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COMPLEX SYSTEMS AND MATHEMATICAL MODELS

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ABSTRACT. We shall present several mathematical models for the various complex systems using the nonlinear diffusion system of equations. We shall also present some simulation results which show excellent correlation with observations.

keywords: complex system, nonlinear diffusion system, chemotaxis-growth.

INTRODUCTION

In recent years many scientists try to formulate nonlinear diffusion systems to describe essential aspcts of the various complex systems. Its general form is in fact written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \nabla \cdot \{u\nabla\chi_1(v)\} + f(u,v), \\ \dfrac{\partial v}{\partial t} = b\Delta v + \nabla \cdot \{v\nabla\chi_2(u)\} + g(u,v). \end{cases}$$

Here, $u = u(x,t)$ denotes the concentration of a substance or a biological species, say A, at a position $x \in \Omega$ and a time $t \in [0,\infty)$ which disperses in a region Ω of $\mathbb{R}^d$ $(d = 1,2,3)$, and $v = v(x,t)$ denotes the concentration of another substance or another biological species, say B, which similarly disperses in the same region Ω.

We incorporate into the model three effects, that is, diffusions, interactions, and reactions. A and B have the nature of random walking in Ω mutually independently with the diffusion rates a and b, respectively. On the other hand, they have mobility directed by interactions, $\chi_1(v)$ and $\chi_2(u)$ are potential functions or sensitivity functions. If $\chi_1'(v) \geq 0$, then A moves to evade B. If $\chi_1(v)' \leq 0$, then A moves to pursue B. Reactions in the model are governed by the two functions $f(u,v)$ and $g(u,v)$. If $f_v(u,v) \geq 0$, then B acts as a activator of A. If $f_v(u,v) \leq 0$, then B acts as an inhibitor of A.

SOME EXAMPLE IN PHYSICS, BIOLOGY, AND ECONOMICS

Let us present some mathematical models.

Absorbate-Induced Phase Transition Model. Hildebrand et al. (1999) have shown that microreactors with submicroreactor and nanometer sizes may spontaneously develop in surface chemical reactions by a nonequilibrium self-organization process. The self-organized micrometers represent localized structure resulting from the interplay between the reaction, diffusion, and an absorbate-induced structure transformation of the surface. They assumed also that the free energy is associated with the first-order surface phase transition due to the adsorption of chemical substance.

Their system is in fact written as follows:

$$\text{(AI)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + du(u+v-1)(1-u), \\ \dfrac{\partial v}{\partial t} = b\Delta v + \nabla \cdot \{v(1-v)\nabla\chi(u)\} - fe^{\alpha\chi(u)}v \\ \qquad\qquad - gv + h(1-v). \end{cases}$$

Here, $u(x,t)$ denotes the order parameter and takes values in $0 \le u \le 1$. The order parameter denotes the thermodynamic state of the surface $\Omega \subset \mathbb{R}^2$. And $v(x,t)$ denote the absorbate coverage rate of the surface by the carbon monoxide, $v(x,t)$ also takes values in $0 \le v \le 1$. The coverage rate v has a tendency to move toward lower values of the chemical potential $\chi(u)$ with the rate $1-v$. We may note from the first-order phase transition that a prototype of $\chi(u)$ is $\chi(u) = -cu^2(3-2u)$. v decreases with the rate $fe^{\alpha\chi(u)}$. The order parameter u has three equilibrium points 0, $1-v$, and 1; 0 and 1 are stable, on the other hand $1-v$ is unstable.

Chemotaxis-Growth Model. Mimura and Tsujikawa (1996) modeled by a simple diffusion system the aggregating pattern formation of biological individuals due to the effects of chemotaxis and growth. Their equations are written as

$$\text{(CG)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla \cdot \{u\nabla\chi(\rho)\} + f(u), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - c\rho + du. \end{cases}$$

Here, $u(x,t)$ denotes the population density of biological individuals which disperse in a region $\Omega \subset \mathbb{R}^2$. And $\rho(x,t)$ denotes the concentration of a chemical substance produced by the amebas. The amebas have the directed mobility toward higher concentration of the chemical substance. This nature is called chemotaxis in biology. $\chi(\rho)$ is the sensitivity function of the individuals; the normalized forms of $\chi(\rho)$ are, for example,

$$\chi(\rho) = \rho, \ \ \rho^2, \ \ \log\rho, \ \ \tfrac{\rho}{\rho+1}$$

and so on. $f(u)$ is the growth term of u; a prototype is $f(u) = fu(u-\alpha)(1-u)$, where $f > 0$ and $0 < \alpha < 1$ are constants, with two stable equilibriums 0 and 1 and one unstable equilibrium $0 < \alpha < 1$.

Labor Mobility and Urbanization Model. Let us now derive a growth model of macro economics incorporating the mobility of labor and of capital in an economic sector $\Omega \subset \mathbb{R}^2$. Let $L(x,t)$ denotes the population density of labor, and $K(x,t)$ denotes the concentration of capital. The product function is denoted by $F = F(L,K)$, here we use the production function of Cobb-Douglas type, namely

$$F(L,K) = \gamma L^{\alpha} K^{\beta},$$

where $\gamma > 0$ is a constant and α and β are two exponents such that $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. Our model centers on two concentrations, cf. Takagi et al. (1995):
(i) Labor have a tendency to move toward higher earnings and the earnings are in proportion to $F(L,K)/L = \gamma(K/L)^{\beta}$.
(ii) Capital move toward higher profit rates which are in proportion to $F(L,K)/K = \gamma(L/K)^{\alpha}$.

Our model is then governed by the following nonlinear system of equations:

$$\text{(LK)} \qquad \begin{cases} \dfrac{\partial L}{\partial t} = a\Delta L - \nabla \cdot \{L \nabla \chi_1(\frac{K}{L})\} + f(L), \\ \dfrac{\partial K}{\partial t} = -\nabla \cdot \{K \nabla \chi_2(\frac{L}{K})\} - \mu K + \gamma(1-\delta) L^{\alpha} K^{\beta}. \end{cases}$$

a is a diffusion rate of the labor. $\chi_1(K/L)$ is a sensitivity function of L toward the capital labor ratio, and $\chi_2(L/K)$ is a sensitivity function of K toward the labor capital ratio. $1-\delta$ is an investment rate of F, where δ is a constant such that $0 < \delta < 1$. μ is a rate of the capital depreciation. Finally, $f(L)$ is a growth term of the labor.

CHEMOTAXIS-GROWTH MODEL

In their paper, Woodward et al. (1995) found remarkable pattern formation by Salmonella typhimurium. They inoculated typhimurium on agar in a shot. Bacteria propagated themselves and spread out over agar forming bold aggregating patterns. By the conditions of medium, different patterns were observed. If agar contained much nourishment, bacteria spread out forming concentric circles. If nourishment was little, they aggregated in a certain number of spot points and the spot points were placed very regularly on concentric circumferences.

To understand theoretically such chemotactic pattern formation, several models have been proposed by Alt (1985), Woodward et al. (1995), and Kawasaki et al. (Preprint). Among them, Mimura and Tsujikawa presented the very simple model (CG) above, centering on the three effects, diffusions, chemotaxis, and growth. The authors of the present article have performed numerical simulations for Mimura-Tsujikawa equations. So far three kinds of pattern solutions were found by choosing various parameters appropriately. Fig.1 shows a solution of the target pattern or the concentric circles. Fig. 2 shows a solution of the spot pattern. Fig.3 shows the network pattern. These simulations show us excellent correlation between the experimental results and the nonlinear diffusion model.

CONCLUSION

Many scientists are trying to model the complex systems as nonlinear diffusion systems incorporating interactions and reactions for restricted two species. In some models very good accordance with observations is reported in physics, biology, and economics. It might now be a good position to import such strategy in order to investigate the complex systems in environmental science.

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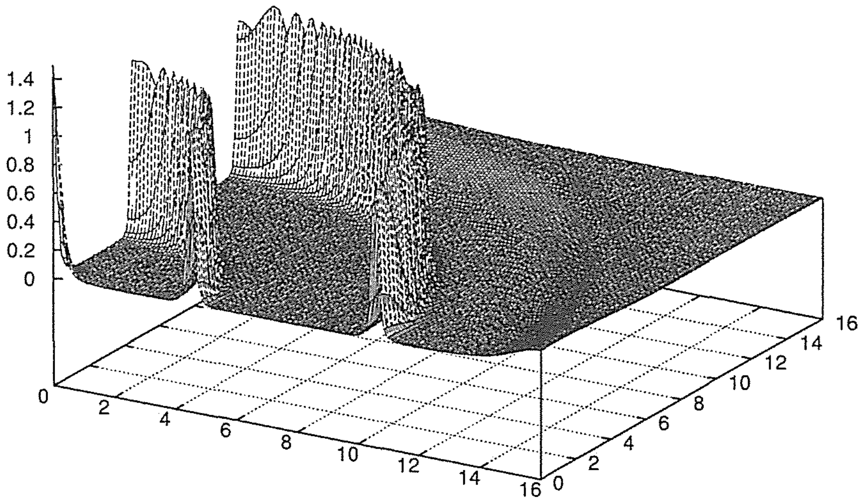

Fig.1 : Target pattern

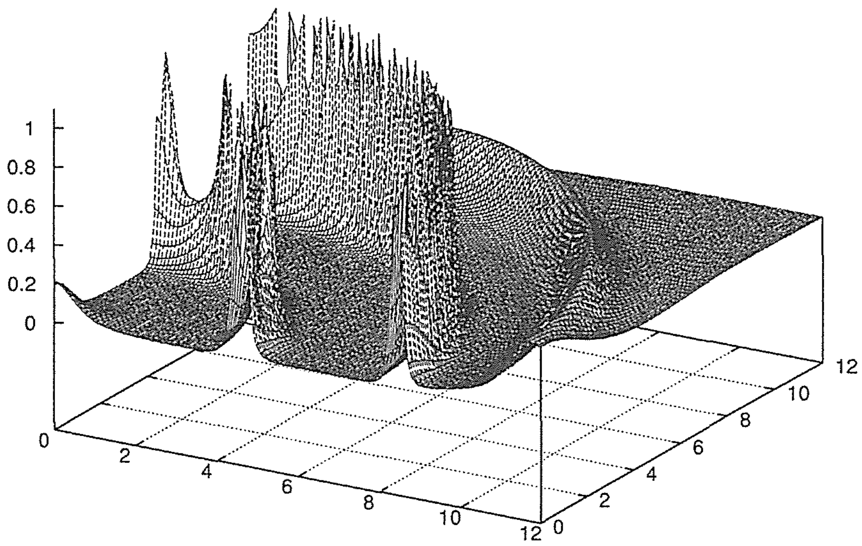

Fig.2 : Spot pattern

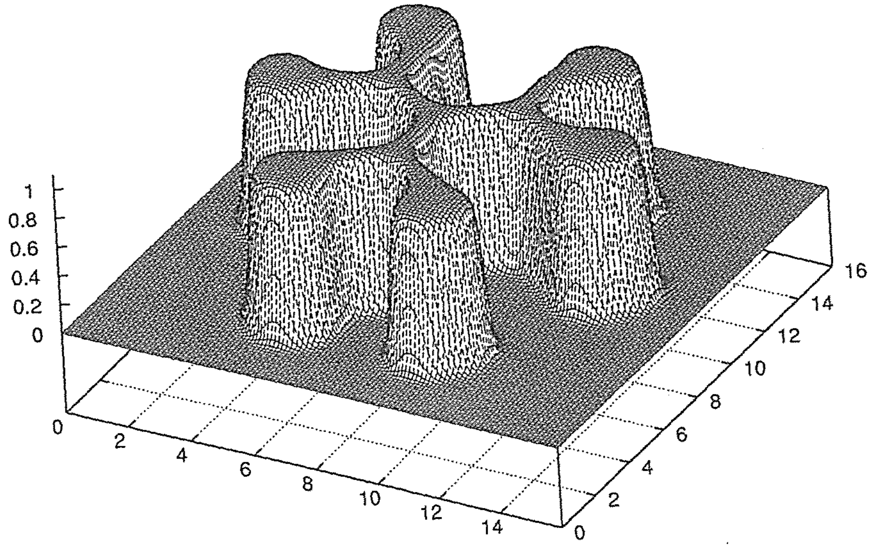

Fig.3 : Network pattern