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Author(s)	Dinh, Cong Huong
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ON THE DYNAMIC OF THE DISCRETE POPULATION MODELS

DINH CONG HUONG

Dept. of Science and Technology, Quy Nhon University,
170 An Duong Vuong, Quynhon, Binhdinh, Vietnam

ABSTRACT. The extinction, persistence and global stability in models of population growth

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m}), \quad n = 0, 1, \cdots$$

is investigated, where G is a function maps $\mathbb{R}_+^{m+1}$ to $\mathbb{R}_+$.

1. INTRODUCTION

Our main motivation in studying properties of solutions of a delay nonlinear difference equation

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m})$$

is the extinction, persistence, global stability and nontrivial periodicity in the model

$$x_{n+1} = \lambda x_n + F(x_{n-m})$$

of population growth in [1,2] and the convergence of solutions of the difference equation

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{m} \alpha_i F(x_{n-i})$$

in [3]. In this paper, we extend some results which is mentioned in [1, 2, 3].

Key words and phrases. Nonlinear difference equation, multiple delay, convergence, equilibrium.

2. The results

Consider the nonlinear difference equation with multiple delay

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m}), \tag{1.1}$$

where $n \in \mathbb{N}_0, x_{-m}, x_{-m+1}, \cdots, x_0$ are positive initial values and the function

$$G(z_0, z_1, \cdots, z_m) : \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \to \mathbb{R}_+.$$

By a solution of (1.1) we mean a sequence $\{x_n\}_{n \geq -m}$ of nonnegative numbers which satisfies (1.1) for all integers $n \geq 0$. Let $a_{-m}, \cdots, a_0$ be $m+1$ given nonnegative numbers. Then (1.1) has a unique solution $\{x_n\}_{n \geq -m}$ which satisfies the initial conditions

$$x_n = a_n, \qquad \text{for } n = -m, \cdots, 0.$$

We give conditions implying that every solution of this equation is extinction, persistence or global stability. First of all we have

Lemma 1. *If $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m < 1$ then there exists a number $s > 1$ such that*

$$\lambda s + \lambda_1 s^2 + \lambda_2 s^3 + \cdots + \lambda_m s^{m+1} < 1.$$

Lemma 2. *Let $\{\beta_n\}_n$ be a sequence which satisfy the following relations:*

$$\beta_0 = \beta_{-1} = \cdots = \beta_{-m} = 1,$$

$$\beta_{n+1} = \lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \cdots + \lambda_m \beta_{n-m}.$$

If $P := \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m > 1$ where $\lambda_i \geq 0$, then $\beta_n > 1, \quad \forall n \in \mathbb{N}_0$ and β_n is monotone increasing for $n \in \mathbb{N}_0$.

2.1. **The extinction.** A positive solution $\{x_n\}_n$ of (1.1) is called extinctive if

$$\lim_{n \to \infty} x_n = 0.$$

The following theorem gives a sufficient and necessary condition for extinctive populations.

Theorem 1. *Assume that $G(z_0, z_1, \cdots, z_m) \leqslant \sum_{i=0}^m \lambda_i z_i$ and $\sum_{i=0}^m \lambda_i < 1$. Then every solution of (1.1) converges to zero.*

Proof. Since $G(z_0, z_1, \cdots, z_m) \leqslant \sum_{i=0}^m \lambda_i z_i$, for a positive number $a > 1$ we have

$$a^{x_{n+1}} = a^{G(x_n, \cdots, x_{n-m})} \leqslant a^{\lambda_0 x_n} a^{\lambda_1 x_{n-1}} \cdots a^{\lambda_m x_{n-m}}.$$

Put $y_n = a^{x_n}$ then we have

$$y_{n+1} \leqslant [y_n]^{\lambda_0} . [y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}.$$

Since $x_n \geq 0, \forall n = -m, -m+1, \cdots$ we have $y_n \geq 1$. Hence, we have $\eta = \max\{y_{-m}, y_{-m+1}, \cdots, y_0\} \geq 1$. Using Lemma 1, we can prove the following estimations by introduction:

$$y_{n+1} \leqslant \eta^{s^{-n}}, \quad n \in \mathbb{N}_0. \tag{1.2}$$

For $n = 0$, we have

$$y_1 \leqslant [y_0]^{\lambda_0}.[y_{-1}]^{\lambda_1} \cdots [y_{-m}]^{\lambda_m} \leqslant \eta^{\lambda_0+\lambda_1+\cdots+\lambda_m} < \eta^1 = \eta^{s^{-0}}.$$

Assume that (1.2) holds for the steps $1, 2, \cdots, n$, we estimate the solution at step $n+1$ as follows:

$$\begin{aligned} y_{n+1} &\leqslant [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m} \\ &\leqslant \eta^{s^{-(n-1)}.\lambda_0}.\eta^{s^{-(n-2)}.\lambda_1} \cdots \eta^{s^{-(n-m+1)}.\lambda_m} \\ &= \eta^{s^{-n}.(\lambda_0 s+\lambda_1 s^2+\lambda_2 s^3+\cdots+\lambda_m s^{m+1})} \\ &\leqslant \eta^{s^{-n}}. \end{aligned}$$

This implies that $\lim\limits_{n\to\infty} y_n \leqslant \eta^0 = 1$. Since $y_n \geq 1$ for all n, we have $\lim\limits_{n\to\infty} y_n = 1$. This follows that $\lim\limits_{n\to\infty} x_n = 0$. The proof is completed.

Assume that equation (1.1) has a unique positive equilibrium $\overline{x}$. We have a sufficient condition for the global stability of the equilibrium $\overline{x}$.

2.2. **The Stability.** A positive solution $\{x_n\}_n$ of (1.1) is called global stability if there exist

$$\lim_{n\to\infty} x_n \in (0, \infty).$$

Theorem 2. *If $G(z_0, z_1, \cdots, z_m)$ satisfies Lipchitz condition in every variable z_i with Lipchitz factors L_i which satisfy $\sum_{i=0}^{m} L_i < 1$ then every solution of (1.1) is convergence to positive equilibrium $\overline{x}$.*

Proof. We have

$$\begin{aligned} |x_{n+1} - \overline{x}| &= |G(x_n, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, \overline{x}, \cdots, \overline{x})| \\ &\leqslant |G(x_n, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, x_{n-1}, \cdots, x_{n-m})| \\ &+ |G(\overline{x}, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, \overline{x}, x_{n-2}, \cdots, x_{n-m})| \\ &\cdots \\ &+ |G(\overline{x}, \overline{x}, \cdots, \overline{x}, x_{n-m}) - G(\overline{x}, \overline{x}, \cdots, \overline{x})| \\ &\leqslant L_0|x_n - \overline{x}| + L_1|x_{n-1} - \overline{x}| + \cdots + L_m|x_{n-m} - \overline{x}|. \end{aligned}$$

Put $y_n = |x_n - \overline{x}|$ then we have

$$y_{n+1} \leqslant L_0 y_n + L_1 y_{n-1} + \cdots + L_m y_{n-m}.$$

Applying Theorem 1, we have $\lim_{n\to\infty} y_n = 0$. It means $\lim_{n\to\infty} x_n = \overline{x}$. The proof is completed.

In converse condition, it means that $G(z_0, z_1, \cdots, z_m) \geq \sum_{i=0}^{m} \lambda_i z_i$ then the following theorem gives a sufficient condition for the non-convergence to zero of the solutions of (1.1)

Theorem 3. *If $\sum_{i=0}^{m} \lambda_i > 1$ then every solution $\{x_n\}$ of (1.1) satisfies*

$$\liminf_{n\to\infty} x_n > 0.$$

Proof. Similar to the above proof, we also put $y_n = a^{x_n}$ then we have

$$y_{n+1} \geq [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}.$$

let us denote $\theta = \min\{y_0, y_{-1}, \cdots, y_{-m}\}$ then $\theta > 1$. We prove $y_n \geq \theta^{\beta_n}$ by induction.

Clearly, $y_1 \geq [y_0]^{\lambda_0}.[y_{-1}]^{\lambda_1} \cdots [y_{-m}]^{\lambda_m} \geq \theta^{\lambda_0+\lambda_1+\lambda_2+\cdots+\lambda_m} = \theta^{\beta(1)}$. Assuming that $y_n \geq \theta^{\beta_n}$ for the steps $1, 2, \cdots, n$, we have

$$\begin{aligned} y_{n+1} &\geq [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m} \\ &\geq \theta^{\lambda_0\beta_n}.\theta^{\lambda_1\beta_{n-1}} \cdots \theta^{\lambda_m\beta_{n-m}} \\ &= \theta^{\lambda_0\beta_n+\lambda_1\beta_{n-1}+\cdots+\lambda_m\beta_{n-m}} \\ &= \theta^{\beta_{n+1}}. \end{aligned}$$

By Lemma 2, we have $y_{n+1} \geqslant \theta^{\beta_{n+1}} \geqslant \theta^{\beta_1} = \theta^P, \quad \forall n \in \mathbb{N}_0$. This implies that $x_{n+1} \geqslant P.\log_a \theta > 0$. Hence, $\liminf\limits_{n\to\infty} x_n \geqslant P.\log_a \theta > 0$.

2.3. **The Persistence.** A positive solution $\{x_n\}_n$ of (1.1) is called persistent if

$$0 < \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n < \infty.$$

The following theorem gives a sufficient condition for persistent (non-extinctive) populations.

Theorem 4. *Assume that*

$$G(x_0, x_1, \cdots, x_m) = H(x_0, x_1, \cdots, x_m, x_0, x_1, \cdots, x_m)$$

where

$$H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m) : [0, \infty)^{2(m+1)} \to [0, \infty)$$

is a continuous function, increasing in x_i but decreasing in y_i and

$$H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m) > 0$$

if $x_i, y_i > 0$. *Suppose further that*

$$\limsup_{x_i, y_i \to \infty} \frac{H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m)}{x_0 + x_1 + \cdots + x_m} < \frac{1}{m+1},$$

$$\liminf_{x_i, y_i \to 0^+} \frac{H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m)}{x_0 + x_1 + \cdots + x_m} > \frac{1}{m+1}.$$

Then every solution $\{x_n\}_{n=-m}^{\infty}$ *of (1.1) is persistent.*

Proof. The proof of this theorem can be obtained similarly as the proof of Theorem 2 in [1].

References

[1] Dang Vu Giang and Dinh Cong Huong, *Extinction, Persistence and Global stability in models of population growth*, *J. Math. Anal. Appl.* **308** (2005), 195-207.

[2] Dang Vu Giang and Dinh Cong Huong, *Nontrivial periodicity in discrete delay models of population growth*, *J. Math. Anal. Appl.* **305** (2005), 291-295.

[3] Dinh Cong Huong, *On the asymptotic behaviour of solutions of a nonlinear difference equation with bounded multiple delay*, Vietnam Journal of Mathematics. **34:2** (2006), 163-170.