ON THE SINGULAR INTEGRAL EQUATIONS WITH CARLEMAN SHIFT IN THE CASE OF THE VANISHING COEFFICIENT

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Abstract

Based on the well-known necessary and sufficient condition for the linear-fractional function to be the generator of a cyclic group of n-terms, this report obtains the general form of all linear-fractional functions being Carleman shifts on unit circle. Our main result is solving in a closed form for a class of singular integral equations with Carleman shifts on the unit circle in the case when the coefficient vanishes on the curve.

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Introduction

The Noetherian theory of singular integral equations of Cauchy's type was started with works of Noether and Carleman in 1921, and then it has been developed by many others (see [1], [2], [6] and reference therein). A reason that this theory attracts a lot of attention is that there is the effective relation between Riemann boundary-value problems of the analytic functions and the singular integral equations of Cauchy's type. In [4], the author studied the singular integral equation with the rotation on the unit circle under the assumption that it's coefficient has no zero-point on the curve. In general, the cases of vanishing coefficients in either differential equations or integral equations always challenge new study. In [7], the second author investigated the solvability in a closed form for a class of singular integral equations with the rotation on the unit circle in the case when the coefficient has isolate zero-point on the curve.
In this report, we study the solvability in a closed form for the class of singular integral equations with the Carleman shifts being the linear-fractional functions in the case when the coefficient has zero-point on the curve.

The cyclic group of linear-fractional functions on the complex plane and on the unit circle

We denote by W the set of all linear-fractional functions of the form

$$\omega(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

satisfying the condition $\alpha\delta - \beta\gamma = 1$.

Theorem 2.1. Let $\omega \in W$ be given. Then ω satisfies

$$\begin{cases} \omega^n \equiv I, \\ \omega^m \not\equiv I, m = 1, 2, \dots, n-1 \end{cases} \qquad (2.0.1)$$

if and only if

$$\begin{cases} \alpha + \delta = 2\cos\dfrac{k\pi}{n}, \quad k \in \{1, \dots, n-1\}, \quad (n, k) = 1, \\ \alpha\delta - \beta\gamma = 1. \end{cases}$$

Let $\Gamma = \{t \in C : |t| = 1\}$ be the unit circle on the complex plane C. In this section, we will determine a general form of the linear-fractional functions $\omega(z)$ satisfying the following condition:

$$\omega(\Gamma) \subset \Gamma, \quad \omega^n \equiv I, \quad \omega^m \not\equiv I, \; m = 1, 2, \dots, n-1, \tag{2.0.2}$$

where $n \in N$, $n \geq 2$ is given.

It is well-known that the linear-fractional function $\omega(z)$ mapping Γ into Γ if and only if that is of the form:

$$\omega(z) = e^{i\theta} \frac{z - \alpha}{\overline{\alpha} z - 1}$$

where $\theta \in R$, α is the zero-point of $\omega(z)$, i.e. $\alpha \in \overline{C}$, $\omega(\alpha) = 0$ (see [3], p.83)
So we have proved the following theorem

Theorem 2.2. Suppose that $\omega(z)$ is a linear-fractional function satisfying condition:

$$\omega(\Gamma) \subset \Gamma, \quad \omega^n \equiv I, \quad \omega^m \not\equiv I, \; m = 1, 2, \dots, n-1,$$

where $n \in N$, $n \geq 2$ is given.

1) If $n = 2$ then ω is of the form either;

$$\omega(z) = e^{i\theta} \frac{1}{\overline{z}}, \qquad \theta \in R$$

or

$$\omega(z) = \frac{z - \alpha}{\overline{\alpha} z - 1} \qquad |\alpha| \neq 1.$$

2) If $n > 2$ then w is of the form:

$$\omega(z) = e^{i\theta} \frac{z - \alpha}{\overline{\alpha} z - 1}$$

where $|\alpha| < 1$, $\cos\theta = 1 - 2(1 - \alpha\overline{\alpha})\cos^2 \frac{k\pi}{n}$, $k \in \{1, 2, \dots, n-1\}$, $(k,n) = 1$.

On the class of singular integral equations with shifts on unit circle

Let $\Gamma = \{t \in C : |t| = 1\}$ be the unit circle on the complex plane C. In this section, we consider the solvability in a closed form of the following equation in $H^\mu(\Gamma)$ $(0 < \mu < 1)$:

$$a(t)\varphi(t) + \frac{b(t)}{n} \sum_{k=0}^{n-1} \varepsilon_l^{n-k} \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \omega^k(t)} d\tau = f(t), \tag{3.0.1}$$

where $1 \leq l \leq n-1$, $\varepsilon_1 = e^{\frac{2\pi i}{n}}$, $e_l = \varepsilon_1^l$, $\omega(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma = 1$ is a linear-fractional function being the Carleman shift on Γ and *a(t), b(t), f(t)* are the given functions is $H^\mu(\Gamma)$.

Consider the following operators in $X := H^{\mu}(\Gamma)$:

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau, \tag{3.0.2}$$

$$(W\varphi)(t) = \varphi(\omega(t)), \tag{3.0.3}$$

$$P_k = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-j-1}W^{j+1}, \quad k = 1, 2, \ldots, n. \tag{3.0.4}$$

In the sequel we shall need the following identities (see [4], [6])

$$\begin{cases} W^k = \sum\limits_{j=1}^{n} \varepsilon_j^k P_j, & k = 1, 2, \ldots, n, \\ P_k P_j = \delta_{kj} P_j, & k, j = 1, 2, \ldots, n, \\ \sum\limits_{j=1}^{n} P_j = I, \end{cases} \tag{3.0.5}$$

where δ_{kj} is the Kronecker's symbol. For every $a \in X$ we write $(K_{a\varphi})(t) = a(t)\varphi(t)$.

Lemma 3.1. Let $\varphi \in X$. Then the following identity holds

$$(SW\varphi)(z) = (WS\varphi)(z) - (S\varphi)(\frac{\alpha}{\gamma}), \tag{3.0.6}$$

where α, γ are coefficient of $\omega(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$.

Lemma 3.2. Let $\varphi \in X$. Then

1) $(SW^k\varphi)(z) = (W^kS\varphi)(z) - (W^{k-1}S\varphi)(\frac{\alpha}{\gamma}), \; k = 1, 2, \ldots, n, \; W^0 = I$

2) $(P_kS\varphi)(z) = (SP_k\varphi)(z) + \frac{1}{\varepsilon_k - 1}(SP_k\varphi)(\frac{\alpha}{\gamma}), \quad k = 1, 2, \ldots, n-1.$

Now we represent the equation (3.0.1) in the following form:

$$a(t)\varphi(t) + b(t)(P_\ell S\varphi)(t) = f(t), \tag{3.0.7}$$

where $a, b, f \in X$ are given and S, P_l, $1 \le l \le n-1$ are the operators defined by (3.0.2), (3.0.3), (3.0.4). Suppose that the function $a(t)$ has isolated zero-point on Γ, i.e.

$$a(t) = \prod_{j=1}^{m}(t - \alpha_i)^{r_j}s(t),$$

where $\alpha_j \in \Gamma$, r_j are positive integers (j = 1, 2, ..., m) and $s(t)$ is a non-vanishing function on Γ. Without loss generality we may assume $s(t) = 1$.

Lemma 3.3. Let $\varphi \in X$. Then φ is a solution of (3.0.7) if and only if $\{\varphi_k = P_k\varphi,\ k = 1, 2, \ldots, n\}$ is a solution of the following system:

$$a^*(t)\varphi_k(t) + b^*_{k\ell}(t)(S\varphi_\ell)(t) + \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)(\frac{\alpha}{\gamma}) = f^*_k(t), \quad k = 1, 2, \ldots, n, \tag{3.0.8}$$

where

$$\begin{aligned} a^*(t) &= \prod_{j=1}^{n} a(\omega^{j+1}(t)), \\ b^*_{k\ell}(t) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon^{\ell-k}_{j+1} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t)), \\ f^*_k(t) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-1-j} f(\omega^{j+1}(t)) \prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t)). \end{aligned} \tag{3.0.9}$$

Lemma 3.4. If $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a solution of the system (3.0.8), then $(P_1\varphi_1, P_2\varphi_2, \ldots, P_n\varphi_n)$ is also its solution.

Theorem 3.1. If $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a solution of (3.0.8), then

$$\varphi = \sum_{i=1}^{n} P_i\varphi_i,$$

is a solution of equation (3.0.7).

Theorem 3.2. The equation (3.0.7) has solutions in X if and only if the equation:

$$a^*(t)\varphi(t) + b^*_{\ell\ell}(t)(S\varphi)(t) + \frac{b^*_{\ell\ell}(t)}{\varepsilon_\ell - 1}(S\varphi)(\frac{\alpha}{\gamma}) = f^*_\ell(t), \tag{3.0.10}$$

has a solution $\varphi_0(t)$ satisfying the following conditions:

$$\left\{f^*_k(t) - b^*_{k\ell}(t)(S\varphi_0)(t) - \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_0)(\frac{\alpha}{\gamma})\right\}_{(j,t_i)} = 0, \quad k = 1, 2, \ldots, n, \tag{3.0.11}$$

where $t_i \in \Omega$, $j=0, 1, \ldots, r_i$, $i=1, 2, \ldots, m$

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