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# ON THE DYNAMIC OF THE DISCRETE POPULATION MODELS

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ABSTRACT. The extinction, persistence and global stability in models of population growth

 $x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m}), \quad n = 0, 1, \cdots$ is investigated, where G is a function maps  $\mathbb{R}^{m+1}_+$  to  $\mathbb{R}_+$ .

## 1. INTRODUCTION

Our main motivation in studying properties of solutions of a delay nonlinear difference equation

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m})$$

is the extinction, persistence, global stability and nontrivial periodicity in the model

$$x_{n+1} = \lambda x_n + F(x_{n-m})$$

of population growth in [1,2] and the convergence of solutions of the difference equation

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^m \alpha_i F(x_{n-i})$$

in [3]. In this paper, we extend some results which is mentioned in [1, 2, 3].

 $Key\ words\ and\ phrases.$  Nonlinear difference equation, multiple delay, convergence, equilibrium.

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### 2. The results

Consider the nonlinear difference equation with multiple delay

$$x_{n+1} = G(x_n, x_{n-1}, \cdots, x_{n-m}),$$
 (1.1)

where  $n \in \mathbb{N}_0, x_{-m}, x_{-m+1}, \cdots, x_0$  are positive initial values and the function

$$G(z_0, z_1, \cdots, z_m) : \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \to \mathbb{R}_+.$$

By a solution of (1.1) we mean a sequence  $\{x_n\}_{n\geq -m}$  of nonnegative numbers which satisfies (1.1) for all integers  $n \geq 0$ . Let  $a_{-m}, \dots, a_0$  be m+1 given nonnegative numbers. Then (1.1) has a unique solution  $\{x_n\}_{n\geq -m}$  which satisfies the initial conditions

$$x_n = a_n,$$
 for  $n = -m, \cdots, 0.$ 

We give conditions implying that every solution of this equation is extinction, persistence or global stability. First of all we have

**Lemma 1.** If  $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m < 1$  then there exists a number s > 1 such that

$$\lambda s + \lambda_1 s^2 + \lambda_2 s^3 + \dots + \lambda_m s^{m+1} < 1.$$

**Lemma 2.** Let  $\{\beta_n\}_n$  be a sequence which satisfy the following relations:

$$\beta_0 = \beta_{-1} = \dots = \beta_{-m} = 1,$$
  
$$_{n+1} = \lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \dots + \lambda_m \beta_{n-m}$$

If  $P := \lambda_0 + \lambda_1 + \lambda_2 + \dots + \lambda_m > 1$  where  $\lambda_i \ge 0$ , then  $\beta_n > 1$ ,  $\forall n \in \mathbb{N}_0$ and  $\beta_n$  is monotone increasing for  $n \in \mathbb{N}_0$ .

2.1. The extinction. A positive solution  $\{x_n\}_n$  of (1.1) is called extinctive if

$$\lim_{n \to \infty} x_n = 0.$$

The following theorem gives a sufficient and necessary condition for extinctive populations.

**Theorem 1.** Assume that  $G(z_0, z_1, \dots, z_m) \leq \sum_{i=0}^m \lambda_i z_i$  and  $\sum_{i=0}^m \lambda_i < 1$ . Then every solution of (1.1) converges to zero.

*Proof.* Since  $G(z_0, z_1, \dots, z_m) \leq \sum_{i=0}^m \lambda_i z_i$ , for a positive number a > 1 we have

$$a^{x_{n+1}} = a^{G(x_n, \cdots, x_{n-m})} \leqslant a^{\lambda_0 x_n} a^{\lambda_1 x_{n-1}} \cdots a^{\lambda_m x_{n-m}}$$

Put  $y_n = a^{x_n}$  then we have

$$y_{n+1} \leq [y_n]^{\lambda_0} \cdot [y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}.$$

Since  $x_n \ge 0, \forall n = -m, -m+1, \cdots$  we have  $y_n \ge 1$ . Hence, we have  $\eta = \max\{y_{-m}, y_{-m+1}, \cdots, y_0\} \ge 1$ . Using Lemma 1, we can prove the following estimations by introduction:

$$y_{n+1} \leqslant \eta^{s^{-n}}, \quad n \in \mathbb{N}_0.$$

For n = 0, we have

$$y_1 \leqslant [y_0]^{\lambda_0} \cdot [y_{-1}]^{\lambda_1} \cdots [y_{-m}]^{\lambda_m} \leqslant \eta^{\lambda_0 + \lambda_1 + \dots + \lambda_m} < \eta^1 = \eta^{s^{-0}}.$$

Assume that (1.2) holds for the steps  $1, 2, \dots, n$ , we estimate the solution at step n + 1 as follows:

$$y_{n+1} \leqslant [y_n]^{\lambda_0} \cdot [y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}$$
  
$$\leqslant \eta^{s^{-(n-1)} \cdot \lambda_0} \cdot \eta^{s^{-(n-2)} \cdot \lambda_1} \cdots \eta^{s^{-(n-m+1)} \cdot \lambda_m}$$
  
$$= \eta^{s^{-n} \cdot (\lambda_0 s + \lambda_1 s^2 + \lambda_2 s^3 + \dots + \lambda_m s^{m+1})}$$
  
$$\leqslant \eta^{s^{-n}} \cdot$$

This implies that  $\lim_{n\to\infty} y_n \leq \eta^0 = 1$ . Since  $y_n \geq 1$  for all n, we have  $\lim_{n\to\infty} y_n = 1$ . This follows that  $\lim_{n\to\infty} x_n = 0$ . The proof is completed.

Assume that equation (1.1) has a unique positive equilibrium  $\overline{x}$ . We have a sufficient condition for the global stability of the equilibrium  $\overline{x}$ .

2.2. The Stability. A positive solution  $\{x_n\}_n$  of (1.1) is called global stability if there exist

 $\lim_{n \to \infty} x_n \in (0, \infty).$ 

**Theorem 2.** If  $G(z_0, z_1, \dots, z_m)$  satisfies Lipchitz condition in every variable  $z_i$  with Lipchitz factors  $L_i$  which satisfy  $\sum_{i=0}^m L_i < 1$  then every solution of (1.1) is convergence to positive equilibrium  $\overline{x}$ .

Proof. We have

$$\begin{aligned} |x_{n+1} - \overline{x}| &= |G(x_n, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, \overline{x}, \cdots, \overline{x})| \\ &\leqslant |G(x_n, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, x_{n-1}, \cdots, x_{n-m})| \\ &+ |G(\overline{x}, x_{n-1}, \cdots, x_{n-m}) - G(\overline{x}, \overline{x}, x_{n-2}, \cdots, x_{n-m})| \\ &\cdots \\ &+ |G(\overline{x}, \overline{x}, \cdots, \overline{x}, x_{n-m}) - G(\overline{x}, \overline{x}, \cdots, \overline{x})| \\ &\leqslant L_0 |x_n - \overline{x}| + L_1 |x_{n-1} - \overline{x}| + \cdots + L_m |x_{n-m} - \overline{x}|. \end{aligned}$$

Put  $y_n = |x_n - \overline{x}|$  then we have

$$y_{n+1} \leq L_0 y_n + L_1 y_{n-1} + \dots + L_m y_{n-m}.$$

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Applying Theorem 1, we have  $\lim_{n\to\infty} y_n = 0$ . It means  $\lim_{n\to\infty} x_n = \overline{x}$ . The proof is completed.

In converse condition, it means that  $G(z_0, z_1, \dots, z_m) \geq \sum_{i=0}^m \lambda_i z_i$  then the following theorem gives a sufficient condition for the non-convergence to zero of the solutions of (1.1)

**Theorem 3.** If  $\sum_{i=0}^{m} \lambda_i > 1$  then every solution  $\{x_n\}$  of (1.1) satisfies  $\liminf_{n \to \infty} x_n > 0.$ 

*Proof.* Similar to the above proof, we also put  $y_n = a^{x_n}$  then we have

$$y_{n+1} \ge [y_n]^{\lambda_0} \cdot [y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_m}.$$

let us denote  $\theta = \min\{y_0, y_{-1}, \cdots, y_{-m}\}$  then  $\theta > 1$ . We prove  $y_n \ge \theta^{\beta_n}$  by induction.

Clearly,  $y_1 \ge [y_0]^{\lambda_0} \cdot [y_{-1}]^{\lambda_1} \cdots [y_{-m}]^{\lambda_m} \ge \theta^{\lambda_0 + \lambda_1 + \lambda_2 + \dots + \lambda_m} = \theta^{\beta(1)}$ . Assuming that  $y_n \ge \theta^{\beta_n}$  for the steps  $1, 2, \cdots, n$ , we have

$$y_{n+1} \ge [y_n]^{\lambda_0} \cdot [y_{n-1}]^{\lambda_1} \cdots [y_{n-m}]^{\lambda_1} \\ \ge \theta^{\lambda_0 \beta_n} \cdot \theta^{\lambda_1 \beta_{n-1}} \cdots \theta^{\lambda_m \beta_{n-m}} \\ = \theta^{\lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \dots + \lambda_m \beta_{n-m}} \\ = \theta^{\beta_{n+1}} \cdot$$

By Lemma 2, we have  $y_{n+1} \ge \theta^{\beta_{n+1}} \ge \theta^{\beta_1} = \theta^P$ ,  $\forall n \in \mathbb{N}_0$ . This implies that  $x_{n+1} \ge P \cdot \log_a \theta > 0$ . Hence,  $\liminf_{n \to \infty} x_n \ge P \cdot \log_a \theta > 0$ .

2.3. The Persistence. A positive solution  $\{x_n\}_n$  of (1.1) is called persistent if

$$0 < \liminf_{n \to \infty} x_n \leqslant \limsup_{n \to \infty} x_n < \infty.$$

The following theorem gives a sufficient condition for persistent (non-extinctive) populations.

Theorem 4. Assume that

$$G(x_0, x_1, \cdots, x_m) = H(x_0, x_1, \cdots, x_m, x_0, x_1, \cdots, x_m)$$

where

 $H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m) : [0, \infty)^{2(m+1)} \to [0, \infty)$ 

is a continuous function, increasing in  $x_i$  but decreasing in  $y_i$  and

$$H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m) > 0$$

if  $x_i, y_i > 0$ . Suppose further that

$$\limsup_{x_i, y_i \to \infty} \frac{H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m)}{x_0 + x_1 + \cdots + x_m} < \frac{1}{m+1},$$
$$\liminf_{x_i, y_i \to 0^+} \frac{H(x_0, x_1, \cdots, x_m, y_0, y_1, \cdots, y_m)}{x_0 + x_1 + \cdots + x_m} > \frac{1}{m+1}.$$

Then every solution  $\{x_n\}_{n=-m}^{\infty}$  of (1.1) is persistent.

*Proof.* The proof of this theorem can be obtained similarly as the proof of Theorem 2 in [1].

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