# THE APPROXIMATE CONTROLLABILITY FOR THE SYSTEM DESCRIBED BY RIGHT INVERTIBLE OPERATORS

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**Abstract.** In this paper, we deal with the approximate controllability for the linear system described by right invertible operators in the infinite dimensional Hilbert space.

**Keywords:** Right invertible operator, initial operator, initial value problem.

## 0 Introduction

By the appearance of the theory of right invertible operators, the initial, boundary and mixed boundary value problems for the linear systems described by right invertible operators and generalized invertible operators were studied by many Mathematicians (see[4, 6]). Nguyen Dinh Quyet, in his articles, has considered the controllability of linear systems described by right invertible operators in the case the resolving operator is invertible (see[8, 10, 11]). These results were generalized by A. Pogorzelec in the case of one-sized invertible resolving operarors (see [5, 6]) and by Nguyen Van Mau for the system described by generalized invertible operators (see [3, 4]). The above mentioned controllability is $F_1$-exactly controllable from one state to another. However, in infinite dimensional space, the exact controllability is not always realized. To overcome these restrictions, we define the so-called $F_1$-approximately controllable, in the sense of: "A system is approximately controllable if any state can be transfered to the neighbourhood of other state by an admissible control". In this article, we consider the approximate controllability for the system $(LS)_0$ of the form (2.1)-(2.2) in infinite dimensional Hilbert space, with $\dim(\ker D) = +\infty$. The necessary and sufficient conditions for the linear system $(LS)_0$ to be approximately reachable, and exactly controllable are also found.

# 1 Preliminaries

Let $X$ be a linear space over a field $\mathcal{F}$ of scalars ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$). Denote by $L(X)$ the set of all linear operators with domains and ranges belonging to $X$. We write

$$L_0(X) = \{A \in L(X) : \mathrm{dom}A = X\}.$$

An operator $D \in L(X)$ is said to be *right invertible* if there exists an operator $R \in L_0(X)$ such that $RX \subset \mathrm{dom}D$ and $DR = I$ on $\mathrm{dom}R$ (where $I$ is an identity operator), in this case $R$ is called a *right inverse* of $D$. The set of all right invertible operators in $L(X)$ will be denoted by $R(X)$. For a given $D \in R(X)$, we will denote by $\mathcal{R}_D$ the set of all right inverses of $D$, i.e.

$$\mathcal{R}_D = \{R \in L_0(X) : DR = I\}.$$

An operator $F \in L_0(X)$ is said to be an *initial operator* for $D$ corresponding to $R \in \mathcal{R}_D$ if $F^2 = F, FX = \ker D$ and $FR = 0$ on $\mathrm{dom}R$. The set of all initial operators for D will be denoted by $\mathcal{F}_D$.

**Proposition 1.1.** [6] *If $D \in R(X)$ then for every $R \in \mathcal{R}_D$, we have*

$$\mathrm{dom}D = RX \oplus \ker D. \tag{1.1}$$

**Theorem 1.1.** [6] *Suppose that $D \in R(X)$. A necessary and sufficient condition for an operator $F \in L(X)$ to be an initial operator for $D$ corresponding to $R \in \mathcal{R}_D$ is that*

$$F = I - RD \quad on \quad \mathrm{dom}D. \tag{1.2}$$

**Theorem 1.2.** [14] *Let $X, Y, Z$ be infinite dimensional Hilbert spaces. Suppose that $F \in L(X, Z)$ and $G \in L(Y, Z)$, then two following conditions are equivalent:*

*(i)* $\mathrm{Im}F \subset \mathrm{Im}G$ ,

*(ii) There exists $c > 0$ such that* $\|G^*f\| \geqslant c\|F^*f\|$ *for all* $f \in Z^*$.

**Theorem 1.3.** (The separation theorem) *Suppose that $M, N$ are convex sets in Banach space $X$ and $M \cap N = \emptyset$.*

*(i) If* $\mathrm{int}M \neq \emptyset$ *then there exists a functional* $x^* \in X^*, x^* \neq 0$ *and* $\lambda \in \mathbb{R}$ *such that*

$$\langle x^*, x\rangle \leq \lambda \leq \langle x^*, y\rangle, \quad \textit{for every} \quad x \in M,\, y \in N\,.$$

*(ii) If $M$ is a compact set, $N$ is a closed set then there exists* $x^* \in X^*, x^* \neq 0$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that*

$$\langle x^*, x\rangle \leq \lambda_1 < \lambda_2 \leq \langle x^*, y\rangle, \quad \textit{for every} \quad x \in M,\, y \in N\,.$$

***The theory of right invertible operators and their applications can be seen in [4, 6]. The proof of Theorems 1.2 and 1.3 can be found in [2, 14].***

# 2 The approximate controllability

Let $X$ and $U$ be infinite dimensional Hilbert spaces over the same field $\mathcal{F}$ of scalars ( $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$). Suppose that $D \in R(X)$, with $\dim(\ker D) = +\infty$, $F \in \mathcal{F}_D$ is an initial operator for $D$ corresponding to $R \in \mathcal{R}_D$, $A \in L_0(X)$, and $B \in L_0(U, X)$.

Now we will consider the linear system $(LS)_0$ of the form:

$$Dx = Ax + Bu\,,\; u \in U\,,\; BU \subset (D - A)\mathrm{dom}D, \tag{2.1}$$

$$Fx = x_0 \quad , \quad x_0 \in \ker D\,. \tag{2.2}$$

The spaces $X$ and $U$ are called the *space of states* and the *space of controls,* respectively. So that, elements $x \in X$ and $u \in U$ are called *states* and *controls,* respectively. The element $x_0 \in \ker D$ is said to be an *initial state.* A pair $(x_0, u) \in (\ker D) \times U$ is called an *input.* If (2.1)-(2.2) has solution $x = \Phi(x_0, u)$ then this solution is called *output* correspondent to input $(x_0, u)$.

Note that, since the inclusion $BU \subset (D - A)\mathrm{dom}D$ is satisfied, if the resolving operator $I - RA$ is invertible then for every fixed pair $(x_0, u) \in (\ker D) \times U$, the initial value problem (2.1)-(2.2) is well-posed and has a unique solution, which is given by (see [4, 6]):

$$\Phi(x_0, u) = E_A(RBu + x_0)\,, \text{ where } \quad E_A = (I - RA)^{-1}\,. \tag{2.3}$$

Write

$$\mathrm{Rang}_{U,x_0}\Phi = \bigcup_{u\in U} \Phi(x_0,u)\ , \quad x_0 \in \ker D\,. \tag{2.4}$$

Clearly, $\mathrm{Rang}_{U,x_0}\Phi$ is the set of all solutions of (2.1)-(2.2) for arbitrarily fixed initial state $x_0 \in \ker D$. This is reachable set from the initial state $x_0$ by means of controls $u \in U$.

**Definition 2.1.** *Let a linear system $(LS)_0$ of the form (2.1) – (2.2) be given.*

*(i) A state $x \in X$ is called approximately reachable from the initial state $x_0 \in \ker D$ if for every $\varepsilon > 0$ there exists a control $u \in U$ such that $\|x - \Phi(x_0,u)\| < \varepsilon$.*

*(ii) The linear system $(LS)_0$ is said to be approximately reachable from the initial state $x_0 \in \ker D$ if*

$$\overline{\mathrm{Rang}_{U,x_0}\Phi} = X\,.$$

**Theorem 2.1.** *The linear system $(LS)_0$ is approximately reachable from zero if and only if*

$$B^*R^*E_A^*h = 0 \quad \textit{implies} \quad h = 0\,. \tag{2.5}$$

*Proof.* By definition, the system $(LS)_0$ is approximately reachable from zero if

$$\overline{E_ARBU} = X\,. \tag{2.6}$$

According to Theorem 1.3, the condition (2.6) is equivalent to for $h \in X^*$ such that

$$\langle h, x\rangle = 0\,,\ \forall x \in \overline{E_ARBU} \Longrightarrow h = 0\,. \tag{2.7}$$

By $E_ARBU$ is subspace of $X$, (2.7) is also equivalent to

$$\langle h, x\rangle = 0\,,\ \forall x \in E_ARBU \Longrightarrow h = 0\,,$$

or equivalently

$$\langle h, E_ARBu\rangle = 0\,,\ \forall u \in U \Longrightarrow h = 0\,.$$

That is

$$\langle B^*R^*E_A^*h, u\rangle = 0\,, \forall u \in U \Longrightarrow h = 0\,.$$

This implies that

$$B^*R^*E_A^*h = 0 \Longrightarrow h = 0\,.$$

Conversely, if the condition (2.5) satisfied, then (2.7) is also satisfied, and therefore we obtain (2.6). □

**Definition 2.2.** [6] *Let be given a linear system* $(LS)_0$ *and* $F_1 \in \mathcal{F}_D$ *be arbitrary initial operator for* $D$.

(i) *A state* $x_1 \in \ker D$ *is said to be* $F_1$*-reachable from the initial state* $x_0 \in \ker D$ *if there exists a control* $u \in U$ *such that* $x_1 = F_1\Phi(x_0, u)$. *The state* $x_1$ *is called a final state.*

(ii) *The system* $(LS)_0$ *is said to be* $F_1$*-controllable if for every initial state* $x_0 \in \ker D$,

$$F_1(\mathrm{Rang}_{U,x_0}\Phi) = \ker D\,.$$

(iii) *The system* $(LS)_0$ *is said to be* $F_1$*-controllable to zero if*

$$0 \in F_1(\mathrm{Rang}_{U,x_0}\Phi)\,,$$

*for every initial state* $x_0 \in \ker D$.

**Definition 2.3.** *Let a linear system* $(LS)_0$ *of the form* $(2.1)-(2.2)$ *be given. Suppose that* $F_1 \in \mathcal{F}_D$ *is arbitrary initial operator for* $D$.

(i) *The system* $(LS)_0$ *is said to be* $F_1$*-approximately reachable from the initial state* $x_0 \in \ker D$ *if*

$$\overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)} = \ker D\,.$$

(ii) *The system* $(LS)_0$ *is said to be* $F_1$*-approximately controllable if for every initial state* $x_0 \in \ker D$, *we have*

$$\overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)} = \ker D.$$

(iii) *The system* $(LS)_0$ *is said to be* $F_1$*-approximately controllable to* $x_1 \in \ker D$ *if*

$$x_1 \in \overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)}\,,$$

*for every initial state* $x_0 \in \ker D$.

**Lemma 2.1.** *Let there be given a linear system* $(LS)_0$ *of the form* (2.1) – (2.2) *and an arbitrary initial operator* $F_1 \in \mathcal{F}_D$. *Suppose that the system* $(LS)_0$ *is* $F_1$*-approximately controllable to zero and*

$$F_1 E_A(\ker D) = \ker D\,. \tag{2.8}$$

*Then every final state* $x_1 \in \ker D$ *is* $F_1$*-approximately reachable from zero.*

*Proof.* By the assumption, $0 \in \overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)}$, for all $x_0 \in \ker D$. Therefore, for every $x_0 \in \ker D$ and $\varepsilon > 0$, there exists a control $u_0 \in U$ such that

$$\|F_1 E_A(RBu_0 + x_0)\| < \varepsilon\,. \tag{2.9}$$

The condition (2.8) implies that with any $x_1 \in \ker D$, there exists $x_2 \in \ker D$ such that

$$F_1 E_A x_2 = -x_1\,.$$

This equality and (2.9) together imply that for every $x_1 \in \ker D$ and $\varepsilon > 0$, there exists a control $u_1 \in U$ such that

$$\|F_1 E_A RBu_1 - x_1\| < \varepsilon\,.$$

This proves that every final state $x_1$ is $F_1$-approximately reachable from zero. □

**Theorem 2.2.** *Suppose that all assumptions of Lemma 2.1 are satisfied. Then the system* $(LS)_0$ *is* $F_1$*-approximately controllable.*

*Proof.* According to our assumption, for every $x_0 \in \ker D$ and $\varepsilon > 0$, there exists a control $u_0 \in U$ such that

$$\|F_1 E_A(RBu_0 + x_0)\| < \frac{\varepsilon}{2}\,. \tag{2.10}$$

By Lemma 2.1, for every $x_1 \in \ker D$ there exists $u_1 \in U$ such that

$$\|F_1 E_A RBu_1 - x_1\| < \frac{\varepsilon}{2}\,. \tag{2.11}$$

From (2.10) and (2.11), it follows that for every $x_0, x_1 \in \ker D$ and $\varepsilon > 0$, there exists a control $u = u_0 + u_1 \in U$ such that

$$\begin{aligned}\|F_1 E_A(RBu + x_0) - x_1\| &= \|F_1 E_A[RB(u_0 + u_1) + x_0] - x_1\|\\ &\leq \|F_1 E_A(RBu_0 + x_0)\| + \|F_1 E_A RBu_1 - x_1\|\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

The arbitrariness of $x_0, x_1 \in \ker D$ and $\varepsilon > 0$ implies $\overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)} = \ker D$. □

**Theorem 2.3.** *Let there be given a linear system $(LS)_0$ and an arbitrary initial operator $F_1 \in \mathcal{F}_D$. Then the system $(LS)_0$ is $F_1$-approximately controllable if and only if it is $F_1$-approximately controllable to every element $y' \in F_1 E_A R X$.*

*Proof.* The necessary condition is easy to be obtained. In order to prove the sufficient condition, first we prove the equality

$$F_1 E_A (RX \oplus \ker D) = \ker D\,. \tag{2.12}$$

Indeed, since $(I - RA)(\mathrm{dom}D) \subset \mathrm{dom}D = RX \oplus \ker D$ (by Proposition 1.1 and property of the right invertible operator), there exists a set $E \subset X$ and $Z \subset \ker D$ such that

$$RE \oplus Z = (I - RA)(\mathrm{dom}D)\,.$$

This implies $E_A(RE \oplus Z) = E_A(I - RA)(\mathrm{dom}D) = \mathrm{dom}D$. Thus, we have

$$\begin{aligned} \ker D = F_1(\mathrm{dom}D) &= F_1 E_A (RE \oplus Z) \\ &\subset F_1 E_A (RX \oplus \ker D) \\ &\subset \ker D\,. \end{aligned}$$

Therefore, the formula (2.12) holds.

Suppose that the system $(LS)_0$ is $F_1$-approximately controllable to every element $y' = F_1 E_A R y$, $y \in X$, i.e. for every $y \in X$ and arbitrary $\varepsilon > 0$ there exists a control $u_0 \in U$ such that

$$\|F_1 E_A (RBu_0 + x_0) - F_1 E_A R y\| < \frac{\varepsilon}{2}\,.$$

That is

$$\|F_1 E_A (RBu_0 + x_0 + x_2) - F_1 E_A (Ry + x_2)\| < \frac{\varepsilon}{2}\,. \tag{2.13}$$

where $x_2 \in \ker D$ is arbitrary.

By the formula (2.12), for every $x_1 \in \ker D$, there exists $y_1 \in X$ and $x_2' \in \ker D$ such that

$$x_1 = F_1 E_A (Ry_1 + x_2')\,.$$

This equality and (2.13) together imply

$$\|F_1 E_A(RBu_0' + x_0 + x_2') - x_1\| < \frac{\varepsilon}{2}\,. \tag{2.14}$$

On the other hand, since $0 \in F_1 E_A RX$ and our assumptions allows that $(LS)_0$ is $F_1$-approximately controllable to zero, i.e.

$$0 \in \overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)}\,, \quad \text{for arbitrary } x_0 \in \ker D\,.$$

Thus, for the element $x_2' \in \ker D$ there exists $u_1 \in U$ such that

$$\|F_1 E_A(RBu_1 - x_2')\| < \frac{\varepsilon}{2}\,. \tag{2.15}$$

From (2.14) and (2.15), it is concluded that for every $x_0, x_1 \in \ker D$ and $\varepsilon > 0$ there exist $u = u_0' + u_1 \in U$ such that

$$\begin{aligned}
&\|F_1 E_A(RBu + x_0) - x_1\| \\
&\quad = \|F_1 E_A[RB(u_0' + u_1) + x_0] - x_1\| \\
&\quad = \|F_1 E_A(RBu_0' + x_0 + x_2') - x_1 + F_1 E_A(RBu_1 - x_2')\| \\
&\quad \le \|F_1 E_A(RBu_0' + x_0 + x_2') - x_1\| + \|F_1 E_A(RBu_1 - x_2')\| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

By the arbitrariness of $x_0, x_1 \in \ker D$ and $\varepsilon > 0$, we obtain $\overline{F_1(\mathrm{Rang}_{U,x_0}\Phi)} = \ker D$. ∎

**Theorem 2.4.** *Let a linear system $(LS)_0$ and an arbitrary initial operator $F_1 \in \mathcal{F}_D$ be given. Then the system $(LS)_0$ is $F_1$-approximately reachable from zero if and only if*

$$B^* R^* E_A^* F_1^* h = 0 \quad \textit{implies} \quad h = 0\,. \tag{2.16}$$

*Proof.* Suppose that the system $(LS)_0$ is $F_1$-approximately reachable from zero, we have

$$\overline{F_1(\mathrm{Rang}_{U,0}\Phi)} = \ker D\,.$$

This means that

$$\overline{F_1 E_A RBU} = \ker D\,. \tag{2.17}$$

According to Theorem 1.3, the equality (2.17) is equivalent to for $h \in (\ker D)^*$ such that

$$\langle h, x\rangle = 0\,,\ \forall x \in \overline{F_1 E_A RBU} \implies h = 0\,. \tag{2.18}$$

Because $F_1E_ARBU$ is a subspace of ker$D$, the condition (2.18) is also equivalent to

$$\langle h, x\rangle = 0\,,\ \forall x \in F_1E_ARBU \implies h = 0\,,$$

or equivalently

$$\langle h, F_1E_ARBu\rangle = 0\,,\ \forall u \in U \implies h = 0\,.$$

It is satisfied if and only if

$$\langle B^*R^*E_A^*F_1^*h, u\rangle = 0\,,\ \forall u \in U \implies h = 0\,. \tag{2.19}$$

Hence, the condition (2.19) allows $B^*R^*E_A^*F_1^*h = 0$ which implies $h = 0$.
Conversely, if (2.16) is satisfied then (2.19) holds. This implies (2.17) and therefore we obtain

$$\overline{F_1(\mathrm{Rang}_{U,0}\Phi)} = \ker D\,.$$

□

**Theorem 2.5.** *A necessary and sufficient condition for the system* $(LS)_0$ *to be* $F_1$*-controllable is that there exists a real number* $\alpha > 0$ *such that*

$$\|B^*R^*E_A^*F_1^*f\| \geq \alpha\|f\|\,, \quad \textit{for all} \quad f \in (\ker D)^*\,. \tag{2.20}$$

*Proof.* ***Necessity.*** Suppose that the system $(LS)_0$ is $F_1$- controllable, we have

$$F_1(\mathrm{Rang}_{U,x_0}\Phi) = \ker D\,, \quad \text{for every} \quad x_0 \in \ker D\,.$$

It implies that $F_1E_ARBU = \ker D$. By Theorem 1.2, there exists a real number $\alpha > 0$ such that

$$\|(F_1E_ARB)^*f\| \geq \alpha\|f\|\,, \quad \text{for all} \quad f \in (\ker D)^*\,,$$

i.e. the condition (2.20) is held.
***Sufficiency.*** Suppose that the condition (2.20) is satisfied, by Theorem 1.2, we obtain

$$F_1E_ARBU \supseteq \ker D$$

Moreover, $F_1E_ARBU \subseteq \ker D$, since $F_1$ is an initial operator for $D$. Consequently, we have $F_1E_ARBU = \ker D$. It implies that

$$F_1(\mathrm{Rang}_{U,x_0}\Phi) = \ker D\,, \quad \text{for every} \quad x_0 \in \ker D\,.$$

□

**Theorem 2.6.** *The linear system* $(LS)_0$ *is* $F_1$*-controllable to zero if and only if there exists* $\beta > 0$ *such that*

$$\|B^* R^* E_A^* F_1^* f\| \geq \beta \|E_A^* F_1^* f\| , \quad \textit{for every} \quad f \in (\ker D)^* . \tag{2.21}$$

*Proof.* Suppose that the system $(LS)_0$ is $F_1$- controllable to zero. This means that

$$0 \in F_1(\mathrm{Rang}_{U,x_0}\Phi) , \quad \text{for all} \quad x_0 \in \ker D .$$

Therefore, for arbitrary $x_0 \in \ker D$, there exists $u \in U$ such that

$$F_1 E_A (RBu + x_0) = 0 .$$

It allows that for every $x'_0 \in \ker D$, there exists $u' \in U$ such that $F_1 E_A x'_0 = F_1 E_A RBu'$. Thus, we obtain $F_1 E_A(\ker D) \subseteq F_1 E_A RBU$. Using Theorem 1.2, there exists $\beta > 0$ such that

$$\|(F_1 E_A RB)^* f\| \geq \beta \|(F_1 E_A)^* f\| , \quad \text{for all} \quad f \in (\ker D)^* .$$

Conversely, Assume that (2.21) is satisfied, according to Theorem 1.2, we conclude that

$$F_1 E_A(\ker D) \subseteq F_1 E_A RBU .$$

Hence, for every $x_0 \in \ker D$, there exists $u \in U$ such that

$$F_1 E_A (RBu + x_0) = 0 ,$$

i.e. the system $(LS)_0$ is $F_1$-controllable to zero. □

**Example.** Consider the control system

$$\frac{\partial x(t,s)}{\partial t} = \lambda x(t,s) + u(t) , \tag{2.22}$$

with initial condition

$$x(0,s) = f(s) . \tag{2.23}$$

Let $X = C(\mathbb{R}^2)$ be the space of all continous functions over $\mathbb{R}^2$. Write $D = \frac{\partial}{\partial t}$, $R = \int_0^t$. It is possible to check that $\mathrm{dom} D = \{x \in X : x(t, s_0) \in C^1(\mathbb{R})$ for every fixed $s_0 \in \mathbb{R}\}$, $\ker D = \{x \in X : x(t,s) =$

$\varphi(s),\ \varphi \in C(\mathbb{R})\}$. Thus, we have dim(ker$D$)=$+\infty$, and dom$R = X$. In addition,

$$(DRx)(t,s) = \frac{\partial}{\partial t}\left(\int_0^t x(\tau,s)d\tau\right) = (Ix)(t,s)\,, \text{ for all } x \in X.$$

So the operator $D$ is right invertible and $R$ is a right inverse of $D$. The initial operator for $D$ corresponding to $R$ is defined by $(Fx)(t,s) = (I - RD)x(t,s) = x(0,s)$.

Moreover, for every $t_i \in \mathbb{R}, i = 1,2,3,...$ let $R_i = \int_{t_i}^{t}$, then $R_i$ are right inverses of $D$, and $F_i x(t,s) = x(t_i, s)$ are the initial operators for $D$ corresponding to $R_i$, respectively (see[6]).

Therefore, the problem (2.22)-(2.23) can be rewritten in the form:

$$Dx = Ax + Bu\,,\ u \in U \tag{2.24}$$

$$Fx = x_0\,, \quad x_0 \in \ker D\,. \tag{2.25}$$

Where $A = \lambda I$, $B = I$ are stationary operators, since $AD = DA, AR = RA, BD = DB$ and $BR = RB$. The set $U = C(\mathbb{R})$ is the space of all continous functions over $\mathbb{R}$. If write

$$(Cx)(t,s) = \int_0^t e^{\lambda(t-\tau)} x(\tau,s)d\tau\,,$$

then

$$(I + \lambda C)(I - \lambda R)x(t,s) = (I - \lambda R)(I + \lambda C)x(t,s) = Ix(t,s)\,.$$

This means that the resolving operator $I - \lambda R$ is invertible and its inverse is given by

$$(E_A x)(t,s) = (I - \lambda R)^{-1}x(t,s) = (I + \lambda C)x(t,s)$$
$$= x(t,s) + \lambda \int_0^t e^{\lambda(t-\tau)} x(\tau,s)d\tau\,.$$

Hence, for every $u(t) \in C(\mathbb{R})$, by formula (2.3), the solution of (2.24)-(2.25) ( which is also the solution of (2.22)-(2.23)) is given by

$$x(t,s) = E_A(RBu + x_0)(t,s) = e^{\lambda t}\left(\int_0^t e^{-\lambda\tau} u(\tau)d\tau + f(s)\right)\,.$$

In addition, it is easy to check that $F_1 E_A x_0 = e^{\lambda(t_1)} x_0 = S(t_1)x_0$, for every $x_0 \in \ker D$. Where $S(t)$ is a semigroup of continuous linear operators generated by $A$.

Since $B$ is a stationary operator and $\ker R = \{0\}$, in this case, the condition (2.16) is equivalent to $B^* E_A^* F_1^* h = 0$ which implies $h = 0$ or $B^*(F_1 E_A)^* h = 0$ implies $h = 0$. This means that

$$B^* S^*(t_1) h = 0 \quad \Longrightarrow \quad h = 0 \,. \tag{2.26}$$

Note that, the condition (2.26) is necessary and sufficient for the linear system in infinite dimensional space to be approximately reachable (see [14]). For the system (2.22)-(2.23), the condition (2.26) is completely satisfied. Hence, by Theorem 2.4, the system (2.24)-(2.25) is $F_1$-approximately reachable from zero.

This example shows that in the case $D$ is a differential operator, the concept and results of $F_1$-approximately controllable are completely coincident with the approximate controllability of the linear control system in infinite dimensional space.

*Acknowledgements.* I would like to take this opportunity to thank Professor Nguyen Dinh Quyet and Professor Nguyen Van Mau for their great contributions to this article in terms of correction and fulfilment.

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