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ON THE GENERALIZED CONVOLUTION WITH A WEIGHT-FUNCTION FOR THE FOURIER AND COSINE-FOURIER INTEGRAL TRANSFORMS

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Abstract. The generralized convolution for Fourier and cosine-Fourier transforms is introduced. Its properties and applications to the heat conduction equations are considered.

I. Introduction

The generalized convolution of the Fourier sine and cosine transforms was first introduced by Chuchill R. V. In 1941 [2]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} f(y)\big[g(|x-y|) - g(x+y)\big]\,dy, \quad x > 0$$

for which the factorization property holds

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y), \quad \forall y > 0.$$

In the 90s decate of the last century, Yakubovic S. B. published some papers on special cases of generalized convolutions for integral transdorm according to index [9, 10, 11]. In 1998, Kakichev V. A. and Nguyen Xuan Thao [3] proposesed a contructive method of defining the generalized convolution for any integral transforms K_1, K_2, K_3 with the weight-function $\gamma(x)$ of functions f, g for which we have the factorization property:

$$K_1(f \overset{\gamma}{*} g)(y) = \gamma(y)(K_2 f)(y)(K_3 g)(y)$$

In recent years, there have been publishied some works on the generalized convoltuion, for instance the generalized convolution for integral transforms Stieltijes Hilbert and the cosine-sine transforms [5]; the generalized convolution for H-transform [4]; the generalized convolution for I-transform [7]. For example, the generalized convolution for the Fourier consine and sine has been defined [6] by the identify

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\big[\sin y(y-x) g(|y-x|) + g(y+x)\big] dy, \quad x > 0$$

for which the factorization property holds

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0$$

In this article we give a notion of the generalized convolution with a weight-function of function f and g for the Fourier and Fourier cosine transforms. We will prove some of its properties as well as point out its relationship to the cosine-Fourier convolution Finaly we will apply this notion to solving systems of integral equations and an the heat conduetion equaiton.

II. Generalized convolution for the Fourier and cosine-Fourier transorms

Definition 1. *Generalized convolution with the weight-function $\gamma(y) = \cos y$ for the Fourier and Fourier cosine transforms of functions f and g is defined by*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(t)\big[g(x+1+t)+ \\ g(|x+1-t|)+g(|x-1+t|)+g(|x-1-t|)\big] dt, \;\; x > 0 \tag{1}$$

Remark.

1) For the Fourier integral transform of even function we have [1]:

$$(Ff)(y) = (F_c f)(y), \quad \forall y \in R. \tag{2}$$

2) The convolution (1) is even function.

We denote by $L(R_+)$ the set of all functions f defined on $(-\infty, \infty)$ such that $\int_{-\infty}^{+\infty} |f(x)| dx < +\infty$. We denote by $L(R_+)$.

Theorem 1. *Let f and g be continuous in $L(R_+)$. Then the generalized convolution (1) belongs to $L(R)$ and the factorization property holds*

$$F(f \overset{\gamma}{*} g)(y) = \cos y (F_c f)(|y|)(F_c g)(|y|), \quad \forall y \in R \tag{3}$$

Proof. Base on (1), (2) and the hypothesis that f and $g \in L(R_+)$ we have

$$\int_{-\infty}^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big| dx = 2 \int_{0}^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big| dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |f(t)| \times$$

$$\times \Big| \big[g(|x+1+t|) + g(|x+1-t|) + g(|x-1+t|) + g(|x-1-t|) \big] dt dx$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |f(t)| \Big[\int_{0}^{+\infty} |g(|x+1+t|) dx + \int_{0}^{+\infty} |g(x+1-t)| dx +$$

$$+ \int_{0}^{+\infty} |g(|x-1+t|) dx + \int_{0}^{+\infty} |g(|x-1-t|) \big] dt$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |f(t)| dt.4 \int_{0}^{+\infty} |g(u)| du < +\infty \Rightarrow (f * g)(x) \in L(R)$$

On the other hand

$$(f * g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(t) \big[g(|x+1+t|) +$$

$$g(|x+1-t|) + (|x-1+t|) + g(|x-1-t|) \big] dt$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(t) \Big[\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x+1+t| (F_c g)(|\tau|) d\tau +$$

$$\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x+1-t| \times F_c g(|\tau|) d\tau + \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x-1+$$

$$t| (F_c g)(|\tau|) d\tau + \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x-1-t| F_c g(|\tau|) d\tau \Big] dt.$$

Because

$$\cos \tau \cos(\tau t) \cos(\tau x) = \frac{1}{4} \big[\cos \tau (x+1+t) +$$

$$\cos \tau (x+1-t) + \cos \tau (x-1+t) + \cos \tau (x-1-t) \big],$$

we have

$$(f \overset{\gamma}{*} g)(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \cos(\tau t) f(t) dt \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \cos(\tau x) \cos\tau (F_c g)(|\tau|) d\tau$$
$$= \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \cos(\tau x) \cos\tau (F_c f)(|\tau|)(F_c g(|\tau|) d\tau$$

From (2) and F_c is symmetrical [8]

$$F(f \overset{\gamma}{*} g)(y) = F_c(\overset{\gamma}{*} g)(y) = F_c\{F_c\{\cos\tau(F_c f)(|\tau|)(F_c g)(|\tau|), x\}, y\}$$
$$= \cos y (F_c f)(|y|)(F_c g)(|y|), \quad \forall y \in R.$$

Theorem 2. *In the space of continuous belonging to $L(R)$, the generalized convolution (1) is commutative, associative and distributive.*

Proof. We prove the generalized convolution with a weight-function for Fourier and cosine-Fourier integral transforms is associative, i.e.

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = f \overset{\gamma}{*} (g \overset{\gamma}{*} h)$$

Indeed,

$$F\big[(f \overset{\gamma}{*} g) \overset{\gamma}{*} h\big](y) = \cos y F_c[f \overset{\gamma}{*} g](|y|)(F_c h)(|y|)$$
$$= \cos y \cos y (F_c f)(|y|)(F_c g)(|y|)(F_c h)(|y|)$$
$$= \cos y (F_c f)(|y|)\big[\cos y (F_c g)(|y|)(F_c h)(|y|)\big]$$
$$= \cos y (F_c f)(|y|) F_c\big[g \overset{\gamma}{*} h\big](|y|) = F\big[f \overset{\gamma}{*} (g \overset{\gamma}{*} h)\big](y), \quad \forall y \in R$$

implies that

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = f \overset{\gamma}{*} (g \overset{\gamma}{*} h).$$

The commutative, distributive properties are similary proved. The theorem is proved.

Theorem 3. *If f and g are cojtinuous functions in $L(R)$, then the following equality hold*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}\big[(f \underset{F_c}{*} g)(|x+1|) + (f \underset{F_c}{*} g)(|x-1|)\big], \quad \forall x \in R \qquad (4)$$

Here $(f \underset{F_c}{*} g)$ is the convolution of $f(x)$ and $g(x)$ for cosine Fourier integral transform [8]

Theorem 4. *In the space of continuous in $L(R)$ there does not exists the unit element for the operation of the generalized convolution* (1). *Set*

$$L(e^{-x}, R) = \{e^{-x} f_1, \quad f_1 \in L(R)\}$$

Theorem 5. (Tichmarch type theorem)

Let $f, g \in L(e^{-x}, R_+)$. *If* $(f \overset{\gamma}{*} g)(x) \equiv 0, \ \ \forall x \in R$, *then either* $f(x) = 0$ *or* $g(x) = 0, \ \ \forall x \in R$.

III. Application to solving the heat conduction equaiton

Consider the heat conduction euqation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} \tag{5}$$

with initial condition $u(x,0) = f(x)$, $x \in R$, $f(x)$ is even function and $u(x,t) \to 0$, $\dfrac{\partial u(x,t)}{\partial x} \to 0$ when $|x| \to \infty$.

Theorem 6. *The above problem has asolution* $u_0(x,t)$ *which is satisfies*

1) $u_0(x,t)$ *is even function.*

2) $u_0(x+1,t) + u_0(x-1,t) = A(t)(f \overset{\gamma}{*} g)(x)$ *with* $A(t) = \sqrt{\frac{2}{t}}, g(v) = e^{-\frac{v^2}{4t}}$.

Proof. In [8], the equation (5) has a solution

$$u_0(x,t) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi$$

+) when $x \geq 0$, we have

$$\begin{aligned} u_0(x,t) &= \frac{1}{2\sqrt{\pi t}} \int\limits_{0}^{+\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi + \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{0} f(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi \\ &= \frac{1}{2\sqrt{\pi t}} \Big\{ \int\limits_{0}^{+\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi - \int\limits_{-\infty}^{0} f(y) e^{\frac{(x+y)^2}{4t}} dy \Big\} \\ &= \frac{1}{2\sqrt{\pi t}} \Big\{ \int\limits_{0}^{+\infty} f(\xi) g(|x-\xi|) d\xi + \int\limits_{-\infty}^{0} f(y)(x+y) dy \Big\} \\ &= \frac{1}{2\sqrt{\pi t}} (f \underset{F_c}{*} g)(x) \end{aligned} \tag{6}$$

+) Similaly, when $x < 0$ we get

$$u_0(x,t) = \frac{1}{\sqrt{2t}} (f \underset{F_c}{*} g)(-x) \tag{7}$$

From (6), (7) we obtain

$$u_0(x,t) = \frac{1}{\sqrt{2t}}(f \underset{F_c}{*} g)(|x|) \tag{8}$$

i.e. $u(x,t)$ is even function for x.

From (8) we have

$$\begin{aligned} u_0(x+1,t) &= \frac{1}{\sqrt{2t}}(f \underset{F_c}{*} g)(|x+1|) \\ u_0(x-1,t) &= \frac{1}{\sqrt{2t}}(f \underset{F_c}{*} g)(|x-1|) \end{aligned} \tag{9}$$

From (9) and Theorem 3 we have

$$u_0(x+1,t) + u_0(x-1,t) = A(t)(f \overset{\gamma}{*} g)(x), \quad \forall x \in R,;\ A(t) = \sqrt{\frac{2}{t}}.$$

The theorem is proved.

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