Osaka University Knowledge Archive

Title	CONVOLUTION FOR INTEGRAL TRANSFORM WITH AIRY FUNCTION IN NUCLEUS
Author(s)	Nguyen, Xuan Thao; Trinh, Tuan
Citation	Annual Report of FY 2005, The Core University Program between Japan Society for the Promotion of Science (JSPS) and Vietnamese Academy of Science and Technology (VAST). 2006, p. 341-347
Version Type	VoR
URL	https://hdl.handle.net/11094/12920
rights	
Note	

Osaka University Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

Osaka University

CONVOLUTION FOR INTEGRAL TRANSFORM WITH AIRY FUNCTION IN NUCLEUS

NGUYEN XUAN THAO, TRINH TUAN

Hanoi Water Resoarces University,
175 Tay Son, Dong Da, Hanoi, Vietnam.

The integral transform is one of the most ancient parts of mathematics that plays an important role in mathematics as well as resolving physical problem and many other natural fields. The different issues of theory and application of the Fourier transform, one and multiple-direction Laplace is related to several works. Mathematicians have studied deep enough many other classical integral transforms, for example the integral transforms of Mellin, Hankel, Stieltzes, Meijer, Hilbert, Kontorovich - Lebedev, Mehler - Fox etc.

In 1941, firstly Churchill R. set out the convolution of f and g functions toward the Fourier integral transform:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-t)g(t)dt, \quad x \in R$$

satisfy the factorization equality:

$$F(f * g)(y) = (Ff)(y)(Fg)(y), \quad \forall y \in R.$$

Then, several integral transforms have found some correlative convolutions, for example in [6]: convolution for cosine - Fourier integral transform, Laplace and Mallin integral transforms. Convolution is applied to resolve the physical problem [6], to get sum of a series and calculate the integral. The convolution itself is an integral transform, hence it is the research object of [2]. In the theory of standard ring, the convolution is used with the status of multiplication of elements. The integral equations in the form of convolution finds some interesting applications and there are many scientific works that are related to these applications (see [7]).

However, known convolutions are also the convolutions without weight and there are many integral transforms that we haven't known their correlative convolution. In 1958, Vilenkin I. Ya studied the convolution of the general transform Mehler - Fox with the weight function [12]:

$$\gamma(x) = \frac{\pi}{xsh(\pi x)}\left|\Gamma\left(p + ix + \frac{1}{2}\right)\right|^2.$$

In 1967, Kakichev V. A, set out the method of building the convolution for any integral transform K with the weight function $\gamma(x)$ that satisfies the factorization equality [4]:

$$K(f \overset{\gamma}{*} g)(y) = \gamma(y)(Kf)(y)Kg(y).$$

This method allows us to find the convolution for the integral transforms when there is a certain binding between nucleus of the given integral transform and the correlative inverse integral transform, (see [8-11]).

Up to the present, almost the convolution for known integral transforms are the convolution with the weight function, and convolution without the weight function is rather rare. In this paper, we build the convolution with the weight function for the integral transform with Airy function in nucleus. We also describe the functional class which exists convolution, affirm that there is not unitary element toward the convolution in this functional class, the property of having no divisor of zero toward the convolution and give the application for resolving the integral equation in the convolution type.

Definition I. [12] Integral transform with the Airy function in the nucleus is determined as follows:

$$g(x) = (A_i f)(x) = \int_{-\infty}^{+\infty} A_i(x+y)f(y)dy, \quad x \in R \tag{1}$$

of which, $A_i(x)$ is the Airy function [2].

Theorem I. [12] *It is supposed that the function f is continuous and belonging to $L(R)$ then the transform* [1] *has inverse formula:*

$$f(y) = \int_{-\infty}^{+\infty} A_i(y+x)g(x)dx, \quad y \in R.$$

Definition 2. Convolution of the integral transform (1) with the weight function $\gamma(x) = \dfrac{tgx}{x}$ defined as follows:

$$(f \overset{\gamma}{*} g)(x) = \pi A_i(x) \int_{-\infty}^{+\infty} A_i(u) f(u) du \int_{-\infty}^{+\infty} A_i(v) f(v) dv, \quad x \in R \tag{2}$$

Theorem 1. *It is supposed that the function f is continuous and belonged to $L(R)$ then we have convolution $(f \overset{\gamma}{*} g)$ belongs to $L(R)$ and we have the following factorization equality:*

$$A_i(f \overset{\gamma}{*} g)(y) = \gamma(y)(A_i f)(y)(A_i g)(y), \quad \forall y \neq (2k\pi)\frac{\pi}{2}.$$

Proof. We have:

$$\begin{aligned} A_i(f \overset{\gamma}{*} g)(y) &= \int_{-\infty}^{+\infty} A_i(x+y)(f \overset{\gamma}{*} g)(x) dx \\ &= \pi \int_{-\infty}^{+\infty} A_i(x+y) A_i(x)(A_i f)(0)(A_i g)(0) dx \\ &= \pi (A_i f)(0)(A_i g)(0) \int_{-\infty}^{+\infty} A_i(x+y) A_i(x) dx \end{aligned} \tag{4}$$

Otherwise [1] due to:

$$A_i(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixt + i\frac{t^3}{3}} dt$$

is even bornological, replace into [4], changing the order to calculate the integral then use formula 3.336 page 173 [5] we have:

$$A_i(\gamma A_i f A_i g)(x) = \int_{-\infty}^{+\infty} \gamma(y) A_i(x+y)(A_i f)(y)(A_i g)(y)dy$$

$$= \int_{-\infty}^{+\infty} \gamma(y) \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x+y)t+i\frac{t^3}{3}} (A_i f)(y)(A_j g)(y)dtdy$$

$$= \int_{-\infty}^{+\infty} \gamma(y) \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x+y)t+i\frac{t^3}{3}} dydt \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iyu+i\frac{u^3}{3}} \int_{-\infty}^{+\infty} e^{ius} f(u)duds$$

$$\times \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iyv+i\frac{v^3}{3}} \int_{-\infty}^{+\infty} e^{ivr} g(v)dvdr$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixt+i\frac{t^3}{3}} \ dt \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\frac{u^3}{3}} \int_{-\infty}^{+\infty} e^{ius} f(x)duds$$

$$\times \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\frac{v^3}{3}} \int_{-\infty}^{+\infty} e^{ivr} g(v)dvdr. \int_{-\infty}^{+\infty} e^{i(t+u+v)y} \frac{tgy}{y} dy$$

$$= \pi A_i(x)(A_i f)(0)(A_i g)(0).$$

Besides, when use (2), formula 9.72 page 199 in [1], we have:

$$\int_{-\infty}^{+\infty} (f \overset{\gamma}{*} g)(x)dx = \pi(A_i f)(0)(A_i g)(0). \int_{-\infty}^{+\infty} A_i(x)dx < +\infty.$$

Therefore, $(f \overset{\gamma}{*} g)$ belongs to $L(R)$. Hence, theorem is proved. □

Theorem 2. *It is supposed that the function f is continuous and belonged to $L(R)$ then the convolution (2) haven't got the unitary element in the $L(R)$.*

Proof. we prove by using the antithesis method. Supposed that the function l is existed and belonged to $L(R)$ so that $(f \overset{\gamma}{*} e) = f$. Due to the factorization equality (3) we have:

$$\gamma(y)(A_i f)(y)(A_i e)(y) = (A_i f)(y), \quad \forall y.$$

Therefore:

$$(A_i e)(y) = \frac{1}{\gamma(y)}$$

Coordinate with the symmetric property of the transform (1) we have:

$$e = A_i\Big(\frac{1}{\gamma(y)}\Big).$$

Using the asymptote formula of the integral that contains the Airy function: 10.4.82, 10.4.83, page 268 [1] and the property of the transform (1) we see that the function -$\dfrac{1}{\gamma(y)}$ is not regrettable. This is contradicted with our antithesis supposition. The theorem is proved.

Theorem 3. (Titchmarsch type theorem) *It is supposed that functions f, g are continuous and belonged to $L(R)$. Then if we have $(f \overset{\gamma}{*} g) \equiv 0$ generalized $f \equiv 0$ or $g \equiv 0$.*

Proof. From the supposition, useing the factorization equality (2) we have $\gamma(y)(A_i f)(y)(A_i g)(y) = 0$, $\forall y \in R$. Then either $A_i f \equiv 0$ or $A_i g \equiv 0$. Coordinating with the continuous property of the f, g functions, we have $f \equiv 0$ or $g \equiv 0$. Theorem is proved.

Theorem 4. *It is supposed that f, g functions are continuous, known and belonged to $L(R)$, λ is the complex number and not zero so that $1 + \lambda\gamma(y)(A_i g)(y) \neq 0$ $\forall y \in R$, then the $\varphi \in L(R)$ so that:*

$$\frac{A_i g(y)}{1 + \lambda\gamma(y)(A_i g)(y)} = (A_i \varphi)(y),$$

then the integral equation in convolution type:

$$f(x) + \lambda\pi A_i(x) \int_{-\infty}^{+\infty} A_i(u) f(u) du. \int_{-\infty}^{+\infty} A_i(v) f(v) dv = h(x) \qquad (5)$$

has solution:

$$f(x) = h(x) - \lambda(h \overset{\gamma}{*} \varphi)(x) \in L(R)$$

Proof. From the equation (5) and the factorization equality (3) we have:

$$(A_i f)(y) + \lambda\gamma(y)(A_i f)(y)(A_i g)(y) = (A_i h)(y).$$

Due to

$$1 + \lambda\gamma(y)(A_i g)(y) \neq 0, \quad \forall y,$$

we have:

$$\begin{aligned}(A_i f)(y) &= \frac{(A_i h)(y)}{1 + \lambda\gamma(y)(A_i g)(y)} \\ &= \Big[1 - \lambda\frac{\gamma(y)(A_i g)(y)}{1 + \lambda\gamma(y)(A_i g)(y)}\Big](A_i h)(y).\end{aligned}$$

According to the theorem Wiener - Levi, the j function existed and belonged to $L(R)$ so that:

$$(A_i\varphi)(y) = \frac{(A_ig)(y)}{1 + \lambda\gamma(y)(A_ig)(y)}$$

From that we have:

$$\begin{aligned}(A_if)(y) &= [1 - \lambda\gamma(y)(A_i\varphi)(y)](A_ih)(y) \\ &= (A_ih)(y) - \lambda\gamma(y)(A_ih)(y)(A_i\varphi)(y)\end{aligned}$$

Here, and from the theorem 1, 2 we have:

$$f(x) = h(x) - \lambda(h \overset{\gamma}{*} \varphi)(x) \in L(R).$$

The theorem is proved.

REFERENCES

1. Abramovich M., Stigan I., Selection of special function, Moscow, Nauk. 1979, 832 pages (Russian).
2. Bateman H., Erdelyi A., Supergeometric function, Moscow, Nauk., Volume 1, 1965, 294 pages. Volume 2, 1966, 295 pages, Volume 3, 1967, 299 pages (Russian).
3. Hirxman I. I Uidder D. V., Transform in convolution type, I*L. 1958 (Russian).
4. Kakivhev.V. A., Convolution of the integral transforms, Izv. ANBSSR. Ser. Fiz., Mat. Nauk, 1967, N. 2, page 48- 57 (Russian).
5. Ruzuk I. M., Gradstein I. S., Table of integrals, sum, series, composition, Moscow, Fizmtagiz, 1951. 464 pages (Russian).
6. Sneddon I., Fourier transforms, Moscow, I*L, 1955, 668 pages.
7. Titchmarsch E., Introductory section of Fourier integral theory, GTI, 1948 (Russian).
8. Nguyen Xuan Thao, Both directional integral transform in Mellin type, Non linear physical problem and applications, AN. USSR, Selection of scientific works, 1999, 296-301 pages (Russian).
9. Nguyen Xuan Thao and Nguyen Thanh Hai, Convolutions for Integral transform and their application, Computer centre of the Russian Academy, Moscow, 1997, 44 pages (Russian).
10. Nguyen Xuan Thao, Trinh Tuan, Convolution for transforms M of the function with many matrix arguments, The IVth National Mathematics Conference, Hue 7/9 to 10/9/2002, 149 pages.
11. Vilenkin. N. Ya., Matrix element of the irreducible mobile group representation Unita in the space Lobachevski and the general transform Mehler - Fox, Scientific report AN. SSSR. 1958, Volume 118, N2, 219 - 222 pages (Russian).

12. Vu Kim Tuan, Integral transform in Fourier type in the new function classes, Scientific report, AN. BSSR, 1985, Volume 28, N7, 584-587 pages (Russian).