_2(x) \in L(\mathbb{R})$ and it is defined by

$$(Fl_1)(y) = \frac{3(Fg)(y)}{1+\lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]},$$

$$(Fl_2)(y) = \frac{(Fg)(-y)}{1+\lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]}.$$

Proof. The equation (12) can be rewritten in the form

$$f + \lambda(f \overset{\gamma}{*} g) = h.$$

Due to Theorem 1

$$(Tf)(y) + \lambda T(f \overset{\gamma}{*} g)(y) = (Th)(y).$$

It follows that

$$(Tf)(y)[1 + \lambda e^{-y^2/2}(Tg)(y)] = (Th)(y).$$

Since $1 + \lambda e^{-y^2/2}(Tg)(y) \neq 0, \forall y \in \mathbb{R}$

$$\begin{aligned}(Tf)(y) =& (Th)(y)\frac{1}{1+\lambda e^{-y^2/2}(Tg)(y)}\\ =& (Th)(y)[1 - \frac{\lambda e^{-y^2/2}(Tg)(y)}{1+\lambda e^{-y^2/2}(Tg)(y)}]\end{aligned}$$

$$\begin{aligned}\frac{(Tg)(y)}{1+\lambda e^{-y^2/2}(Tg)(y)} =& \Big[\frac{3(Fg)(y)}{1+\lambda e^{-y^2/2}[3(Fg)(y) + (F^{-1}g)(y)]}\\ &+ \frac{(F^{-1}g)(y)}{1+\lambda e^{-y^2/2}[3(Fg)(y) + (F^{-1}g)(y)]}\Big].\end{aligned}$$

Due to Wiener-Levi's theorem , there exists a function $l_1 \in L \in \mathbb{R}$ such that

$$(Fl_1)(y) = \frac{3(Fg)(y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]}.$$

Similarly, there exists $l_2 \in L \in \mathbb{R}$ such that

$$(Fl_2)(y) = \frac{(Fg)(-y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]}.$$

Put $l = l_1 + l_2$. It implies that

$$\frac{(Tg)(y)}{1 + \lambda e^{-y^2/2}(Tg)(y)} = F(l).$$

Hence

$$Tf)(y) = (Th)(y)[1 - \lambda e^{-y^2/2}(Fl)(y)].$$

It follows that

$$(Tf)(y) = (Th)(y) - \lambda T(h \underset{T}{*} l)(y).$$

Thus

$$f = h - \lambda(h \underset{T}{*} l).$$

□

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