LINEAR EQUATIONS AND INITIAL VALUE PROBLEMS WITH GENERALIZED RIGHT INVERTIBLE OPERATORS

NGUYEN VAN MAU, PHAM THI BACH NGOC
Hanoi University of Science, VNUH

Abstract

Consider the linear equation with generalized right invertible operator V of the form

$$Q[V]x := \sum_{m=0}^{M}\sum_{n=0}^{N} V^m A_{mn} V^n x = y, \ \ y \in X, \qquad (0.1)$$

where $M, N \in I\!N$, $A_{mn} \in L_0(X)$, $A_{MN} = I$, $A_{mn}X_{M+N-n} \subset X_m$, $X_j :=$ dom V^j, and consider an initial value problem for liner equation (0.1) satisfying the following initial conditions

$$FV^j x = y_j, \ \ y_j \in \ker V \ \text{ are given}, \quad j = 0, \dots, M+N-1, \qquad (0.2)$$

The similar equation with right invertible operatorswere studied by Przeworska- Rolewicz D., and others (see [1], [2], [3]). In [4], the first author and Nguyen Minh Tuanconstructed the generalized right invertible operators. In this paper, we present some new properties of generalized right invertible operators and construct general forms of resolving operator for the equation (0.1) and for the problem (0.1) – (0.2). Then we find all solutions of the equation (0.1) and the problem (0.1) – (0.2) in a closed form.

1 Generalized right invertible operators

Let X be a linear space over a field $\mathcal{K}$ of scalars. Denote by $L(X)$ the set of all linear operators with domains and ranges in X. The set of all right (left, generalized) invertible operators in $L(X)$ will be denoted by $R(X)(\Lambda(X), W(X))$ (see [1] - [3]).

Definition 1.1 [4]. An operator $V \in L(X)$ is said to be generalized right invertible if there exists a $W \in L(X)$ such that $\operatorname{Im} W \subset \operatorname{dom} V$, $\operatorname{Im} V \subset \operatorname{dom} W$, $VWV = V$, $V^2W = V$. Then we shall write $V \in R_1(X)$ and $W \in \mathcal{R}_V^1$.

It is easy to see $R(X) \subset R_1(X) \subset W(X)$, $\Lambda(X) \subset W(X)$.

Lemma 1.1. For every $V \in R_1(X)$ there exists $W \in \mathcal{R}_V^1$ such that

$$WVW = V, \ VW^2 = W \ \text{on} \ \operatorname{dom} W.$$

Write: $\mathcal{R}_V^{(1)} = \{W \in \mathcal{R}_V^1 : \ WVW = W, \ VW^2 = W\}.$

Definition 1.2. An operator $F \in L(X)$ $(G \in L(X))$ such that $FX = \ker V$, $F^2 = F$ and $\exists W \in \mathcal{R}_V^{(1)} : \ FW = 0$, $(G^2 = G, \ GV = 0$ and $\exists W \in \mathcal{R}_V^{(1)} : \ GX = \ker W)$. Then $F(G)$ is said to be a right (left) initial operator corresponding t o a right inverse W of V.

Denote by $\mathcal{F}_V$ $(\mathcal{G}_V)$ the set of all right (left) initial operators of $V \in R_1(X)$.

Lemma 1.2. For every $V \in R_1(X)$ and $W \in \mathcal{R}_V^{(1)}$, then $F \in \mathcal{F}_V$ $(G \in \mathcal{G}_V)$ corresponding to a W if and only if $F(G)$ is of the form : $F = I - WV$, $(G = I - VW)$.

Theorem 1.1. A necessary and sufficient condition for an operator $F \in L(X)(G \in L(X))$ to be a right (left) initial operator for $V \in R_1(X)$ corresponding to a $W \in R_V^{(1)}$ is that

$$F = I - WV \ \text{on dom} \ V, (G = I - VW \ \text{on dom} \ W). \tag{1.2}$$

Theorem 1.2.*(Taylor-Gontcharov formula induced by generalized right initial operators).*

Suppose that $V \in R_1(X)$ and $\mathcal{F}_V = \{F_\beta\}_{\beta \in \Gamma}$ is a family of right initial operators corresponding to $\{W_\beta\}_{\beta\in\Gamma} \subset \mathcal{R}_V^{(1)}$. Let $\{\beta_n\} \subset \Gamma$, $n \in \mathbb{N}_0$ be an arbitrary sequence of indices. Then for every positive integer N the follwing identity holds on dom V^N.

$$I = F_{\beta_0} + \sum_{k=1}^{N-1} W_{\beta_0} \dots , W_{\beta_{k-1}} F_{\beta_k} V^k + W_{\beta_0} \dots , W_{\beta_{N-1}} V^N. \tag{1.3}$$

Puting in (1.3) $W_{\beta_k} = W$ and $F_{\beta_k} = F$ for $k = 0, \dots, N$, we obtain the Taylor formula

$$I = \sum_{k=0}^{N-1} W^k F V^k + W^N V^N \quad \text{on dom} \quad V^N. \tag{1.4}$$

Lemma 1.2. Let $V \in R_1(X)$, $W \in \mathcal{R}_V^{(1)}$. Then

$$\ker V^N = \left\{ x \in X : x = \sum_{k=0}^{N-1} W^k z_k, \;\; z_0, \dots, z_{N-1} \in \ker V \right\},$$

$$\text{dom } V^N = \left\{ x \in X : x = W^N y + \sum_{k=0}^{N-1} W^k z_k, \;\; y \in V^N X_N, \;\; z_0, \dots, z_{N-1} \in \ker V \right\}.$$

2 Linear equations with generalized right invertible operators

To begin with, we consider the equation

$$V^N x = y, \;\; y \in X, \;\; N \in I\!N^+. \tag{2.1}$$

Theorem 2.1. Suppose that $V \in R_1(X)$, $\dim \ker V \neq 0$, $\dim \operatorname{coker} V \neq 0$ and $W \in \mathcal{R}_V^{(1)}$. Then the equation (2.1) has solutions if and only if $y \in \operatorname{Im} V^N$. If this condition is satisfied then all solutions of (2.1) are given by

$$x = W^N y + \sum_{k=0}^{N-1} W^k z_k, \;\; z_0, \dots, z_{N-1} \in \ker V \tag{2.2}$$

Proof. If $y \in \operatorname{Im} V^N$ then there is $y_1 \in \text{dom } V^N$ such that $y = V^N y_1$. Hence, (2.1) can be written in the form $V^N x = V^N y_1$. Since $V^N = V^N W^N V^N$, the last equation is equivalent to $V^N(x - W^N V^N y_1) = 0$. Lemma 1.2 implies the formula (2.2).

Now we consider the equation (0.1)

Definition 2.1 Suppose that $Q[V]$ is of the form (0.1). Write

$$Q(A) = \sum_{m=0}^{M}\sum_{n=0}^{N} W^{M-m} A_{mn} W^{N-n} - \sum_{m=0}^{M} W^{M-m}\tilde{A}_{mN} G, \tag{2.3}$$

where

$$\widetilde{A}_{mn} := \begin{cases} 0 & \text{if } m = M, \ n = N, \\ A_{mn} & \text{if } m+n < M+N. \end{cases} \tag{2.4}$$

Then $Q(A)$ is said to be a resolving operator for the equation (0.1).

Lemma 2.1 Write

$$\tilde{Q}(A) := \sum_{m=0}^{M}\sum_{n=0}^{N} W^{M-m}\tilde{A}_{mn} V^{n}, \tag{2.5}$$

$Q(A)$ is right invertible (left invertible, invertible, generalized invertible) on X_M if and only if $I + W^N\tilde{Q}(A)$ is right invertible (left invertible, invertible, generalized invertible) on X_{M+N}. Moreover, if we denote by $R_{\tilde{Q}}(L_{\tilde{Q}}, W_{\tilde{Q}})$ is a right (left, generalized) inverse of $I+W^N\tilde{Q}(A)$ then there exists $R_Q(L_Q, W_Q)$ is a right (left, generalized) inverse of $Q(A)$ such that, respectively

$$\begin{aligned} &R_{\tilde{Q}} := I - W^N R_Q\tilde{Q}(A) \quad L_{\tilde{Q}} := I - W^N L_Q\tilde{Q}(A) \\ &W_{\tilde{Q}} := I - W^N W_Q\tilde{Q}(A) \quad (I + W^N\tilde{Q}(A))^{-1} := I - W^N Q^{-1}(A)\tilde{Q}(A) \end{aligned} \tag{2.6}$$

Proof. It is easy to see that $Q(A) = I + \tilde{Q}(A)W^N$, $Q(A) \subset L_0(X_M)$ and $I + W^N\tilde{Q}(A) \in L_0(X_{M+N})$.

(i) Suppose that $Q(A)$ is right invertible on X_M, i.e. there exists $R_Q \in \mathcal{R}_{Q(A)}$ such that $R_Q X_M \subset X_M$ and $Q(A)R_Q = I$. Write $R_{\tilde{Q}} := I - W^N R_Q\tilde{Q}(A)$. It to check that $R_{\tilde{Q}}$ is well defined on X_{M+N} and $R_{\tilde{Q}}X_{M+N} \subset X_{M+N}$. On X_{M+N} we have

$$\begin{aligned} \left[I + W^N\tilde{Q}(A)\right] R_{\tilde{Q}} &= \left[I + W^N\tilde{Q}(A)\right]\left[I - W^N R_Q\tilde{Q}(A)\right] \\ &= I + W^N\tilde{Q}(A) - W^N\left[I + \tilde{Q}(A)W^N\right] R_Q\tilde{Q}(A) \\ &= I + W^N\tilde{Q}(A) - W^N\tilde{Q}(A) = I, \end{aligned}$$

which proves that $I + W^N\tilde{Q}(A)$ is right invertible on X_{M+N}.

Conversely, suppose that $I + W^N\tilde{Q}(A)$ is right invertible on X_{M+N}, i.e. there exists $R_{\tilde{Q}} \in \mathcal{R}_{I+W^N\tilde{Q}(A)}$ such that $R_{\tilde{Q}}X_{M+N} \subset X_{M+N}$ and $[I + W^N\tilde{Q}(A)]R^Q = I$ on X_{M+N}. Write $R_Q = I - \tilde{Q}(A)R_{\tilde{Q}}W^N$. If $x \in X_M$ then $u = W^N x \in X_{M+N}$, $y = R_{\tilde{Q}}u \in X_{M+N}$ and $R_Q x = \left[I - \tilde{Q}(A)R_{\tilde{Q}}W^N\right] x = x - \tilde{Q}(A)y \in X_M$. On X_M, we have

$$\begin{aligned}\left[I + \tilde{Q}(A)W^N\right] R_Q &= \left[I + \tilde{Q}(A)W^N\right]\left[I - \tilde{Q}(A)R_{\tilde{Q}}W^N\right] \\ &= I + \tilde{Q}(A)W^N - \tilde{Q}(A)\left[I + W^N\tilde{Q}(A)\right] R_{\tilde{Q}}W^N \\ &= I + \tilde{Q}(A)W^N - \tilde{Q}(A)W^N = I,\end{aligned}$$

which proves that $I + \tilde{Q}(A)W^N$ is a right invertible on X_M. In the same way, we can get proofs for the other cases.

Theorem 2.2. Suppose that $V \in R_1(X)$, $W \in \mathcal{R}_V^{(1)}$ and $Q[V]$, $Q(A)$, $\tilde{Q}(A)$ are given by (0.1) – (2.3) – (2.5), respectively.

(i) If $Q(A)$ is invertible then all solutions of the equation (0.1) are given by

$$x = \left[I - W^N Q^{-1}(A)\tilde{Q}(A)\right]\left(W^{M+N}y + \sum_{j=0}^{M+N-1} W^j z_j\right), \quad z_0, \dots, z_{M+N-1} \in \ker V. \tag{2.7}$$

(ii) If $Q(A) \in R(X)$ then all solutions of the equation (0.1) are given by

$$x = \left[I - W^N R_Q\tilde{Q}(A)\right]\left(W^{M+N}y + \sum_{j=0}^{M+N-1} W^j z_j\right) + z, \tag{2.8}$$

where $R_Q \in \mathcal{R}_{Q(A)}$, $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker[I + W^N\tilde{Q}(A)]$ are arbitrary.

(iii) If $Q(A) \in \Lambda(X)$ and $L_Q \in \mathcal{L}_{Q(A)}$ then (0.1) is solvable if and only if there exist $z_0, \dots, z_{M+N-1} \in \ker V$ and $y \in X_{M+N}$ such that

$$W^{M+N}y + \sum_{j=0}^{M+N-1} W^j z_j \in [I + W^N\tilde{Q}(A)]X_{M+N}. \tag{2.9}$$

If this condition is satisfied, then all solutions of the equation (0.1) are given by

$$x = \left[I - W^N L_Q \tilde{Q}(A)\right] \left(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\right), \quad z_0, \dots, z_{M+N-1} \in \ker V \tag{2.10}$$

(iv) If $Q(A) \in W(X)$ and $W_Q \in \mathcal{W}_{Q(A)}$ then (0.1) is solvable if and only if the condition (2.9) is satisfied and then all solutions of the equation (0.1) are given by

$$x = \left[I - W^N W_Q \tilde{Q}(A)\right] \left(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\right) + z, \tag{2.11}$$

where $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker[I + W^N \tilde{Q}(A)]$ are arbitrary.
Proof. We have

$$Q[V]x := \left(\sum_{m=0}^{M} \sum_{n=0}^{N} V^m A_{mn} V^n\right) x = y$$

$$V^{M+N} \left(I + \sum_{m=1}^{M} \sum_{n=0}^{N} W^{M+N-m} \tilde{A}_{mn} V^n\right) x = y - \sum_{n=0}^{N} \tilde{A}_{0N} V^n x.$$

By Theorem (2.1), we imply

$$\left(I + \sum_{m=1}^{M} \sum_{n=0}^{N} W^{M+N-m} \tilde{A}_{mn} V^n\right) x = W^{M+N} \left(y - \sum_{n=0}^{N} \tilde{A}_{0N} V^n x\right) + \sum_{j=0}^{M+N-1} W^j z_j$$

$$\left(I + W^N \sum_{m=0}^{M} \sum_{n=0}^{N} W^{M-m} \tilde{A}_{mn} V^n\right) x = W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j$$

$$\left[I + W^N \tilde{Q}(A)\right] x = W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j. \tag{2.12}$$

We have $Q(A) = I + \tilde{Q}(A) W^N$.

(i) If $Q(A)$ is invertible then $Q(A)$ is invertible on X_M. Lemma 2.1 implies that $\left[I + W^N \tilde{Q}(A)\right]^{-1} := I - W^N Q^{-1}(A) \tilde{Q}(A)$. This and (2.12) together imply (2.7).

(ii) If $Q(A)$ is right invertible then $Q(A)$ is right invertible on X_M. Lemma 2.1 implies that $I + W^N\tilde{Q}(A)$ is right invertible on X_{M+N} and $R_{\tilde{Q}} := I - W^N R_Q \tilde{Q}(A)$ is a right inverse of $I + W^N\tilde{Q}(A)$. This and (2.12) together imply (2.8).

(iii)((iv)) If $Q(A)$ is left (generalized) invertible then $Q(A)$ is left (generalized) invertible on X_M. Lemma 2.1 implies that $I + W^N\tilde{Q}(A)$ is left (generalized) invertible on X_{M+N} and $L_{\tilde{Q}} := I - W^N L_Q \tilde{Q}(A)(W_{\tilde{Q}} := I - W^N W_Q \tilde{Q}(A))$ is a left (generalized)inverse of $I + W^N\tilde{Q}(A)$. This and (2.12) imply that (0.1) has solutions if only if the condition (2.9) is satisfied. If this is the case then all solutions of the equation (0.1) are of the form (2.10)((2.11)). The theorem is complete.

3 Initial value problems for equations with generalized right invertible operators

To begin with, we consider the initial value problem for the operator V^N : Find all solutions of the problem

$$V^N x = y, \quad y \in V^N X_N, \tag{3.1}$$

$$FV^j x = y_j, \quad y_j \in \ker V \quad \text{are given} \quad (j = 0, \ldots, N-1). \tag{3.2}$$

Theorem 3.1. Let $V \in R_1(X)$, $\dim\ker V \neq 0$, $\dim\ \operatorname{coker} V \neq 0$, $\ker V \subset \operatorname{Im} V$ and let $F \in \mathcal{F}_V$ be the right initial operator corresponding to a $W \in \mathcal{R}_V^{(1)}$. Then the problem (3.1) – (3.2) has a unique solution of the form

$$x = W^N y + \sum_{j=0}^{N-1} W^j y_j. \tag{3.3}$$

Proof. Theorem 2.1 implies that every solution of (3.1) is of the form

$$x = W^N y + \sum_{k=0}^{N-1} W^k z_k, \quad z_0, \ldots, z_{N-1} \in \ker V.$$

Since $FW = 0$ and $Vz_k = 0$, $(k = 0, \ldots, N-1)$, we have

If $j = 0$ then $y_0 = Fx = Fz_0 = z_0$.

If $j = 1, \ldots, N-1$ then $y_j = FV^j x = FVWz_j = VWz_j$. Since $\ker V \subset \operatorname{Im} V$, $y_j \in \ker V$ then $z_j = VWz_j$. So $y_j = z_j$. Which implies the required formula (3.3).

Definition 3.1(cf. Przeworska-Rolewicz [1])

(i) The initial value problem (0.1) – (0.2) is well-posed if it has a unique solution for every $y \in Q[V]X_{M+N}$, $y_0, \ldots, y_{M+N-1} \in \ker V$.

(ii) The initial value problem (0.1) – (0.2) is ill-posed if either there exist $y \in Q[V]X_{M+N}$, $y_0, \ldots, y_{M+N-1} \in \ker V$ such that this problem has no solutions or the homogeneous problem induced by (0.1) – (0.2) (i.e. $y = y_0 = \cdots = y_{M+N-1} = 0$) has non- trivial solution.

Definition 3.2. Suppose that $Q[V]$ is of the form (0.1). Write

$$\tilde{Q} := \sum_{m=0}^{M} \sum_{n=0}^{N} W^{M-m} B_{mn} W^{N-n} - \sum_{m=0}^{M} W^{M-m} B_{mN} G, \tag{3.4}$$

where

$$B_{mn} := \begin{cases} \tilde{A}_{0n} & \text{if } m = 0 \\ \tilde{A}_{mn} - \sum_{k=m}^{M} FV^{k-m+1} W \tilde{A}_{kn} & \text{otherwise,} \end{cases} \tag{3.5}$$

$\tilde{A}_{mn}$ is given by formula (2.4).

Then $I + \tilde{Q}$ is the resolving operator for the problem (0.1) – (0.2).

Lemma 3.1. Write

$$Q := \sum_{m=0}^{M} \sum_{n=0}^{N} W^{M+N-m} B_{mn} V^n, \tag{3.6}$$

$\tilde{Q}$, B_{mn} are defined by (3.4) – (3.5). Then

$$(i) \;\; QW^N = W^N \tilde{Q}. \tag{3.7}$$

$$(ii) \;\; FV^j (I + Q) = FV^j, \;\; (j = 0, \ldots, M+N-1). \tag{3.8}$$

Proof. (i) We have

$$QW^N = W^N \left(\sum_{m=0}^{M} \sum_{n=0}^{N-1} W^{M-m} B_{mn} W^{N-n} + \sum_{m=0}^{M} W^{M-m} B_{mN} VW \right) = W^N \tilde{Q}.$$

(ii) For $j = 0, \dots, N-1$, we obtain

$$FV^j(I+Q) = FV^j + \sum_{m=0}^{M}\sum_{n=0}^{N} FW^{M+N-m-j}B_{mn}V^n = FV^j.$$

If $j = N+i, \ \ i = 0, \dots, M-1$, then

$$FV^{N+i}(I+Q) = FV^{N+i} + \sum_{m=1}^{M}\sum_{n=0}^{N} FV^{N+i}W^{M+N-m}$$
$$\left(\tilde{A}_{mn} - \sum_{k=m}^{M} FV^{k-m+1}W\tilde{A}_{kn}\right)V^n + \sum_{n=0}^{N} FV^{N+i}W^{M+N}\tilde{A}_{0n}V^n$$
$$= FV^{N+i} + \sum_{m=M-i}^{M}\sum_{n=0}^{N} FV^{i-M+m+1}W\left(\tilde{A}_{mn} - \sum_{k=m}^{M} FV^{k-m+1}W\tilde{A}_{kn}\right)V^n = FV^{N+i}.$$

The Lemma is complete.

Lemma 3.2. Let Q be defined by (3.6). Then the initial value problem (0.1) – (0.2) is well-posed if and only if $I+Q$ is invertible on X_{M+N}.
Proof. By Theorem 2.1, we can write the equation (0.1) in the form

$$(I+Q)x = W^{M+N}y + \sum_{j=0}^{M+N-1} W^j z_j, \quad z_0, \dots, z_{M+N-1} \in \ker V \tag{3.9}$$

The formulae (3.8) – (3.9) and Theorem 3.1 together imply that the initial value problem (0.1) – (0.2) is equivalent to the equation

$$(I+Q)x = W^{M+N}y + \sum_{j=0}^{M+N-1} W^j y_j, \tag{3.10}$$

(i) If $\lambda = -1$ is an eigenvalue of Q, then the corresponding homogeneous equation $(I+Q)x = 0$ has a non-trivial solution, i.e. the problem (0.1) – (0.2) is ill-posed and $I+Q$ is not invertible.

(ii) If $\lambda = -1$ is not an eigenvalue of Q and $I+Q$ is not invertible on X_{M+N}, i.e. $(I+Q)X_{M+N} \not\subseteq X_{M+N}$, then (3.10) is solvable if and only if

$$W^{M+N}y + \sum_{j=0}^{M+N-1} W^j y_j \in (I+Q)X_{M+N}. \tag{3.11}$$

Fix $u \in X_{M+N} \setminus (I+Q)X_{M+N}$ and let $y := V^{M+N}u$, $y_j := FV^ju$ ($j = 0, \ldots, M+N-1$). Then by the Taylor formula,

$$W^{M+N}y + \sum_{j=0}^{M+N-1} W^j y_j = W^{M+N}V^{M+N}u + \sum_{j=0}^{M+N-1} W^j F V^j u = u \notin (I+Q)X_{M+N},$$

i.e. the initial value problem (0.1) – (0.2) is ill-posed.

(iii) If $I+Q$ is invertible on X_{M+N} then from (3.10) we find a unique solution of the problem (0.1) – (0.2)

$$x = (I+Q)^{-1}\left(W^{M+N}y + \sum_{j=0}^{M+N-1} W^j y_j\right).$$

Now we can prove the main result for the initial value problem(0.1) – (0.2).

Theorem 3.2. Let $V \in R_1(X)$, $\ker V \subset \operatorname{Im} V$, $y \in Q[V]X_{M+N}$ and let $F \in \mathcal{F}_V$ ($G \in \mathcal{G}_V$) be a right (left) initial operator corresponding to $W \in \mathcal{R}_V^{(1)}$. Suppose that Q and $\tilde{Q}$ are given by (3.4) – (3.6), respectively. Write

$$H := \sum_{m=0}^{M}\sum_{n=0}^{N} W^{M-m}B_{mn}V^n.$$

(i) If the resolving operator $I+\tilde{Q}$ is invertible then the problem (0.1) – (0.2) is well-posed and its unique solution is of the form

$$x = \left[I - W^N(I+\tilde{Q})^{-1}H\right]\left(W^{M+N}y + \sum_{j=0}^{M+N-1} W^j y_j\right),$$

(ii) If the resolving operator $I+\tilde{Q}$ is right invertible and $\dim\ker(I+\tilde{Q}) \neq 0$ then the problem (0.1) – (0.2) is ill-posed. However, this problem has solutions of the form

$$x = \left(I - W^N R_{\tilde{Q}} H\right)\left(W^{M+N}y + \sum_{j=0}^{M+N-1} W^j y_j\right) + z,$$

where $R_{\tilde{Q}} \in \mathcal{R}_{I+\tilde{Q}}$ and $z \in \ker(I+Q)$ is arbitrary.

(iii) If the resolving operator $I+\tilde{Q}$ is left invertible, dim coker $(I+\tilde{Q}) \neq 0$ and $L_{\tilde{Q}} \in \mathcal{L}_{I+\tilde{Q}}$ then the problem (0.1) – (0.2) is ill-posed and has a solution if and only if the condition (3.11) is satisfied. If this is the case then a unique solution is

$$x = (I - W^N L_{\tilde{Q}} H) \left(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j y_j \right).$$

(iv) If the resolving operator $I + \tilde{Q}$ is generalized invertible, $\dim \ker(I + \tilde{Q}) \neq 0$ and $\dim \ \mathrm{coker}\,(I + \tilde{Q}) \neq 0$ then the problem $(0.1) - (0.2)$ is ill-posed and has solutions if and only if the condition (3.11) is satisfied. If this in the case then all solutions are given by

$$x = (I - W^N W_{\tilde{Q}} H) \left(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j y_j \right) + z,$$

where $W_{\tilde{Q}} \in \mathcal{W}_{I+\tilde{Q}}$ and $z \in \ker(I + Q)$ is arbitrary.
Proof. It is easy to check that

$$I + \tilde{Q} = I + HW^N \quad I + Q = I + W^N H.$$

Proving this in the same way as used in the Theorem 2.2 and Lemma 2.1, we get the results.

References

[1] D. Przeworska - Rolewicz, *Algebraic analysis,* PWN - Polish scientific Publishers and D. Reided Publishing Company, Warszawa - Dordrecht, 1988.

[2] D. Przeworska - Rolewicz and S. Rolewicz, *Equations in linear spaces,* Monografie Matematyczne 47, PWN - Polish Scientific Publishers, Warszawa 1968.

[3] Nguyen Van Mau, *Boundary value problems and controllability of linear systems with right invertible operators,* Dissertationes Math., CCCXVI, Warszawa 1992.

[4] Nguyen Van Mau and Nguyen Minh Tuan, *Algebraic properties of generalized right invertible operatos,* Demonstratio Math., 3(1997) 495 - 508.