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ON THE GENERALIZED CONVOLUTION
WITH A WEIGHT-FUNCTION FOR THE FOURIER
AND COSINE-FOURIER INTEGRAL TRANSFORMS

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Abstract. The generalized convolution for Fourier and cosine-Fourier transforms is introduced. Its properties and applications to the heat conduction equations are considered.

I. Introduction

The generalized convolution of the Fourier sine and cosine transforms was first introduced by Chuchill R. V. in 1941 [2]

\[(f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) [g(|x - y|) - g(x + y)] dy, \quad x > 0\]

for which the factorization property holds

\[F_s(f \ast g)(y) = (F_s f)(y)(F_c g)(y), \quad \forall y > 0.\]

In the 90s decade of the last century, Yakubovic S. B. published some papers on special cases of generalized convolutions for integral transforms according to index [9, 10, 11]. In 1998, Kakichev V. A. and Nguyen Xuan Thao [3] proposed a constructive method of defining the generalized convolution for any integral transforms \(K_1, K_2, K_3\) with the weight-function \(\gamma(x)\) of functions \(f, g\) for which we have the factorization property:
In recent years, there have been published some works on the generalized convolution, for instance the generalized convolution for integral transforms Stieltjes Hilbert and the cosine-sine transforms [5]; the generalized convolution for H-transform [4]; the generalized convolution for I-transform [7]. For example, the generalized convolution for the Fourier cosine and sine has been defined [6] by the identity

\[ (f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y) \left[ \sin y (y - x) + g(y + x) \right] dy, \quad x > 0 \]

for which the factorization property holds

\[ F_c(f \ast g)(y) = (Fsf)(y)(Fsg)(y), \quad \forall y > 0 \]

In this article we give a notion of the generalized convolution with a weight-function of function \( f \) and \( g \) for the Fourier and Fourier cosine transforms. We will prove some of its properties as well as point out its relationship to the cosine-Fourier convolution. Finally we will apply this notion to solving systems of integral equations and the heat conduction equation.

II. Generalized convolution for the Fourier and cosine-Fourier transforms

**Definition 1.** Generalized convolution with the weight-function \( \gamma(y) = \cos y \) for the Fourier and Fourier cosine transforms of functions \( f \) and \( g \) is defined by

\[
(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(t) \left[ g(x+1+t) + g(x-1-t) \right] dt, \quad x > 0
\]

**Remark.**

1) For the Fourier integral transform of even function we have [1]:

\[ (Ff)(y) = (F_c f)(y), \quad \forall y \in R. \]

2) The convolution (1) is even function.

We denote by \( L(R_+) \) the set of all functions \( f \) defined on \((-\infty, \infty)\) such that \( \int_{-\infty}^{+\infty} |f(x)| dx < +\infty \). We denote by \( L(R_+) \).

**Theorem 1.** Let \( f \) and \( g \) be continuous in \( L(R_+) \). Then the generalized convolution (1) belongs to \( L(R) \) and the factorization property holds
Proof. Base on (1), (2) and the hypothesis that \( f \) and \( g \) \( \in \mathcal{L}(R_+) \) we have

\[
\int_{-\infty}^{+\infty} |(f \ast g)(x)| \, dx = 2 \int_{0}^{+\infty} |(f \ast g)(x)| \, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |f(t)| \times \]
\[
\times \left[ g(|x + 1 + t|) + g(|x + 1 - t|) + g(|x - 1 + t|) + g(|x - 1 - t|) \right] \, dt \, dx
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |f(t)| \left[ \int_{0}^{+\infty} |g(|x + 1 + t|)| \, dx + \int_{0}^{+\infty} |g(x + 1 - t)| \, dx + 
\int_{0}^{+\infty} |g(|x - 1 + t|)| \, dx + \int_{0}^{+\infty} |g(|x - 1 - t|)| \, dt \right]
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |f(t)| \, dt \cdot \int_{0}^{+\infty} |g(u)| \, du < +\infty \Rightarrow (f \ast g)(x) \in \mathcal{L}(R)
\]

On the other hand

\[
(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(t) \left[ g(|x + 1 + t|) + g(|x + 1 - t|) + g(|x - 1 + t|) + g(|x - 1 - t|) \right] \, dt
\]
\[
= \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(t) \left[ \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x + 1 + t| \times F_{cg}(|\tau|) \, d\tau + 
\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x + 1 - t| \times F_{cg}(|\tau|) \, d\tau + 
\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x - 1 + t| \times F_{cg}(|\tau|) \, d\tau + 
\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos \tau |x - 1 - t| \times F_{cg}(|\tau|) \, d\tau \right] \, dt.
\]

Because

\[
\cos \tau \cos(\tau t) \cos(\tau x) = \frac{1}{4} \left[ \cos \tau (x + 1 + t) + 
\cos \tau (x + 1 - t) + \cos \tau (x - 1 + t) + \cos \tau (x - 1 - t) \right],
\]
we have

\[
(f \ast g)(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos(\tau t) f(t) dt \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos(\tau x) \cos(\tau C g)(|\tau|) d\tau
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos(\tau x) \cos(\tau C f)(|\tau|)(C g)(|\tau|) d\tau
\]

From (2) and \( F_c \) is symmetrical [8]

\[
F((f \ast g))(y) = F_c((f \ast g))(y) = F_c\{\cos(\tau C f)(|\tau|)(C g)(|\tau|), x, y\}, y\}
\]

\[
= \cos y (F_c f)(|y|)(F_c g)(|y|), \forall y \in R.
\]

**Theorem 2.** In the space of continuous belonging to \( L(R) \), the generalized convolution (1) is commutative, associative and distributive.

**Proof.** We prove the generalized convolution with a weight-function for Fourier and cosine-Fourier integral transforms is associative, i.e.

\[
(f \ast g) \ast h = f \ast (g \ast h)
\]

Indeed,

\[
F[(f \ast g) \ast h](y) = \cos y F_c[f \ast g](|y|)(F_c h)(|y|)
\]

\[
= \cos y \cos y (F_c f)(|y|)(F_c g)(|y|)(F_c h)(|y|)
\]

\[
= \cos y (F_c f)(|y|)[\cos y (F_c g)(|y|)(F_c h)(|y|)]
\]

\[
= \cos y (F_c f)(|y|) F_c[g \ast h](|y|) = F[f \ast (g \ast h)](y), \forall y \in R
\]

implies that

\[
(f \ast g) \ast h = f \ast (g \ast h).
\]

The commutative, distributive properties are similary proved. The theorem is proved.

**Theorem 3.** If \( f \) and \( g \) are contiguous functions in \( L(R) \), then the following equality hold

\[
(f \ast g)(x) = \frac{1}{2} [(f \ast F_c g)(|x + 1|) + (f \ast F_c g)(|x - 1|)], \forall x \in R
\]

(4)

Here \( (f \ast g) \) is the convolution of \( f(x) \) and \( g(x) \) for cosine Fourier integral transform [8]

**Theorem 4.** In the space of continuous in \( L(R) \) there does not exists the unit element for the operation of the generalized convolution (1). Set
Theorem 5. (Tichmarch type theorem)

Let \( f, g \in L(e^{-x}, R_+) \). If \( (f \ast g)(x) \equiv 0 \), \( \forall x \in R \), then either \( f(x) = 0 \) or \( g(x) = 0 \), \( \forall x \in R \).

III. Application to solving the heat conduction equation

Consider the heat conduction equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}
\]

with initial condition \( u(x, 0) = f(x), x \in R \), \( f(x) \) is even function and \( u(x, t) \to 0, \frac{\partial u(x, t)}{\partial x} \to 0 \) when \( |x| \to \infty \).

Theorem 6. The above problem has a solution \( u_0(x, t) \) which is satisfies

1) \( u_0(x, t) \) is even function.
2) \( u_0(x + 1, t) + u_0(x - 1, t) = A(t)(f \ast g)(x) \) with \( A(t) = \sqrt{\frac{2}{t}}, g(v) = e^{-\frac{v^2}{4t}} \).

Proof. In [8], the equation (5) has a solution

\[
u_0(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4t}} d\xi
\]

+) when \( x \geq 0 \), we have

\[
u_0(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4t}} d\xi + \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{0} f(\xi)e^{-\frac{(x-\xi)^2}{4t}} d\xi
\]

\[
= \frac{1}{2\sqrt{\pi t}} \left\{ \int_{0}^{+\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4t}} d\xi - \int_{-\infty}^{0} f(y)e^{-\frac{(x+y)^2}{4t}} dy \right\}
\]

\[
= \frac{1}{2\sqrt{\pi t}} \left\{ \int_{0}^{+\infty} f(\xi)g(|x-\xi|)d\xi + \int_{-\infty}^{0} f(y)(x+y)dy \right\}
\]

\[
= \frac{1}{2\sqrt{\pi t}} (f \ast g)(x)
\]

+) Similarly, when \( x < 0 \) we get

\[
u_0(x, t) = \frac{1}{\sqrt{2t}} (f \ast g)(-x)
\]

From (6), (7) we obtain
\[ u_0(x, t) = \frac{1}{\sqrt{2t}} (f \ast_{\mathcal{F}_c} g)(|x|) \]  

(8)

e.i. \( u(x, t) \) is even function for \( x \).

From (8) we have

\[ u_0(x + 1, t) = \frac{1}{\sqrt{2t}} (f \ast_{\mathcal{F}_c} g)(|x + 1|) \]
\[ u_0(x - 1, t) = \frac{1}{\sqrt{2t}} (f \ast_{\mathcal{F}_c} g)(|x - 1|) \]  

(9)

From (9) and Theorem 3 we have

\[ u_0(x + 1, t) + u_0(x - 1, t) = A(t)(f \ast g)(x), \quad \forall x \in \mathbb{R}, ; A(t) = \sqrt{\frac{2}{t}}. \]

The theorem is proved.

References


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