ON AN INVERSE SOURCE PROBLEM FOR 3-DIMENSIONAL WAVE EQUATION

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Abstract We discuss an inverse source problem for three-dimensional wave equation. The source term is assumed to be a point source. The location of a point source does not move, and the magnitude changes with time. For such a problem, we propose a direct identification method of a point source without a priori information about the location and magnitude. The effectiveness of the method is shown by numerical examples.

Keywords: Inverse source problem, Numerical method, Wave equation, Point source

1 Introduction

Inverse problems for partial differential equations have attracted the attention of many researchers in recent years [3]. Especially, inverse source problem appears in various fields of science and engineering, such as cracks in structure, electrical activity of human brain, acoustic source identification, and so on. Investigation of numerical method for inverse source problem becomes very important [1, 4, 5, 6]. Usual approach is the combination of forward analysis and least squares method. However, this approach needs a priori information about unknown parameters and requires much computation time.

In this paper, we consider an inverse source problem for three-dimensional wave equation where the source term is expressed by a point source. Assume that the location of a point source does not move, and that the magnitude changes with time. For such a problem, we propose a numerical method without using forward analysis and a priori information about unknowns. At first, special case of initial and boundary conditions is discussed. Next, our method is applied to general case using weighted integral. Numerical examples illustrate the effectiveness of the proposed method.

2 Inverse source problem for wave equation

We consider an inverse source problem for the following wave equation:

$$\Delta u(\boldsymbol{x},t) - \frac{1}{c^2}\partial_t^2 u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad c > 0, \quad \partial_t = \frac{\partial}{\partial t}. \tag{1}$$

The source term $f(\boldsymbol{x},t)$ is assumed to be a point source.

$$f(\boldsymbol{x},t) = -q(t)\delta(\boldsymbol{x}-\boldsymbol{p}), \qquad \boldsymbol{p} \in \Omega, \tag{2}$$

where $\delta(\cdot)$ is Dirac's delta function, $\boldsymbol{p}$ and $q(t)$ are the location and magnitude of a point source, respectively. The magnitude $q(t)$ is written by

$$q(t) = \begin{cases} q_0(t), & t \in (a,b), \quad 0 < a < b, \\ 0, & \text{otherwise}. \end{cases}$$

The following function satisfies eq.(1).

$$u_s(\boldsymbol{x},t) = \frac{q(t_d)}{4\pi|\boldsymbol{x}-\boldsymbol{p}|}, \quad t_d = t - |\boldsymbol{x}-\boldsymbol{p}|/c. \tag{3}$$

As shown in eq.(3), u_s gives the information of a point source at a retarded time $t_d = t - |\boldsymbol{x}-\boldsymbol{p}|/c$. In the next section, we consider the identification of $\boldsymbol{p}$ and $q(t_d)$ in a special case of $u = u_s$.

3 Identification for special case of initial and boundary conditions

We consider the identification for the special case of eq.(1) such that

$$\begin{array}{lll} \text{Initial condition :} & u(\boldsymbol{x},0)=\partial_t u(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\Omega, \\ \text{Boundary condition :} & u(\boldsymbol{x},t)=u_s(\boldsymbol{x},t), & \boldsymbol{x}\in\Gamma=\partial\Omega. \end{array}$$

In this case, the solution of eq.(1) is

$$u(\boldsymbol{x},t)=u_s(\boldsymbol{x},t)=\frac{q(t_d)}{4\pi|\boldsymbol{x}-\boldsymbol{p}|}. \tag{4}$$

In the following, we show the identification of the location and magnitude using observations of u and its derivatives at a point outside of the convex hull of Ω.

From eq.(4), $\partial_t u(\boldsymbol{x},t)$ and $\nabla u(\boldsymbol{x},t)$ are written by

$$\begin{aligned} \partial_t u(\boldsymbol{x},t) &= \frac{\dot{q}(t_d)}{4\pi|\boldsymbol{x}-\boldsymbol{p}|}, \\ \nabla u(\boldsymbol{x},t) &= -\frac{q(t_d)}{4\pi}\frac{\boldsymbol{x}-\boldsymbol{p}}{|\boldsymbol{x}-\boldsymbol{p}|^3}-\frac{\dot{q}(t_d)}{4\pi c}\frac{\boldsymbol{x}-\boldsymbol{p}}{|\boldsymbol{x}-\boldsymbol{p}|^2} \\ &= -\left(\frac{q(t_d)}{4\pi|\boldsymbol{x}-\boldsymbol{p}|^2}+\frac{1}{c}\frac{\dot{q}(t_d)}{4\pi|\boldsymbol{x}-\boldsymbol{p}|}\right)\frac{\boldsymbol{x}-\boldsymbol{p}}{|\boldsymbol{x}-\boldsymbol{p}|}, \end{aligned}$$

where $\dot{q}(t)=dq(t)/dt$. Therefore, the observations at $\boldsymbol{x}=\boldsymbol{\xi}$ and $t=\tau$ are

$$\begin{aligned} u(\boldsymbol{\xi},\tau) &= \frac{q(\tau_d)}{4\pi|\boldsymbol{\xi}-\boldsymbol{p}|}, \quad \partial_t u(\boldsymbol{\xi},\tau)=\frac{\dot{q}(\tau_d)}{4\pi|\boldsymbol{\xi}-\boldsymbol{p}|}, \\ \nabla u(\boldsymbol{\xi},\tau) &= -\left(\frac{q(\tau_d)}{4\pi|\boldsymbol{\xi}-\boldsymbol{p}|^2}+\frac{1}{c}\frac{\dot{q}(\tau_d)}{4\pi|\boldsymbol{\xi}-\boldsymbol{p}|}\right)\frac{\boldsymbol{\xi}-\boldsymbol{p}}{|\boldsymbol{\xi}-\boldsymbol{p}|}, \end{aligned} \qquad \tau_d=\tau-|\boldsymbol{\xi}-\boldsymbol{p}|/c. \tag{5}$$

Substituting $u(\boldsymbol{\xi},\tau)$ and $\partial_t u(\boldsymbol{\xi},\tau)$ into $\nabla u(\boldsymbol{\xi},\tau)$, we have

$$\nabla u(\boldsymbol{\xi},\tau)=\left(\frac{u(\boldsymbol{\xi},\tau)}{|\boldsymbol{\xi}-\boldsymbol{p}|}+\frac{1}{c}\partial_t u(\boldsymbol{\xi},\tau)\right)\boldsymbol{e}_p, \quad \boldsymbol{e}_p=-\frac{\boldsymbol{\xi}-\boldsymbol{p}}{|\boldsymbol{\xi}-\boldsymbol{p}|}. \tag{6}$$

Suppose that $|\nabla u(\boldsymbol{\xi},\tau)|\neq 0$. Then, the following equation holds:

$$\frac{\nabla u(\boldsymbol{\xi},\tau)}{|\nabla u(\boldsymbol{\xi},\tau)|}=\operatorname{sgn}\left(\frac{u(\boldsymbol{\xi},\tau)}{|\boldsymbol{\xi}-\boldsymbol{p}|}+\frac{1}{c}\partial_t u(\boldsymbol{\xi},\tau)\right)\boldsymbol{e}_p. \tag{7}$$

We cannot evaluate the signum function in eq.(7). Therefore, two unit vectors $\pm\nabla u/|\nabla u|$ are obtained as candidates of $\boldsymbol{e}_p$. However, $\boldsymbol{e}_p$ can be uniquely determined since an observation point $\boldsymbol{\xi}$ exists outside of the convex hull of Ω.

Multiply both sides of eq.(6) by the unit vector $\boldsymbol{e}_p$, then we have

$$\nabla u(\boldsymbol{\xi},\tau)\cdot\boldsymbol{e}_p=\frac{u(\boldsymbol{\xi},\tau)}{|\boldsymbol{\xi}-\boldsymbol{p}|}+\frac{1}{c}\partial_\tau u(\boldsymbol{\xi},\tau).$$

In case of $u(\boldsymbol{\xi},\tau)\neq 0$, the distance $|\boldsymbol{\xi}-\boldsymbol{p}|$ is identified by

$$|\boldsymbol{\xi}-\boldsymbol{p}|=\left(\frac{\nabla u(\boldsymbol{\xi},\tau)}{u(\boldsymbol{\xi},\tau)}\cdot\boldsymbol{e}_p-\frac{1}{c}\frac{\partial_t u(\boldsymbol{\xi},\tau)}{u(\boldsymbol{\xi},\tau)}\right)^{-1}. \tag{8}$$

We can identify the direction $\boldsymbol{e}_p = -(\boldsymbol{\xi} - \boldsymbol{p})/|\boldsymbol{\xi} - \boldsymbol{p}|$ and distance $|\boldsymbol{\xi} - \boldsymbol{p}|$. Therefore, the location $\boldsymbol{p}$ is written by

$$\boldsymbol{p} = \boldsymbol{\xi} + |\boldsymbol{\xi} - \boldsymbol{p}|\boldsymbol{e}_p = \boldsymbol{\xi} + \left(\frac{\nabla u(\boldsymbol{\xi}, \tau)}{u(\boldsymbol{\xi}, \tau)} \cdot \boldsymbol{e}_p - \frac{1}{c} \frac{\partial_t u(\boldsymbol{\xi}, \tau)}{u(\boldsymbol{\xi}, \tau)} \right)^{-1} \boldsymbol{e}_p.$$

From eqs.(5) and (8), the magnitude $q(\tau_d)$ becomes

$$q(\tau_d) = 4\pi \left(\frac{\nabla u(\boldsymbol{\xi}, \tau)}{u(\boldsymbol{\xi}, \tau)} \cdot \boldsymbol{e}_p - \frac{1}{c} \frac{\partial_t u(\boldsymbol{\xi}, \tau)}{u(\boldsymbol{\xi}, \tau)} \right)^{-1} u(\boldsymbol{\xi}, \tau).$$

4 Identification for general case of initial and boundary conditions

This section considers the general case of eq.(1) such that

$$\begin{aligned} &\text{Initial condition :} \quad && u(\boldsymbol{x}, 0) = \phi(\boldsymbol{x}), \quad \partial_t u(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}), \quad && \boldsymbol{x} \in \Omega, \\ &\text{Boundary condition :} \quad && u(\boldsymbol{x}, t) = h(\boldsymbol{x}, t), && \boldsymbol{x} \in \Gamma, \end{aligned}$$

where Ω is a bounded domain, and ϕ and ψ are unknown functions. In the following, we generate u_s, $\partial_t u_s$, and ∇u_s at a pseudo observation point $\boldsymbol{\xi}$ using a weighted integral.

Multiply both sides of eq.(1) by a weighting function g and integrate over Ω and $T = [0, t_F]$, then we have [2]

$$\int_T \int_\Omega g(\Delta u - \frac{1}{c^2}\partial_t^2 u) dV(\boldsymbol{x})\, dt = \int_T \int_\Omega f g dV(\boldsymbol{x})\, dt. \tag{9}$$

Here, we use a fundamental solution of wave equation as a weighting function g.

$$g(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = \frac{1}{4\pi r}\delta(t - t_r), \qquad r = |\boldsymbol{x} - \boldsymbol{\xi}|, \quad t_r = \tau - r/c.$$

Suppose that $\boldsymbol{\xi}$ does not belong to the closure of Ω, and that τ exists in the interval

$$\max_{\boldsymbol{\eta} \in \overline{\Omega}} |\boldsymbol{\xi} - \boldsymbol{\eta}|/c + \varepsilon \le \tau \le t_F + \min_{\boldsymbol{\eta} \in \overline{\Omega}} |\boldsymbol{\xi} - \boldsymbol{\eta}|/c - \varepsilon,$$

where $\varepsilon (> 0)$ is sufficiently small constant. Under the above assumption, t_r exists in T. Then, the right-hand side of eq.(9) becomes

$$\begin{aligned} \int_T \int_\Omega f g dV(\boldsymbol{x})\, dt &= \int_T \int_\Omega q(t) \delta(\boldsymbol{x} - \boldsymbol{p}) \frac{1}{4\pi r} \delta(t - t_r) dV(\boldsymbol{x})\, dt \\ &= -\frac{q(\tau_d)}{4\pi |\boldsymbol{\xi} - \boldsymbol{p}|} = -u_s(\boldsymbol{\xi}, \tau). \end{aligned} \tag{10}$$

From $\boldsymbol{\xi} \notin \overline{\Omega}$, weighting function g satisfies

$$\Delta g(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) - \frac{1}{c^2}\partial_t^2 g(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = 0, \qquad \boldsymbol{x} \in \Omega.$$

The left-hand side of eq.(9) is written by

$$\begin{aligned} &\int_T \int_\Omega g(\Delta u - \frac{1}{c^2}\partial_t^2 u) dV(\boldsymbol{x})\, dt \\ &= \int_T \int_\Omega \left\{ g(\Delta u - \frac{1}{c^2}\partial_t^2 u) - u(\Delta g - \frac{1}{c^2}\partial_t^2 g) \right\} dV(\boldsymbol{x})\, dt \\ &= \int_T \int_\Omega (g\Delta u - u\Delta g) dV(\boldsymbol{x})\, dt - \frac{1}{c^2} \int_T \int_\Omega (g\partial_t^2 u - u\partial_t^2 g) dV(\boldsymbol{x})\, dt \\ &= \int_T \int_\Gamma (g\partial_n u - u\partial_n g) dS(\boldsymbol{x})\, dt - \frac{1}{c^2} \int_\Omega \Big[g\partial_t u - u\partial_t g \Big]_0^{t_F} dV(\boldsymbol{x}), \end{aligned} \tag{11}$$

where ∂_n denotes the outward normal differentiation $\partial/\partial \boldsymbol{n}$. The second term of eq.(11) vanishes since $t_r \neq 0, t_F$. The first term of eq.(11) becomes

$$\begin{aligned}&\int_T\int_\Gamma (g\partial_n u - u\partial_n g)dS(\boldsymbol{x})\,dt\\&= \frac{1}{4\pi}\int_\Gamma \frac{1}{r}\left\{\frac{n_r}{r}u(\boldsymbol{x},t_r) + \partial_n u(\boldsymbol{x},t_r) + \frac{n_r}{c}\partial_t u(\boldsymbol{x},t_r)\right\}dS(\boldsymbol{x}), \qquad n_r = \partial_n r.\end{aligned} \tag{12}$$

From eqs.(9), (10), (11), and (12), the following equation holds:

$$u_s(\boldsymbol{\xi},\tau) = -\frac{1}{4\pi}\int_\Gamma \frac{1}{r}\left\{\frac{n_r}{r}u(\boldsymbol{x},t_r) + \partial_n u(\boldsymbol{x},t_r) + \frac{n_r}{c}\partial_t u(\boldsymbol{x},t_r)\right\}dS(\boldsymbol{x}). \tag{13}$$

Equation (13) shows that $u_s(\boldsymbol{\xi},\tau)$ can be calculated from the observations of u, $\partial_n u$, and $\partial_t u$ on Γ. Using $\partial_t g$ instead of g as weighting function, $\partial_t u_s(\boldsymbol{\xi},\tau)$ are written by

$$\partial_t u_s(\boldsymbol{\xi},\tau) = -\frac{1}{4\pi}\int_\Gamma \frac{1}{r}\left\{\frac{n_r}{r}\partial_t u(\boldsymbol{x},t_r) + \partial_t\partial_n u(\boldsymbol{x},t_r) + \frac{n_r}{c}\partial_t^2 u(\boldsymbol{x},t_r)\right\}dS(\boldsymbol{x}). \tag{14}$$

For weighting function ∇g, we have

$$\begin{aligned}\nabla u_s(\boldsymbol{\xi},\tau) = -\frac{1}{4\pi}\int_\Gamma \frac{1}{r}\Bigg\{&\frac{3n_r\nabla r - \boldsymbol{n}}{r^2}u(\boldsymbol{x},t_r) + \frac{\nabla r}{r}\partial_n u(\boldsymbol{x},t_r) + \frac{\nabla r}{c}\partial_t\partial_n u(\boldsymbol{x},t_r)\\&+\frac{3n_r\nabla r - \boldsymbol{n}}{cr}\partial_t u(\boldsymbol{x},t_r) + \frac{n_r\nabla r}{c^2}\partial_t^2 u(\boldsymbol{x},t_r)\Bigg\}dS(\boldsymbol{x}).\end{aligned} \tag{15}$$

As shown in eqs.(13), (14), and (15), pseudo observations u_s, $\partial_t u_s$, and ∇u_s can be obtained from observations of u, $\partial_n u$, $\partial_t u$, $\partial_t\partial_n u$, and $\partial_t^2 u$ on Γ. Moreover, we also assume that $\boldsymbol{\xi}$ is a point outside of the convex hull of Ω. Then, the location and magnitude of a point source are identified from pseudo observations of u_s, $\partial_t u_s$, and ∇u_s as shown in Section 3.

Numerical examples are shown in the presentation.

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