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ON THE DYNAMIC OF THE DISCRETE POPULATION MODELS

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ABSTRACT. The extinction, persistence and global stability in models of population growth

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m}), \quad n = 0, 1, \cdots$$

is investigated, where G is a function maps $\mathbb{R}_+^{m+1}$ to $\mathbb{R}_+$.

1. INTRODUCTION

Our main motivation in studying properties of solutions of a delay nonlinear difference equation

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m})$$

is the extinction, persistence, global stability and nontrivial periodicity in the model

$$x_{n+1} = \lambda x_n + F(x_{n-m})$$

of population growth in [1,2] and the convergence of solutions of the difference equation

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{m} \alpha_i F(x_{n-i})$$

in [3]. In this paper, we extend some results which is mentioned in [1, 2, 3].

Key words and phrases. Nonlinear difference equation, multiple delay, convergence, equilibrium.

2. The results

Consider the nonlinear difference equation with multiple delay

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m}), \qquad (1.1)$$

where $n \in \mathbb{N}_0, x_{-m}, x_{-m+1}, \cdots, x_0$ are positive initial values and the function

$$G(z_0, z_1, \cdots, z_m) : \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \to \mathbb{R}_+.$$

By a solution of (1.1) we mean a sequence $\{x_n\}_{n\geq -m}$ of nonnegative numbers which satisfies (1.1) for all integers $n \geq 0$. Let $a_{-m}, \cdots, a_0$ be $m+1$ given nonnegative numbers. Then (1.1) has a unique solution $\{x_n\}_{n\geq -m}$ which satisfies the initial conditions

$$x_n = a_n, \qquad \text{for } n = -m, \cdots, 0.$$

We give conditions implying that every solution of this equation is extinction, persistence or global stability. First of all we have

Lemma 1. *If $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m < 1$ then there exists a number $s > 1$ such that*

$$\lambda s + \lambda_1 s^2 + \lambda_2 s^3 + \cdots + \lambda_m s^{m+1} < 1.$$

Lemma 2. *Let $\{\beta_n\}_n$ be a sequence which satisfy the following relations:*

$$\beta_0 = \beta_{-1} = \cdots = \beta_{-m} = 1,$$

$$\beta_{n+1} = \lambda_0\beta_n + \lambda_1\beta_{n-1} + \cdots + \lambda_m\beta_{n-m}.$$

If $P := \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m > 1$ where $\lambda_i \geq 0$, then $\beta_n > 1, \quad \forall n \in \mathbb{N}_0$ and β_n is monotone increasing for $n \in \mathbb{N}_0$.

2.1. The extinction.

A positive solution $\{x_n\}_n$ of (1.1) is called extinctive if

$$\lim_{n\to\infty} x_n = 0.$$

The following theorem gives a sufficient and necessary condition for extinctive populations.

Theorem 1. *Assume that $G(z_0, z_1, \cdots, z_m) \leqslant \sum_{i=0}^m \lambda_i z_i$ and $\sum_{i=0}^m \lambda_i < 1$. Then every solution of (1.1) converges to zero.*

Proof. Since $G(z_0, z_1, \cdots, z_m) \leqslant \sum_{i=0}^m \lambda_i z_i$, for a positive number $a > 1$ we have

$$a^{x_{n+1}} = a^{G(x_n, \cdots, x_{n-m})} \leqslant a^{\lambda_0 x_n} a^{\lambda_1 x_{n-1}} \cdots a^{\lambda_m x_{n-m}}.$$

Put $y_n = a^{x_n}$ then we have

$$y_{n+1} \leqslant [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}.$$

Since $x_n \geq 0, \forall n = -m, -m+1, \cdots$ we have $y_n \geq 1$. Hence, we have $\eta = \max\{y_{-m}, y_{-m+1}, \cdots, y_0\} \geq 1$. Using Lemma 1, we can prove the following estimations by introduction:

$$y_{n+1} \leqslant \eta^{s^{-n}}, \quad n \in \mathbb{N}_0. \tag{1.2}$$

For $n = 0$, we have

$$y_1 \leqslant [y_0]^{\lambda_0}.[y_{-1}]^{\lambda_1} \cdots [y_{-m}]^{\lambda_m} \leqslant \eta^{\lambda_0+\lambda_1+\cdots+\lambda_m} < \eta^1 = \eta^{s^{-0}}.$$

Assume that (1.2) holds for the steps $1, 2, \cdots, n$, we estimate the solution at step $n+1$ as follows:

$$\begin{aligned} y_{n+1} &\leqslant [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m} \\ &\leqslant \eta^{s^{-(n-1)}.\lambda_0}.\eta^{s^{-(n-2)}.\lambda_1} \cdots \eta^{s^{-(n-m+1)}.\lambda_m} \\ &= \eta^{s^{-n}.(\lambda_0 s+\lambda_1 s^2+\lambda_2 s^3+\cdots+\lambda_m s^{m+1})} \\ &\leqslant \eta^{s^{-n}}. \end{aligned}$$

This implies that $\lim_{n\to\infty} y_n \leqslant \eta^0 = 1$. Since $y_n \geq 1$ for all n, we have $\lim_{n\to\infty} y_n = 1$. This follows that $\lim_{n\to\infty} x_n = 0$. The proof is completed.

Assume that equation (1.1) has a unique positive equilibrium $\overline{x}$. We have a sufficient condition for the global stability of the equilibrium $\overline{x}$.

2.2. **The Stability.** A positive solution $\{x_n\}_n$ of (1.1) is called global stability if there exist

$$\lim_{n\to\infty} x_n \in (0, \infty).$$

Theorem 2. *If $G(z_0, z_1, \cdots, z_m)$ satisfies Lipchitz condition in every variable z_i with Lipchitz factors L_i which satisfy $\sum_{i=0}^{m} L_i < 1$ then every solution of (1.1) is convergence to positive equilibrium $\overline{x}$.*

Proof. We have

$$\begin{aligned} |x_{n+1} - \overline{x}| &= |G(x_n, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, \overline{x}, \cdots, \overline{x})| \\ &\leqslant |G(x_n, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, x_{n-1}, \cdots, x_{n-m})| \\ &+ |G(\overline{x}, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, \overline{x}, x_{n-2}, \cdots, x_{n-m})| \\ &\cdots \\ &+ |G(\overline{x}, \overline{x}, \cdots, \overline{x}, x_{n-m}) - G(\overline{x}, \overline{x}, \cdots, \overline{x})| \\ &\leqslant L_0|x_n - \overline{x}| + L_1|x_{n-1} - \overline{x}| + \cdots + L_m|x_{n-m} - \overline{x}|. \end{aligned}$$

Put $y_n = |x_n - \overline{x}|$ then we have

$$y_{n+1} \leqslant L_0 y_n + L_1 y_{n-1} + \cdots + L_m y_{n-m}.$$

Applying Theorem 1, we have $\lim_{n\to\infty} y_n = 0$. It means $\lim_{n\to\infty} x_n = \overline{x}$. The proof is completed.

In converse condition, it means that $G(z_0, z_1, \cdots, z_m) \geq \sum_{i=0}^{m} \lambda_i z_i$ then the following theorem gives a sufficient condition for the non-convergence to zero of the solutions of (1.1)

Theorem 3. *If $\sum_{i=0}^{m} \lambda_i > 1$ then every solution $\{x_n\}$ of (1.1) satisfies*

$$\liminf_{n\to\infty} x_n > 0.$$

Proof. Similar to the above proof, we also put $y_n = a^{x_n}$ then we have

$$y_{n+1} \geq [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}.$$

let us denote $\theta = \min\{y_0, y_{-1}, \cdots, y_{-m}\}$ then $\theta > 1$. We prove $y_n \geq \theta^{\beta_n}$ by induction.

Clearly, $y_1 \geq [y_0]^{\lambda_0}.[y_{-1}]^{\lambda_1} \cdots [y_{-m}]^{\lambda_m} \geq \theta^{\lambda_0+\lambda_1+\lambda_2+\cdots+\lambda_m} = \theta^{\beta(1)}$. Assuming that $y_n \geq \theta^{\beta_n}$ for the steps $1, 2, \cdots, n$, we have

$$\begin{aligned} y_{n+1} &\geq [y_n]^{\lambda_0}.[y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m} \\ &\geq \theta^{\lambda_0\beta_n}.\theta^{\lambda_1\beta_{n-1}} \cdots \theta^{\lambda_m\beta_{n-m}} \\ &= \theta^{\lambda_0\beta_n+\lambda_1\beta_{n-1}+\cdots+\lambda_m\beta_{n-m}} \\ &= \theta^{\beta_{n+1}}. \end{aligned}$$

By Lemma 2, we have $y_{n+1} \geqslant \theta^{\beta_{n+1}} \geqslant \theta^{\beta_1} = \theta^P, \quad \forall n \in \mathbb{N}_0$. This implies that $x_{n+1} \geqslant P.\log_a \theta > 0$. Hence, $\liminf_{n\to\infty} x_n \geqslant P.\log_a \theta > 0$.

2.3. **The Persistence.** A positive solution $\{x_n\}_n$ of (1.1) is called persistent if

$$0 < \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n < \infty.$$

The following theorem gives a sufficient condition for persistent (non-extinctive) populations.

Theorem 4. *Assume that*

$$G(x_0, x_1, \cdots, x_m) = H(x_0, x_1, \cdots, x_m, x_0, x_1, \cdots, x_m)$$

where

$$H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m) : [0, \infty)^{2(m+1)} \to [0, \infty)$$

is a continuous function, increasing in x_i but decreasing in y_i and

$$H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m) > 0$$

if $x_i, y_i > 0$. *Suppose further that*

$$\limsup_{x_i,y_i\to\infty} \frac{H(x_0,x_1,\cdots,x_m,y_0,y_1,\cdots,y_m)}{x_0+x_1+\cdots+x_m} < \frac{1}{m+1},$$

$$\liminf_{x_i,y_i\to 0^+} \frac{H(x_0,x_1,\cdots,x_m,y_0,y_1,\cdots,y_m)}{x_0+x_1+\cdots+x_m} > \frac{1}{m+1}.$$

Then every solution $\{x_n\}_{n=-m}^{\infty}$ *of (1.1) is persistent.*

Proof. The proof of this theorem can be obtained similarly as the proof of Theorem 2 in [1].

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