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Convolution for the transform induced by Fourier integral transform and its inverse

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Abstract

The convolutions for $T = 3F + F^{-1}$ transforms are formulated, its properties and applications to solving integral equations are considered.

1 Introduction

The convolution for integral transforms were studied in the 19th century, at first the convolutions for Fourier transform, for the Laplace transform, for the Mellin transform and after that for Hilbert transform [13], Hankel transform [12], Kontorovich- Lebedev transform and Stieltjes transform.

The convolution for the Fourier integral transform [10]

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y)g(y)dy, \quad (1)$$

for which satisfies the factorization equality

$$F(f * g)(y) = (Ff)(y)(Fg)(y), \quad y \in \mathbb{R}$$

in which F is the Fourier transform [4]and is defined as follows

$$\tilde{f}(y) \equiv (Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} f(x)dx.$$

$$f(x) \equiv (F^{-1}\tilde{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy} \tilde{f}(y) dy.$$

In 1941, Churchill R. V. introduced the convolution of two functions f and g for the Fourier cosine transforms [3]

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(|x-y|) + g(x+y)] dy, \quad x \in \mathbb{R}_+$$

for which satisfies the factorization equality

$$F_c(f * g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y \in \mathbb{R}_+,$$

here F_c is the Fourier cosine transform.

Afterwards, in 1967, V. A. Kakichev proposed a constructive method for defining the convolution with a weight- function which is more general than the convolution (1). And as by- products, convolutions of many integral transforms such as the Meijer, Hankel, Fourier- sine were found. For instance, the convolution with the weight- function $\gamma(y) = \sin y$ of the functions f and g for the Fourier- sine integral transform F_s was studied in [1], [5]

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(t)[\text{sign}(x-t+1)g(|x-t+1|) - \text{sign}(x-t-1)g(|x-t-1|) - g(x+t+1) + \text{sign}(x+t-1)g(|x+t-1|)] dt. \quad (2)$$

for which the factorization property holds

$$F_s(f \overset{\gamma}{*} g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$

In 1998, Kakichev V. A and Nguyen xuan Thao proposed a construction method for defining the generalized convolutions of three arbitrary integral transforms [5]. In recents years, several generalized convolutions of integral transforms were published [6]-[8].

In this talk, we define the convolution with the weight- function for the $T = 3F + F^{-1}$ transform, study some its properties and apply them to solving integral equation.

2 Convolution

We consider T transform

$$\begin{aligned}(Tf)(y) &= 3(Ff)(y) + (F^{-1}f)(y) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} [4 \cos(yx) - 2i \sin(yx)] f(x) dx, \quad y \in \mathbb{R}.\end{aligned}$$

Definition 1. The convolution with the weight- function $\gamma(x) = e^{-\frac{x^2}{2}}$ of two function f, g for the T transform is defined as follows

$$\begin{aligned}(f \overset{\gamma}{*} g)(x) &= \frac{1}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v) [13e^{-\frac{(x-u-v)^2}{2}} + 3e^{-\frac{(x+u-v)^2}{2}} \\ &\quad - 3e^{-\frac{(x+u+v)^2}{2}} + 3e^{-\frac{(x-u+v)^2}{2}}] dudv. \quad (3)\end{aligned}$$

Theorem 1. Let f, g be function in $L(\mathbb{R})$. Then the convolution with the weight- function $\gamma(y) = e^{-\frac{y^2}{2}}$ of them for the T transform belongs to $L(\mathbb{R})$ and the factorization property holds

$$T(f \overset{\gamma}{*} g)(y) = \gamma(y)(Tf)(y)(Tg)(y).$$

Proof. We prove $(f \overset{\gamma}{*} g)(x) \in L(\mathbb{R})$.

We have

$$\begin{aligned}\int_{-\infty}^{+\infty} |(f \overset{\gamma}{*} g)(x)| dx &\leq \frac{1}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(u)||g(v)| \left| 13e^{-\frac{(x-u-v)^2}{2}} \right. \\ &\quad \left. + 3e^{-\frac{(x+u-v)^2}{2}} - 3e^{-\frac{(x+u+v)^2}{2}} + 3e^{-\frac{(x-u+v)^2}{2}} \right| dudv dx \\ &\leq N_1 + N_2 + N_3 + N_4,\end{aligned}$$

where

$$N_1 = \frac{13}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(u)||g(v)| e^{-\frac{(x-u-v)^2}{2}} dudv dx.$$

$$N_2 = \frac{3}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(u)||g(v)|e^{-\frac{(x+u-v)^2}{2}} dudvdx.$$

$$N_3 = \frac{3}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(u)||g(v)|e^{-\frac{(x+u+v)^2}{2}} dudvdx.$$

$$N_4 = \frac{3}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(u)||g(v)|e^{-\frac{(x-u+v)^2}{2}} dudvdx.$$

By using the formula $\int_{-\infty}^{+\infty} e^{-u^2/2} du = \sqrt{2\pi}$ and since $f, g \in L(\mathbb{R})$ then $N_i < +\infty, i = 1 \dots 4.$

So

$$\int_{-\infty}^{+\infty} |(f * \gamma g)(x)| dx < +\infty.$$

We prove the factorization property

$$T(f * \gamma g)(y) = \gamma(y)(Tf)(y)(Tg)(y).$$

We have

$$\begin{aligned} & \gamma(y)(Tf)(y)(Tg)(y) = \\ &= \gamma(y) \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)(3e^{-iyu} + e^{iyu}) du \int_{-\infty}^{+\infty} g(v)(3e^{-iyv} + e^{iyv}) dv \\ &= \frac{1}{8\pi\sqrt{2\pi}} 4\sqrt{2\pi} e^{-\frac{y^2}{2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v)(3e^{-iyu} + e^{iyu})(3e^{-iyv} + e^{iyv}) dudv. \end{aligned}$$

Put

$$B_1 = \sqrt{2\pi} e^{-\frac{y^2}{2}} [39e^{-iyu} e^{-iyv} + 13e^{iyu} e^{iyv}].$$

$$B_2 = \sqrt{2\pi} e^{-\frac{y^2}{2}} [9e^{iyu} e^{-iyv} + 3e^{-iyu} e^{iyv}].$$

$$B_3 = -\sqrt{2\pi} e^{-\frac{y^2}{2}} [9e^{iyu} e^{iyv} + 3e^{-iyu} e^{-iyv}].$$

$$B_4 = \sqrt{2\pi} e^{-\frac{y^2}{2}} [9e^{-iyu} e^{iyv} + 3e^{iyu} e^{-iyv}].$$

We consider

$$\begin{aligned} B_1 &= 39\sqrt{2\pi} e^{-\frac{y^2}{2}} e^{-iyu} e^{-iyv} + 13\sqrt{2\pi} e^{iyu} e^{iyv} e^{-\frac{y^2}{2}} \\ &= 39 \int_{-\infty}^{+\infty} e^{-i(x-u-s)y} e^{-\frac{(x-u-v)^2}{2}} e^{-iyu} e^{-iyv} dx + 13 \int_{-\infty}^{+\infty} e^{i(x-u-v)y} e^{-\frac{(x-u-v)^2}{2}} e^{iyu} e^{iyv} dx \\ &= 13 \int_{-\infty}^{+\infty} 3e^{-iyx} e^{-\frac{(x-u-v)^2}{2}} dx + 13 \int_{-\infty}^{+\infty} e^{iyx} e^{-\frac{(x-u-v)^2}{2}} dx \\ &= 13 \int_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) e^{-\frac{(x-u-v)^2}{2}} dx. \end{aligned} \quad (4)$$

Similarly, we have

$$B_2 = 3 \int_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) e^{-\frac{(x+u-v)^2}{2}} dx. \quad (5)$$

$$B_3 = -3 \int_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) e^{-\frac{(x+u+v)^2}{2}} dx. \quad (6)$$

$$B_4 = 3 \int_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) e^{-\frac{(x-u+v)^2}{2}} dx. \quad (7)$$

From (4), (5), (6), (7) we obtain

$$\begin{aligned} 4\sqrt{2\pi} e^{-\frac{t^2}{2}} (3e^{-iyu} + e^{iyu})(3e^{-its} + e^{its}) &= \int_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) [13e^{-\frac{(x-u-v)^2}{2}} \\ &\quad + 3e^{-\frac{(x+u-v)^2}{2}} - 3e^{-\frac{(x+u+v)^2}{2}} + 3e^{-\frac{(x-u+v)^2}{2}}] dx. \end{aligned}$$

Hence

$$\begin{aligned}
\gamma(y)(Tf)(y)(Tg)(y) &= \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{8\pi} \int_{-\infty}^{+\infty} (3e^{-iyx} + e^{iyx}) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v) [13e^{\frac{-(x-u-v)^2}{2}} \\
&\quad + 3e^{\frac{-(x+u-v)^2}{2}} - 3e^{\frac{-(x+u+v)^2}{2}} + 3e^{\frac{-(x-u+v)^2}{2}}] dudvdx \\
&= T(f \overset{\gamma}{*} g)(y).
\end{aligned}$$

□

Remark 1. In the $L(\mathbb{R})$ space, the convolution (3) is comutative, associative and distributive.

Theorem 2. In the space $L(\mathbb{R})$, there does not exist the unit element for the operation of the convolution with a weight function for the T transform.

Proof. Suppose that exists e , the unit element of the operation of convolution in the space $L(\mathbb{R})$: $e \overset{\gamma}{*} g = g \overset{\gamma}{*} e = g$ for any function g belonging to $L(\mathbb{R})$. Then we have

$$T(e \overset{\gamma}{*} g)(y) = (Tg)(y), \quad \forall y \in \mathbb{R}.$$

Hence

$$e^{-y^2/2}(Te)(y)(Tg)(y) = (Tg)(y), \quad \forall y \in \mathbb{R}.$$

The last is equivalent to the equality

$$(Tg)(y)[e^{-y^2/2}(Te)(y) - 1] = 0, \quad \forall y \in \mathbb{R}.$$

Choosing g so that $(Tg)(y) \neq 0, \forall y \in \mathbb{R}$, we see that $e^{-y^2/2}(Te)(y) - 1 = 0$ or $(Te)(y) = e^{y^2/2}$. Since assumption that $e \in L_1(\mathbb{R})$ then

$$(Te)(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (3e^{-ixy} + e^{ixy})e(x)dx.$$

$$|(Te)(y)| \leq \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |e(x)|dx < +\infty, \forall y.$$

So $(Te)(y)$ is bounded function and $e^{y^2/2}$ is not bounded function. This is a contradiction. Hence there does not exist unit element for the operation of convolution with a weight function for the T transform in the space $L(\mathbb{R})$. \square

Definition 2. The norm in the space $L(\mathbb{R})$ is defined by

$$\|f\| = \frac{11}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(x)| dx.$$

Theorem 3. If f, g are functions in to $L(\mathbb{R})$, then the following inequality holds

$$\|f \overset{\gamma}{*} g\| \leq \|f\| \cdot \|g\|.$$

Put

$$L(e^{|\cdot|}, \mathbb{R}) = \left\{ h, \int_{-\infty}^{+\infty} e^{|\cdot|} |h(x)| dx < \infty, h \in L(\mathbb{R}) \right\}.$$

Theorem 4 (A Titchmarsh theorem). Let $f, g \in L(e^{|\cdot|}, \mathbb{R})$.

If $(f \overset{\gamma}{*} g)(x) = 0, \forall x \in \mathbb{R}$ then either $f(x) = 0$ or $g(x) = 0, \forall x \in \mathbb{R}$.

Proof. Under the hypothesis $(f \overset{\gamma}{*} g)(x) = 0, \forall x \in \mathbb{R}$, it follows that

$$T(f \overset{\gamma}{*} g)(x) = \gamma(x)(Tf)(x)(Tg)(x), \forall x \in \mathbb{R}.$$

$$\gamma(x)(Tf)(x)(Tg)(x) = 0, \quad x \in \mathbb{R}.$$

Consider

$$(Tf)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (3e^{-ixy} + e^{ixy})f(x) dx, \quad \forall x \in \mathbb{R}.$$

Since

$$\begin{aligned} \left| \frac{d^n}{dy^n} [3e^{-ixy} + e^{ixy}]f(x) \right| &= |(ix)^n [(-1)^n 3e^{-ixy} + e^{ixy}]f(x)| \\ &\leq |4x^n f(x)| = |4x^n e^{-|x|} \cdot h(x)| \\ &\leq c|h(x)|. \end{aligned}$$

Due to Weierstrass' criterion, the integral $\int_{-\infty}^{+\infty} \frac{d^n}{dy^n} [3e^{-ixy} + e^{ixy}] f(x) dx$ uniformly converges on \mathbb{R} . Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(Tf)(y)$ is analytic for $y \in \mathbb{R}$. Similarly, $(Tg)(y)$ is analytic for $y \in \mathbb{R}$. So we have $(Tf)(y) = 0$ or $(Tg)(y) = 0$. It follows that either $f(x) = 0$ or $g(x) = 0$. \square

Theorem 5. *If f and g are functions in $L(\mathbb{R})$, then the following equality holds*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{8\pi} \left[13[f \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](x) + 3[f_1 \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](x) \right. \\ \left. - 3[f \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](-x) + 3[f_1 \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](-x) \right]$$

where $f(-x) = f_1(x)$.

Proof. We have

$$\frac{13}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x-u-v)}{2}} dudv = \frac{13}{8\pi} \int_{-\infty}^{+\infty} f(u) \int_{-\infty}^{+\infty} e^{\frac{-(x-u-v)}{2}} g(v)dv \\ = \frac{13}{8\pi} \int_{-\infty}^{+\infty} f(u)(e^{-s^2/2} \overset{*}{F} g(v))(x-u)du \\ = \frac{13}{8\pi} [f \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](x). \quad (8)$$

Similarly, we obtain

$$\frac{3}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x+u-v)}{2}} dudv = \frac{3}{8\pi} \int_{-\infty}^{+\infty} f(u)(e^{-v^2/2} \overset{*}{F} g(v))(x+u)du \\ = \frac{3}{8\pi} [f_1 \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](x). \quad (9)$$

$$-\frac{3}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v)e^{\frac{-(x+u+v)}{2}} dudv = -\frac{3}{8\pi} \int_{-\infty}^{+\infty} f(u)(e^{-v^2/2} \overset{*}{F} g(v))(-x-u)du \\ = -\frac{3}{8\pi} [f \overset{*}{F} (e^{-v^2/2} \overset{*}{F} g(v))](-x). \quad (10)$$

$$\begin{aligned} \frac{3}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v)e^{-\frac{(x-u+v)}{2}} dudv &= \frac{3}{8\pi} \int_{-\infty}^{+\infty} f(u)(e^{-v^2/2} *_F g(v))(-x+u)du \\ &= \frac{3}{8\pi} [f_1 *_F (e^{-v^2/2} *_F g(v))](-x) \end{aligned} \quad (11)$$

in which $f(-x) = f_1(x)$.

From (8), (9), (10), (11) we obtain

$$\begin{aligned} (f \overset{\gamma}{*} g)(x) &= \frac{1}{8\pi} \left[13f *_F (e^{-v^2/2} *_F g(v))(x) + 3[f_1 *_F (e^{-v^2/2} *_F g(v))](x) \right. \\ &\quad \left. - 3[f *_F (e^{-v^2/2} *_F g(v))](-x) + 3[f_1 *_F (e^{-v^2/2} *_F g(v))](-x) \right], \end{aligned}$$

here $f(-x) = f_1(x)$. □

3 Application to solving integral equations

Consider the integral equation

$$f(x) + \frac{\lambda}{8\pi} \int_{-\infty}^{+\infty} f(u)\psi(x, u)du = f(x). \quad (12)$$

Here

$$\psi(x, u) = \int_{-\infty}^{+\infty} g(v) \left[13e^{-\frac{(x-u-v)^2}{2}} + 3e^{-\frac{(x+u-v)^2}{2}} - 3e^{-\frac{(x+u+v)^2}{2}} + 3e^{-\frac{(x-u+v)^2}{2}} \right] dv$$

$\lambda \in \mathbb{C}$, g, h are functions in $L(\mathbb{R})$, f is unknown function. To solving the integral equation we introduce the following definition

Definition 3. The generalized convolution of two function f, g for the T, F transforms with the weight- function $\gamma(x) = e^{-x^2/2}$ is defined as follows

$$\begin{aligned} (f *_T g)(x) &= \frac{1}{16\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)g(v) \left[9e^{-\frac{(x-u-v)^2}{2}} + 3e^{-\frac{(x+u-v)^2}{2}} \right. \\ &\quad \left. - 3e^{-\frac{(x+u+v)^2}{2}} - e^{-\frac{(x-u+v)^2}{2}} \right] dudv. \end{aligned} \quad (13)$$

Lema 1. Let f, g be function in $L(\mathbb{R})$. Then the generalized convolution (13) belongs to $L(\mathbb{R})$ and the factorization property holds

$$T(f *_T g)(y) = \gamma(y)(Tf)(y)(Fg)(y).$$

Theorem 6. With the condition $1 + \lambda e^{-y^2/2}(Tg)(y) \neq 0, \forall y \in \mathbb{R}$, there exists a solution in $L(\mathbb{R})$ of (12) which is defined by

$$f = h - \lambda(h *_T l),$$

here $l = l_1 + l_2$, $l_1(x), l_2(x) \in L(\mathbb{R})$ and it is defined by

$$(Fl_1)(y) = \frac{3(Fg)(y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]},$$

$$(Fl_2)(y) = \frac{(Fg)(-y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]}.$$

Proof. The equation (12) can be rewritten in the form

$$f + \lambda(f *_T g) = h.$$

Due to Theorem 1

$$(Tf)(y) + \lambda T(f *_T g)(y) = (Th)(y).$$

It follows that

$$(Tf)(y)[1 + \lambda e^{-y^2/2}(Tg)(y)] = (Th)(y).$$

Since $1 + \lambda e^{-y^2/2}(Tg)(y) \neq 0, \forall y \in \mathbb{R}$

$$\begin{aligned} (Tf)(y) &= (Th)(y) \frac{1}{1 + \lambda e^{-y^2/2}(Tg)(y)} \\ &= (Th)(y) \left[1 - \frac{\lambda e^{-y^2/2}(Tg)(y)}{1 + \lambda e^{-y^2/2}(Tg)(y)} \right] \end{aligned}$$

$$\begin{aligned} \frac{(Tg)(y)}{1 + \lambda e^{-y^2/2}(Tg)(y)} &= \left[\frac{3(Fg)(y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (F^{-1}g)(y)]} \right. \\ &\quad \left. + \frac{(F^{-1}g)(y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (F^{-1}g)(y)]} \right]. \end{aligned}$$

Due to Wiener-Levi's theorem , there exists a function $l_1 \in L \in \mathbb{R}$ such that

$$(Fl_1)(y) = \frac{3(Fg)(y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]}.$$

Similarly, there exists $l_2 \in L \in \mathbb{R}$ such that

$$(Fl_2)(y) = \frac{(Fg)(-y)}{1 + \lambda e^{-y^2/2}[3(Fg)(y) + (Fg)(-y)]}.$$

Put $l = l_1 + l_2$. It implies that

$$\frac{(Tg)(y)}{1 + \lambda e^{-y^2/2}(Tg)(y)} = F(l).$$

Hence

$$Tf(y) = (Th)(y)[1 - \lambda e^{-y^2/2}(Fl)(y)].$$

It follows that

$$(Tf)(y) = (Th)(y) - \lambda T(h *_T l)(y).$$

Thus

$$f = h - \lambda(h *_T l).$$

□

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