Controllability of Linear Systems with Generalized Invertible Operators

Nguyen Van Mau
Hanoi University of Science, VNUH

1 Controllability of first order linear systems with right invertible operators

Let X, Y and U be linear spaces (all over the same field $\mathcal{F}$, where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$). Suppose that $D \in R(X)$, $\dim \ker D \neq 0$, $F \in \mathcal{F}_D$ corresponds to an $R \in \mathcal{R}_D$, $A \in L_0(X)$, $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$ (cf. Section 1). By a first order linear system (shortly: (LS)) we mean the system

$$Dx = Ax + Bu, \;\; RBU \oplus \{x_0\} \subset (I - RA)(\mathrm{dom\,D}), \tag{1.1}$$

$$Fx = x_0, \;\; x_0 \in \ker D, \tag{1.2}$$

$$y = A_1 x + B_1 u. \tag{1.3}$$

The spaces X and U are called the space of states and the space of controls, respectively. The element $x_0 \in \ker D$ is called an initial state. A pair $(x_0, u) \in (\ker D) \times U$ is called an input. The space $(\ker D) \times U$ is called the input space, and the corresponding set of y's in Y the output space. Very often there are considered linear systems with $A_1 = I$ and $B_1 = 0$, i.e. with $Y = X$ and the output $y = x$. We shall denote such systems by $(LS)_0$.

The properties of linear systems depend on the properties of the resolving operators $I - RA$ and $I - AR$, respectively. In a series of papers (cf. [54-56]) Nguyen Dinh Quyet studied some properties of linear systems in the case $I - RA$ invertible. His results concerning controllability were generalized by Pogorzelec [84-85] in the case $I - RA$ and $I - AR$ either left or right invertible, and in the case $I - AR$ invertible.

Hence, there are six cases to deal with:

(i) $I - RA \in R(X)$, (ii) $I - RA \in \mathbb{L}(X)$, (iii) $I - RA$ is invertible,
(iv) $I - AR \in R(X)$, (v) $I - AR \in \mathbb{L}(X)$, (vi) $I - AR$ is invertible.

We show that $I - RA$ is right invertible (left invertible, invertible) if and only if so is $I - AR$, i.e. it is sufficient to consider the first three cases. On the other hand, since every one-sided invertible operator and every invertible operator are generalized almost invertible, we can reduce those cases to the case of $I - RA$ being generalized almost invertible.

Suppose that we are given a linear system $(LS)_0$. The initial value problem (1.1)-(1.2) is equivalent to the equation

$$(I - RA)x = RBu + x_0. \tag{1.4}$$

Hence, the inclusion

$$RBU \oplus \{x_0\} \subset (I - RA)(\mathrm{dom\,D}) \tag{1.5}$$

is a necessary and sufficient condition for the problem (1.1)-(1.2) to have solutions for every $u \in U$.

Denote by G_i $(i = 1, 2, 3, 4)$ following sets defined for every $x_0 \in \ker D$, $u \in U$:

(i) If $I - RA \in R(X)$ and $T_1 \in \mathcal{R}_{\mathcal{I}-\mathcal{RA}}$, then

$$G_1(x_0, u) := \{x = R_1(RBu + x_0) + z : \ z \in \ker(I - RA)\}. \tag{1.6}$$

(ii) If $I - RA \in \mathbb{L}(X)$ and $T_2 \in \mathrm{L}_{I-RA}$, then

$$G_2(x_0, u) := \{x = T_2(RBu + x_0)\}. \tag{1.7}$$

(iii) If $I - RA$ is invertible, then

$$G_3(x_0, u) := \{x = T_3(RBu + x_0)\}, \ T_3 = (I - RA)^{-1}. \tag{1.8}$$

(iv) If $I - RA \in W(X)$ and $T_4 \in \mathcal{W}_{\mathcal{I}-\mathcal{RA}}$, then

$$G_4(x_0, u) := \{x = T_4(RBu + x_0) + z : \ z \in \ker(I - RA)\}. \tag{1.9}$$

Note that the G_i are the sets of all solutions of the problem (1.1)- (1.2) in the corresponding cases. Therefore, to every fixed input (x_0, u) there corresponds an output $x \in G_i(x_0, u)$ for each case.

Definition 1.1. Suppose that we are given a system $(LS)_0$ and the sets $G_i(x_0, u)$ of the forms (1.6)-(1.9). A state $x \in X$ is said to be (i)-reachable $(i = 1, 2, 3, 4)$ from an initial state $x_0 \in \ker D$ if for every T_i ($T_1 \in \mathcal{R}_{\mathcal{I}-\mathcal{RA}}$, $T_2 \in \mathrm{L}_{I-RA}$, $T_3 = (I - RA)^{-1}$, $T_4 \in \mathcal{W}_{\mathcal{I}-\mathcal{RA}}$) there exists a control $u \in U$ such that $x \in G_i(x_0, u)$.

Write

$$\text{Rang}_{U,x_0} G_i = \bigcup_{u \in U} G_i(x_0, u), \quad x_0 \in \ker D \quad (i = 1, 2, 3, 4).$$

It is easy to see that $\text{Rang}_{U,x_0} G_i$ is (i)- reachable from $x_0 \in \ker D$ by means of controls $u \in U$ and it is contained in dom D.

Lemma 1.1. Suppose that T_i (i = 1,2,3,4) are defined as in (1.6)- (1.9). Then

$$T_i(RBU \oplus \{x_0\}) + \ker(I - RA) = T_i RBU \oplus \{T_i x_0\} \oplus \ker(I - RA). \quad (1.10)$$

Remark 1.1. If either $I - RA \in \mathbb{L}(X)$ or $I - RA$ is invertible then $\ker(I - RA) = \{0\}$, and (1.10) takes the form $T_i(RBU \oplus \{x_0\}) = T_i RBU \oplus \{T_i x_0\}$.

The formulae (1.5)-(1.9) imply

Corollary 1.1.

$$\text{Rang}_{U,x_0} G_i = T_i RBU \oplus \{T_i x_0\} \oplus \ker(I - RA). \quad (1.11)$$

Corollary 1.2. A state x is (i)-reachable from a given initial state $x_0 \in \ker D$ if and only if

$$x \in T_i RBU \oplus \{T_i x_0\} \oplus \ker(I - RA), \quad i = 1, 2, 3, 4. \quad (1.12)$$

Lemma 1.2. Write

$$E_i := T_i RB, \quad X_i := T_i(I - RA)(\text{dom D}) - \{\text{x}_0\}.$$

Then the operator E_i maps U into X_i.

Proof. By our assumption, $RBU \oplus \{x_0\} \subset (I - RA)(\text{dom D})$, thus for every $u \in U$ there exist $v \in X$ and $x_1 \in \ker D$ such that

$$RBu + x_0 = (I - RA)(Rv + x_1),$$

i.e. $T_i RBu = T_i[(I - RA)(Rv + x_1) - x_0]$.

Theorem 1.1. Suppose that $B \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$ $R \in L_0(X, X')$ and $T_i \in L_0(X, X')$ $(i = 1, 2, 3, 4)$. Then the generalized Kalman condition

$$\ker B^* R^* T_i^* = \{0\} \quad (1.13)$$

holds if and only if for every initial state $x_0 \in \ker D$, every state $x \in RX \oplus \{x_0\} + \ker(I - RA)$ is (i)-reachable from x_0.

Proof. By Lemma 1.2, the condition (1.13) holds if and only if for every $x_1 \in \ker D$ and $v \in X$ there exists $u \in U$ such that $RBu + x_0 = (I - RA)(Rv + x_1)$. This means that for every $x_1 \in \ker D$, $v \in X$ and $z \in \ker(I - RA)$ there exists $u \in U$ such that

$$T_i(RBu + x_0) + z = T_i(I - RA)(Rv + x_1) + z. \tag{1.14}$$

It is sufficient to consider $i = 4$, i.e. the case when $(I - RA)$ is generalized almost invertible. Write $F' := I - T_4(I - RA)$. It is easy to check that $(I - RA)F' = 0$, $F_2' = F'$ and $F'X = \ker(I - RA)$. Choosing $x_1 := x_0$, $z := F'(Rv + x_1) \in \ker(I - RA)$, we get from (1.14) the equalities

$$T_4(RBu + x_0) + z = (I - F')(Rv + x_0) + F'(Rv + x_0) = Rv + x_0.$$

This means that for every $v \in X$, $z_1 \in \ker(I - RA)$ there exist $z' = z_1 + F'(Rv + x_0) \in \ker(I - RA)$ and $u \in U$ such that

$$T_4(RBu + x_0) + z' \in RX \oplus \{x_0\} + \ker(I - RA),$$

i.e.

$$\mathrm{Rang}_{U,x_0} G_4 = RX \oplus \{x_0\} + \ker(I - RA).$$

Note that the generalized Kalman condition (1.13) in the case of $(I - RA)$ invertible was introduced and applied by Nguyen Dinh Quyet [54-56]. Theorem 1.1 in the case of $I - RA$ one-sided invertible was obtained by Pogorzelec [84].

Now we give another condition for every state $x \in RX + \{T_i x_0\} + \ker(I - RA)$ to be (i)-reachable from any $x_0 \in \ker D$. To begin with, note that

$$T_i RX \subset RX \quad (i = 1, 2, 3, 4). \tag{1.15}$$

Indeed, there exist T_i' $(i = 1, 2, 3, 4)$ such that $T_i = I + RT_i'A$. Thus

$$T_i RX = (I + RT_i'A)RX = R(I + T_i'AR)X \subset RX.$$

Therefore, $T_i RB$ map U into RX. Corollary 6.1 gives the following

Theorem 1.2. A necessary and sufficient condition for every element

$$x \in RX + \{T_i x_0\} + \ker(I - RA)$$

to be (i)-reachable from any initial state $x_0 \in \ker D$ is that $T_i RBU = RX$.

Definition 1.2. Let there be given a linear system $(LS)_0$ of the form (1.1)-(1.2). Let $F_i \in \mathcal{F}_\mathcal{D}$ $(i = 1, 2, 3, 4)$ be arbitrary initial operators (not necessarily different).

(i) A state $x_1 \in \ker D$ is said to be F_i-reachable from an initial state $x_0 \in \ker D$ if there exists a control $u \in U$ such that $x_1 \in F_i G_i(x_0, u)$. The state x_1 is then called a final state.

(ii) The system $(LS)_0$ is said to be F_i-controllable if for every initial state $x_0 \in \ker D$,

$$F_i(\text{ Rang}_{U,x_0} G_i) = \ker D. \tag{1.16}$$

(iii) The system $(LS)_0$ is said to be F_i-controllable to $x_1 \in \ker D$ if

$$x_i \in F_i(\text{ Rang}_{U,x_0} G_i) \tag{1.17}$$

for every initial state $x_0 \in \ker D$.

Lemma 1.3. Let there be given a linear system $(LS)_0$ and an initial operator $F_i \in \mathcal{F}_\mathcal{D}$. Suppose that the system $(LS)_0$ is F_i-controllable to zero and that

$$F_i(T_i \ker D + \ker(I - RA)) = \ker D. \tag{1.18}$$

Then every final state $x_1 \in \ker D$ is F_i-reachable from zero.

Theorem 1.3. Suppose that all assumptions of Lemma 1.3 are satisfied. Then the system $(LS)_0$ is F_i-controllable.

Proof. Suppose that $I - RA \in W(X)$. By our assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I - RA)$ such that

$$F_4[T_4(RBu_0 + x_0) + z_0] = 0. \tag{1.21}$$

By Lemma 1.3, for every $x_1 \in \ker D$ there exist $u_0' \in U$ and $z_1 \in \ker(I - RA)$ such that

$$F_4(T_4 RBu_0' + z_1) = x_1. \tag{1.22}$$

Add (1.21) and (1.22) to find

$$F_4\{T_4[RB(u_0 + u_0') + x_0] + (z_0 + z_1)\} = x_1,$$

i.e. x_1 is F_4-reachable from x_0, which was to be proved.

Corollary 1.4 (cf. Pogorzelec [84]). Let $T_1' \in \mathcal{R}_{\mathcal{I}-\mathcal{AR}}$, $T_2' \in \mathrm{L}_{I-AR}$, $T_3' = (I - AR)^{-1}$ and $T_4' \in \mathcal{W}_{\mathcal{I}-\mathcal{AR}}$ for $I - AR \in R(X)$, $I - AR \in \mathbb{L}(X)$, $I - AR$ invertible and $I - AR \in W(X)$, respectively. If the system $(LS)_0$ is F_i-controllable to zero and

$$F_i(I + RT_i' A)(\ker D) = \ker D, \tag{1.23}$$

then $(LS)_0$ is F_i-controllable.

Indeed, by (6.10)-(6.12), $I + RT_i'A = T_i$. Therefore (1.23) takes the form $F_iT_i(\ker D) = \ker D$ and we get a sufficient condition for F_i-controllability.

Corollary 1.5 (cf. Pogorzelec [84-85]). If the system $(LS)_0$ is F_i-controllable to zero and $F_iT_i(\ker D) = \ker D$, then $(LS)_0$ is F_i-controllable.

So the conditions $F_iT_i(\ker D) = \ker D$ and $F_i(I + RT_i'A)(\ker D) = \ker D$, found by Pogorzelec for the one-sided invertible resolving operators, are identical.

Theorem 1.4. Let a linear system $(LS)_0$ of the form (1.1)-(1.2) and an initial operator $F_i \in \mathcal{F}_D$ be given. Let $T_1 \in \mathcal{R}_{\mathcal{I}-\mathcal{R}\mathcal{A}}$ if $I - RA \in R(X)$ is invertible,

$T_2 \in \mathrm{L}_{I-RA}$ if $I - RA$ is left invertible,

$T_3 = (I - RA)^{-1}$ if $I - RA$ is invertible and

$T_4 \in \mathcal{W}_{\mathcal{I}-\mathcal{R}\mathcal{A}}$ if $I - RA$ is generalized almost invertible.

Suppose that $B \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$, $A, R \in L_0(X, X')$. Then the system $(LS)_0$ is F_i-controllable if and only if

$$\ker B^*R^*T_i^*F_i^* = \{0\}. \tag{1.24}$$

Theorem 1.5. Let there be given a linear system $(LS)_0$ and an initial operator $F_i \in \mathcal{F}_D$. Then the system $(LS)_0$ is F_i-controllable if and only if it is F_i-controllable to every element $v' \in F_iT_iRX$.

Corollary 1.6. The system $(LS)_0$ is F_i-controllable if and only if it is F_i-controllable to every element $v_0 \in F_iRX$.

Indeed, it is easy to check that $T_iRX \subset RX$. Thus $F_iT_iRX \subset F_iRX$.

Theorem 1.6. Suppose that the system $(LS)_0$ is F_i-controllable. Then it is F_i'-controllable for every initial operator $F_i' \in \mathcal{F}_D$.

Proof. Let $R_i \in \mathcal{R}_D$ be the right inverse of D corresponding to F_i, i.e. $F_iR_i = 0$. For every $x_1 \in \ker D$ and $v \in X$ there exists $x_2 \in \ker D$ such that $x_1 = x_2 + F_i'R_iv$. By the assumption, the system $(LS)_0$ is F_i-controllable.

Hence for every $x_0, x_2 \in \ker D$ there exist $u \in U$ and $z \in \ker(I - RA)$ such that $F_i[T_i(RBu + x_0) + z] = x_2$, or equivalently

$$T_i(RBu + x_0) + z = x_2 + R_iv$$

for some $v \in X$. Thus

$$F_i'[T_i(RBu + x_0) + z] = x_2 + F_i'R_iv = x_1.$$

The arbitrariness of $x_0, x_1 \in \ker D$ implies the assertion.

Example 1.1. Let $X = (s)$ be the space of all real sequences. Write

$$\{e_n\} = \{1, 1, 1, \ldots\}, \quad \{0_n\} = \{0, 0, 0, \ldots\},$$

$$D\{x_n\} := \{x_{n+1} - x_n\}, \quad F\{x_n\} := x_1\{e_n\},$$

$$R\{x_n\} := \{y_n\}, \quad y_1 := 0, \quad y_n = \sum_{j=1}^{n-1} x_j \quad (n = 2, 3, \ldots),$$

$$A\{x_n\} := \{z_n\}, \quad z_1 := 2x_2 - x_1, \quad z_n := x_{n+1} - x_n \quad (n = 2, 3, \ldots),$$

$$B := \beta I, \quad \text{where} \quad \beta \in \mathbb{R},$$

$$U := \{\{u_n\} : \quad u_n = 0 \quad \text{for} \quad n = 2, 3, \ldots\}.$$

It is easy to check that $D \in R(X)$, dom $D = X$, $R \in \mathcal{R}_{\mathcal{D}}$ and F is an initial operator for D corresponding to R. Moreover, $\ker D = \{\{ce_n\} : \quad c \in \mathbb{R}\}$.

Consider the following linear system $(LS)_0$

$$Dx = Ax + Bu, \quad Fx = x_0', \quad x_0' \in \ker D. \tag{1.30}$$

Since $(I - RA)\{x_n\} = \{x_1 + x_2, x_3, x_3, \ldots\}$, we conclude that $\ker\,(I - RA) \neq \{0\}$, $(I - RA)X \neq X$. Therefore, $I - RA$ is not one-sided invertible. Write $T_4\{x_n\} := \{x_1, 0, x_3, 0, 0, \ldots\}$. Then

$$T_4(I - RA)\{x_n\} = T_4\{x_1 + x_2, x_3, x_3, \ldots\} = \{x_1 + x_2, 0, x_3, 0, 0, \ldots\},$$

$$(I - RA)T_4(I - RA)\{x_n\} = \{x_1 + x_2, x_3, x_3, \ldots\},$$

i.e. $(I - RA)T_4(I - RA) = I - RA$. Hence, the resolving operator is generalized almost invertible, but it is neither invertible nor one-sided invertible.

Let $x_0' = \{be_n\} \in \ker D$. Then

$$RBU \oplus \{x_0'\} = \{\{x_n\} : \quad x_1 = b, \quad x_k = b + c \quad (k \geqslant 2), \quad c \in \mathbb{R}\}. \tag{1.31}$$

Hence $RBU \oplus \{x_0'\} \subset (I - RA)(\text{dom D})$, i.e. the system (1.30) has solutions for every control $u \in U$.

If $x_1' = \{se_n\}$, $v = \{v_1, v_2, \ldots\} \in X$ then

$$(I - RA)(Rv + x_1') = \{2s, s + v_1 + v_2, s + v_1 + v_2, \ldots\}. \tag{1.32}$$

Now (1.31) and (1.32) together imply $\ker B^* R^* T_4^* \neq \{0\}$, i.e. not every state x in $(RX \oplus \{x_0'\} + \ker(I - RA)$ is reachable from x_0'.

By simple calculation, we also have

$$T_4 RBU = \{\{0, 0, c, 0, 0, \ldots\} : \quad c \in \mathbb{R}\},$$

$$RX + \ker(I - RA) = \{\{\beta, x_1 - \beta, x_1 + x_2 - \beta, y_4, y_5, \ldots\} : \ \beta \in \mathbb{R},$$
$$x = \{x_n\} \in X, \ \ y_k = x_1 + \cdots + x_{k-1} \ (k \geqslant 4)\}.$$

Hence $T_4 RBU \neq RX + \ker(I - RA)$. By Theorem 1.2, there is

$$x \in RX + \{x_0'\} + \ker(I - RA),$$

which is not reachable from x_0'.

Let $F_4\{x_n\} = x_3\{e_n\}$. Then

$$F_4 T_4(\ker D) = \{\beta, \beta, \ldots\},$$

i.e. $F_4 T_4(\ker D) = \ker D$. Corollary 1.5 implies that the system (1.30) is F_4-controllable.

If we put $F_4'\{x_n\} = x_2\{e_n\}$, then $F_4' T_4(\ker D) = \{0\}$. Hence $F_4' T_4(\ker D) \neq \ker D$. However, $F_4'(\ker(I - RA)) = \ker D$, so that

$$F_4' T_4(\ker D) + \ker(I - RA) = \ker D.$$

By Theorem 1.3, the system (1.30) is F_4'-controllable.

Example 1.2. Suppose that X, D, R, F are defined as in Example 1.1 and that

$$A\{x_n\} := \{0, x_3, x_4 - x_3, x_5 - x_4, \ldots\}, \ \ U := X, \ \ B := I.$$

It is easy to check that

$$(I - RA)\{x_n\} = \{x_1, x_2, 0, 0, \ldots\}. \tag{1.33}$$

Hence $I - RA$ is a projection, and so it is not one-sided invertible, but it is generalized almost invertible. The kernel of $I - RA$ is

$$\ker(I - RA) = \{\{0, 0, x_3, x_4, x_5, \ldots\} : \ x_n \in \mathbb{R} \ (n \geqslant 3)\}. \tag{1.34}$$

Fix $x_0' = \{be_n\} \in \ker D$. Then

$$RBU \oplus \{x_0'\} = RX \oplus \{x_0'\}. \tag{1.35}$$

Since $(I - RA)^2 = I - RA$, we get $T_4 = I \in \mathcal{W}_{\mathcal{I}-\mathcal{RA}}$, and

$$T_4 RBU = RX. \tag{1.36}$$

Now (1.34) and (1.36) yield

$$T_4 RBU = RX + \ker(I - RA).$$

Theorem 1.2 implies that every state $x \in RX + \{T_4x'_0\} + \ker(I - RA)$ is (4)-reachable from $x_0 \in \ker D$.

Let $F_4 \in \mathcal{F}_{\mathcal{D}}$, $F_4\{x_n\} := x_3\{e_n\}$. Then $F_4T_4(\ker D) = \ker D$. Hence, by Corollary 1.5, the system (1.30) is F_4-controllable.

Suppose now that $T'_4 = I - RA$. Then $I - RA \in \mathcal{W}_{\mathcal{I}-\mathcal{RA}}$ since $(I - RA)^3 = I - RA$. In this case, we obtain

$$T_4RBU = \{0, \beta, 0, 0, \ldots\}, \;\; T_4(\ker D) = \{\{\beta, \beta, 0, 0, \ldots\} : \;\; \beta \in \mathbb{R}\},$$

$$F_4T_4(\ker D) = \{\{\beta, \beta, 0, 0, \ldots\} : \;\; \beta \in \mathbb{R}\}$$

and $F_4(T_4(\ker D) + \ker(I - RA)) = \{\{ce_n\} : c \in \mathbb{R}\}$. Thus $F_4T_4(\ker D) \not\subseteq \ker D$. However,

$$F_4(T_4(\ker D) + \ker(I - RA)) = \ker D.$$

Theorem 1.3 implies that the system (1.30) is F'_4-controllable for the given generalized almost inverse $T_4 = I - RA$.

2 Controllability of general systems with right invertible operators

Let X, Y and U be linear spaces (all over the same field $\mathcal{F}$, where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$). Let $D \in R(X)$, $R \in \mathcal{R}_{\mathcal{D}}$ and let F be an initial operator corresponding to R. Write

$$X_k := \mathrm{dom\, D^k}, \;\; \mathrm{Z_k} := \mathrm{ker\, D^k} \;\; (\mathrm{k} \in \mathbb{N}). \tag{2.0}$$

Suppose that we are given $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$.

Definition 2.1. A linear system (shortly (LS)) is any system

$$Q[D] = Bu, \;\; FD^jx = x_j, \;\; x_j \in \mathbb{Z}_1 \;\; (j = 0, \ldots, M + N - 1), \tag{2.1}$$

$$y = A_1x + B_1u, \tag{2.2}$$

where

$$Q[D] := \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A_{mn} D^n, \tag{2.3}$$

$A_{mn} \in L(X)$, $A_{mn}X_{M+N-n} \subset X_m$ $(m = 0, \ldots, M;\; n = 0, \ldots, N;\; m + n < M + N)$, $A_{MN} = I$.

Herein we assume that

$$R^{M+N}BU \oplus \{x^0\} \subset (I+Q)X_{M+N}, \tag{2.4}$$

where

$$x^0 := \sum_{j=0}^{M+N-1} R^j x_j \in Z_{M+N}, \tag{2.5}$$

$$Q := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m} B_{mn} D^n, \tag{2.6}$$

where

$$B_{mn} := \begin{cases} A'_{0n} & \text{if } m=0, \\ A'_{mn} - \sum_{\mu=m}^{M} F D^{\mu-m} A'_{\mu n} & \text{otherwise,} \end{cases}$$

$$A'_{mn} := \begin{cases} 0 & \text{if } m=M \text{ and } n=N, \\ A_{mn} & \text{otherwise } (m=0,\dots,M; n=0,\dots,N). \end{cases}$$

The assumption (2.4) is a necessary and sufficient condition for the initial value problem (2.1) to have solutions for every $u \in U$.

If $A_1 = I$ and $B_1 = 0$ then we shall denote the system (2.1)-(2.2) by $(LS)_0$.

Definition 2.2. The linear system (2.1)-(2.2) is said to be well-defined if for every fixed $u \in U$ the corresponding initial value problem (2.1) is well-posed. If there is $u \in U$ such that the initial value problem (2.1) is ill-posed, then the linear system is said to be ill-defined.

Theorem 2.1. Suppose that the condition (2.4) is satisfied. Then the system (2.1)-(2.2) is well-defined if and only if the corresponding resolving operator $I+Q'$, where

$$Q' := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m} B_{mn} R^{N-n} \tag{2.7}$$

is either invertible or left invertible.

Indeed, if $I+Q'$ is either invertible or left invertible, then for every $u \in U$, the initial value problem (2.1) has a unique solution of the form $x = G(x^0, u)$, where

$$G(x^0, u) = E_Q(R^{M+N}Bu + x^0), \tag{2.8}$$

$$E_Q := \begin{cases} I - R^N E_{Q'} Q_1 & \text{if } I+Q' \text{ is invertible,} \\ I - R^N L_{Q'} Q_1 & \text{if } I+Q' \text{ is left invertible,} \end{cases} \tag{2.9}$$

$$E_{Q'} := (I+Q')^{-1}, \;\; L_{Q'} \in \mathrm{L}_{I+Q'},$$

$$Q_1 := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}B_{mn}D^n. \tag{2.10}$$

So, according to (2.2), the output y is uniquely determined by any $u \in U$ and $x^0 \in Z_{M+N}$, and is of the form $y = A_1G(x^0,u) + B_1u$. If we consider a linear system $(LS)_0$, then $y = x = G(x^0,u)$.

Definition 2.3. Write

$$G_0 := A_1E_Q, \;\; G_1 := G_0R^{M+N}B + B_1, \tag{2.11}$$

where E_Q is defined by (2.9). The matrix operator $G^0 = (G_0, G_1)$ defined on the input space $Z_{M+N} \times U$ is said to be the transfer operator for the linear system with the resolving operator $I + Q'$ invertible.

Therefore, to every input (x^0, u) there corresponds a uniquely determined output y, which can be written as

$$y = G_0(x^0,u) = G_0x^0 + G_1u.$$

Consider now the linear system $(LS)_0$, i.e. the system (2.1)-(2.2) with $A_1 = I$, $B_1 = 0$:

$$Q[D]x = Bu, \;\; FD^jx = x_j, \;\; x_j \in Z_1 \;\; (j = 0,\dots,M+N-1), \tag{2.12}$$

$$R^{M+N}BU \oplus \{x^0\} \subset (I+Q)X_{M+N}. \tag{2.13}$$

Write this system in an equivalent form

$$(I+Q)x = R^{M+N}Bu + x^0. \tag{2.14}$$

Denote by H_i (i = 1,2,3,4) the following sets defined for any $x^0 \in Z_{M+N}$, $u \in U$.

(1) If $I + Q' \in R(X)$, then

$$H_1(x^0,u) := \{T_1(R^{M+N}Bu + x^0) + z : \;\; z \in \ker(I+Q)\}, \tag{2.15}$$

where

$$T_1 := I - R^NR_{Q'}Q_1, \;\; R_{Q'} \in \mathcal{R}_{\mathcal{I}+Q'}, \tag{2.16}$$

Q_1 is given by (21.10).

(2) If $I + Q' \in \Lambda(X)$ and $L_{Q'} \in \mathrm{L}_{I+Q'}$, then

$$H_2(x^0,u) := \{T_2(R^{M+N}Bu + x^0)\}, \tag{2.17}$$

where

$$T_2 := I - R^N L_{Q'} Q_1, \quad Q_1 \quad \text{is defined by (2.10).} \tag{2.18}$$

(3) If $I + Q'$ is invertible, then

$$H_3(x^0, u) := \{T_3(R^{M+N} Bu + x^0)\}, \tag{2.19}$$

where

$$T_3 := I - R^N (I + Q')^{-1} Q_1. \tag{2.20}$$

(4) If $I + Q' \in W(X)$ and $W_{Q'} \in \mathcal{W}_{\mathcal{I}+Q'}$, then

$$H_4(x^0, u) := \{T_4(R^{M+N} Bu + x^0) + z : \quad z \in \ker(I + Q)\}, \tag{2.21}$$

where

$$T_4 := I - R^N W_{Q'} Q_1. \tag{2.22}$$

Note that H_i ($i = 1, 2, 3, 4$) are the sets of all solutions of the system $(LS)_0$ in the respective cases.

As in Section 33, we need the following

Definition 2.5. A state $x \in X$ is said to be (i)-reachable ($i = 1, 2, 3, 4$) from an initial state $x^0 \in Z_{M+N}$ if for every T_i ($T_1 \in \mathcal{R}_{\mathcal{I}+Q}$, $T_2 \in \mathrm{L}_{I+Q}$, $T_3 = (I + Q)^{-1}$, $T_4 \in \mathcal{W}_{\mathcal{I}+Q}$) there exists a control $u \in U$ such that $x \in H_i(x^0, u)$.

In the following we only deal with the above four cases. Write

$$\mathrm{Rang}_{U,x^0} H_i = \bigcup_{u \in U} H_i(x^0, u), \quad x^0 \in Z_{M+N}. \tag{2.23}$$

It is easy to see that $\mathrm{Rang}_{U,x^0} H_i$ is (i)-reachable from x^0 by means of controls $u \in U$ and it is contained in X_{M+N}.

Lemma 2.1. Suppose that T_i (i = 1,2,3,4) are given by (2.16), (2.18), (2.20) and (2.22), respectively. Then

$$\begin{aligned} &T_i(R^{M+N} BU \oplus \{x^0\}) + \ker(I + Q) \\ &= T_i R^{M+N} BU \oplus \{T_i x^0\} \oplus \ker(I + Q). \end{aligned} \tag{2.24}$$

Remark 2.1. If $I + Q'$ is either invertible or left invertible, the formula (2.24) is of the form

$$T_i(R^{M+N} BU \oplus \{x_0\}) = T_i R^{M+N} BU \oplus \{T_i x_0\}.$$

Corollary 2.1.

$$\mathrm{Rang}_{U,x^0} H_i = T_i R^{M+N} BU \oplus \{T_i x^0\} \oplus \ker(I + Q). \tag{2.25}$$

Corollary 2.2. The state $x \in X_{M+N}$ is (i)-reachable from $x_0 \in Z_{M+N}$ if and only if

$$x \in T_i R^{M+N} BU \oplus \{T_i x^0\} \oplus \ker(I+Q).$$

Lemma 2.2. Write

$$\begin{aligned} E_i &:= T_i R^{M+N} B, \\ X_{0i} &:= T_i(R^N(I+Q')R^M X + (I+Q)Z_{M+N} - \{x^0\}). \end{aligned} \tag{2.26}$$

Then the operator E_i maps the space U into X_{0i}.

Theorem 2.3. Let there be given a system $(LS)_0$ described by (2.12)-(2.13). Suppose that $B \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$, $T_i \in L_0(X_{M+N}, X'_{M+N})$, $i = 1,2,3,4$; $R \in L_0(X, X')$. Then the generalized Kalman condition

$$\ker B^*(R^*)^{M+N} T_i^* = \{0\} \tag{2.28}$$

holds if and only if for every initial state $x^0 \in Z_{M+N}$, every state $x \in R^{M+N}X$ $+x^0 + \ker(I+Q)$ is reachable from x^0.

Definition 2.6. Let there be given a linear system $(LS)_0$ of the form (2.12)-(2.13) and let $F'_i \in \mathcal{F}_{\mathcal{D}^{M+N}}$.

(i) The state $x^1 \in Z_{M+N}$ is said to be F_i-reachable from an initial state $x^0 \in Z_{M+N}$ if there exists a control $u \in U$ such that $x^1 \in F'_i H_i(x^0, u)$. The state x^1 is then called a final state.

(ii) The system $(LS)_0$ is said to be F_i-controllable if for every initial state $x^0 \in Z_{M+N}$,

$$F'_i(\mathrm{Rang}_{U,x^0} H_i) = Z_{M+N}. \tag{2.30}$$

(iii) The system $(LS)_0$ is said to be F_i-controllable to $x^1 \in Z_{M+N}$ if

$$x^1 \in F_i(\mathrm{Rang}_{U,x^0} H_i) \tag{2.31}$$

for every initial state $x^0 \in Z_{M+N}$.

Lemma 2.3. Let there be given a linear system $(LS)_0$ of the form (2.12)-(2.13) and an initial operator $F'_i \in \mathcal{F}_{\mathcal{D}^{M+N}}$. Suppose that $(LS)_0$ is F'_i -controllable to zero and that

$$F'_i T_i Z_{M+N} = Z_{M+N}. \tag{2.32}$$

Then every final state $x^1 \in Z_{M+N}$ is F'_i -reachable from zero.

Proof. It is sufficient to deal with the case $i = 4$. Since the system is F'_4- controllable to zero, there exists a control $u' \in U$ such that $0 \in F'_4 H_4(x^0, u')$, i.e. there exists $z_0 \in \ker(I+Q)$ such that $F_4'(T_4(R^{M+N}Bu' + x^0) + z_0) = 0$, or equivalently

$$F_4'(T_4(R^{M+N}Bu' + z_0) = -F_4'T_4x^0.$$

By the assumption (2.32), for every given state $x^1 \in Z_{M+N}$ we find $x^2 \in Z_{M+N}$ such that $-F_4'T_4x^2 = x^1$. Hence, there are $u \in U$ and $z_0 \in \ker(I+Q)$ such that

$$F_4'(T_4(R^{M+N}Bu) + z_0) = -F_4'T_4x^2 = x^1.$$

This proves that an arbitrary final state x^1 is reachable from the initial state 0.

Theorem 2.4. Suppose that all assumptions of Lemma 2.3 are satisfied. Then the linear system $(LS)_0$ is F_i'-controllable.

Proof. It is sufficient to deal with the case of a generalized almost invertible resolving operator. By the assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I+Q)$ such

$$F_4'[T_4(R^{M+N}Bu_0 + x^0) + z_0] = 0. \tag{2.33}$$

On the other hand, by Lemma 2.3, for every given $x^1 \in Z_{M+N}$ there exist $u_2 \in U$, that $z_2 \in \ker(I+Q)$ such that

$$F_4'[T_4(R^{M+N}Bu_2 + 0) + z_2] = x^1. \tag{2.34}$$

If we add (2.33) and (2.34), we obtain $F_4'[T_4(R^{M+N}Bu_1 + x^0) + z_1] = x^1$, where $u_1 := u_0 + u_2 \in U$, $z_1 := z_0 + z_2 \in \ker(I+Q)$. Thus every final state $x^1 \in Z_{M+N}$ is F_4-reachable from the initial state $x^0 \in Z_{M+N}$.

Note that Theorem 2.4 was given by Nguyen Dinh Quyet [54-56] and Pogorzelec [84] for systems of the first order with invertible and one-sided invertible resolving operators (cf. Section 33). Theorem 2.4 can be generalized as follows:

Theorem 2.5. Let there be given a system $(LS)_0$ of the form (2.12)-(2.13) and an initial operator $F_i' \in \mathcal{F}_{\mathcal{D}^{M+N}}$. Suppose that $(LS)_0$ is F_i'-controllable to zero and that

$$F_i'[T_i(Z_{M+N} + \ker(I+Q)] = Z_{M+N}. \tag{2.35}$$

Then $(LS)_0$ is F_i'-controllable.

Note that the conditions of Theorem 2.4 and 2.5 are sufficient but not necessary.

Theorem 2.6. Let there be given a system $(LS)_0$ of the form (2.12)-(2.13) and an initial operator $F_i' \in \mathcal{F}_{\mathcal{D}^{M+N}}$. Then $(LS)_0$ is F_i'-controllable if and only if it is F_i'-controllable to every element $v^0 \in F_i'(T_iR^{M+N}X_{M+N})$.

Note that the operator $F_i'T_iR^{M+N}B$ maps U into Z_{M+N}. The following theorem shows that this operator determines the properties of the system $(LS)_0$.

Theorem 2.7. Let a linear system $(LS)_0$ of the form (2.12)- (2.13) and an initial operator $F_i' \in \mathcal{F}_{\mathcal{D}^{M+N}}$ be given. Suppose that $B \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$, $R \in L_0(X, X')$ and $T_i \in L_0(X_{M+N}, X_{M+N})$. Then $(LS)_0$ is F_i'-controllable if and only if

$$\ker B^*(R^*)^{M+N}T_i^*(F_i')^* = \{0\}. \tag{2.42}$$

Theorem 2.7. Suppose that the system $(LS)_0$ is F_i'-controllable. Then it is F'-controllable for every initial operator $F' \in \mathcal{F}_{\mathcal{D}^{M+N}}$.

Example 2.1. Let $X := C[0,1]$ over $\mathbb{C}$. Let $D := d/dt$,

$$R := \int_{t_0}^{t}, \quad (Fx)(t) := x(t_0), \quad t_0 \in [0,1].$$

Consider the system

$$[D^N + P_0(D,I) + P_1(D,I)F' + R^kP_2(D,I)]x = Bu, \tag{2.46}$$

$$FD^jx = x_j, \quad x_j \in \mathbb{C} \ (j = 0, \dots, N-1), \tag{2.47}$$

where $F' \in \mathcal{F}_{\mathcal{D}^N}$, $U = X$, $B \in L_0(X)$, $k \in \mathbb{N}_0$,

$$P_\mu(t,s) := \sum_{i=0}^{N-1} a_{\mu i}t^is^{N-1-i}, \quad a_{\mu i} \in \mathbb{C} \ (\mu = 0,1,2). \tag{2.48}$$

As before, we write

$$Q_1 := P_0(D,I) + P_1(D,I)F' + R^kP_2(D,I),$$

$$Q := R^NQ_1, \quad Q' := P_0(I,R) + R^kP_2(I,R).$$

Since $R \in V(X)$, the resolving operator $I + Q'$ is invertible (Theorem I in Section 6). On the other hand, it is easy to check that $Q' = Q_1R^N$, so that by Theorem 2.1, $I + Q$ is also invertible, and

$$(I+Q)^{-1} = I - R^N(I+Q')Q_1. \tag{2.49}$$

Write the system (2.46)-(2.47) in the following equivalent form:

$$(I+Q)x = R^NBu + x^0, \quad x^0 = \sum_{j=0}^{N-1} R^jx_j. \tag{2.50}$$

From (2.49), we conclude that $I+Q \in L_0(X_N)$ and $(I+Q)^{-1}X_N \subset X_N$. Hence, (2.50) has solutions for every $u \in X$. This means that the condition (2.13) is satisfied. A unique solution of the system (2.46)- (2.47) is

$$x = [I - R^N(I+Q')^{-1}Q_1](R^N Bu + x^0) \in X_N. \tag{2.51}$$

Thus, every state $x \in [I - R^N(I+Q')^{-1}Q_1](R^N Bu \oplus \{x^0\})$ is reachable from $x^0 \in Z_N$.

Let $F_1', F_2' \in \mathcal{F}_{D^N}$ be initial operators for D^N given by

$$F_1' := I - R_1^N D^N, \quad F_2' := I - R_1 R^{N-1} D^N \quad \text{on} \quad \mathrm{dom\, D^N},$$

where $R_1 := \int_{t_1}^{t}$, $t_1 \neq t_0$; $t_0, t_1 \in [0,1]$. Let $T_3 := (I+Q)^{-1}$. It is easy to check that $F_1' R^N X = Z_N$, $F_2' R^N X \neq Z_N$, so that for every $B \in L_0(X)$, we find

$$F_2'(I - R^N(I+Q')^{-1}Q_1)R^N BU = F_2' R^N (I - (I+Q')^{-1}Q_1 R^N)BX \neq Z_N,$$

i.e. $\ker B^*(R^*)^N T_3^* F_2'^* \neq \{0\}$. This means that the system (2.46)- (2.47) is not F_2' -controllable.

Let $B = I$. Since $I - (I+Q')^{-1}Q_1 R^N$ is invertible because $I - R^N(I + Q')^{-1}Q_1$ is invertible, we conclude that

$$[I - (I+Q')^{-1}Q_1 R^N]X = X.$$

This implies

$$F_1' T_3 R^N BU = F_1' T_3 R^N X = F_1'(I - R^N(I+Q')^{-1}Q_1)R^N X$$

$$= F_1' R^N [I - (I+Q')^{-1}Q_1 R^N]X = F_1' R^N X = Z_N.$$

Hence $\ker B^*(R^*)^N T_3{}^* F_1'^* = \{0\}$. Thus, by Theorem 2.7, the system (2.46)-(2.47) is F_1'- controllable.

Example 2.2. Let $X = (s)$ be the space of all real sequences. Write $\{e_n\} := \{1, 1, \ldots\}$, $\{0_n\} := \{0, 0, \ldots\}$. Define the following operators:

$$D\{x_n\} := \{x_{n+1} - x_n\}, \quad F\{x_n\} := x_1\{e_n\},$$

$$R\{x_n\} := \{y_n\}, \quad y_1 := 0, \quad y_n := x_1 + \cdots + x_{n-1} \ (n \geqslant 2),$$

$$A\{x_n\} = \{x_2, x_3 - x_2, 0, 0, \ldots\}, \quad B\{x_n\} = \{x_2, -x_2 - x_2, 0, 0, \ldots\},$$

$$C\{x_n\} = \{x_2 - x_1, 0, 0, \ldots\}.$$

Consider the system

$$\begin{aligned} (D^2 - AD - DB - C)x &= Bu, \\ Fx = x'_0, \quad FDx &= x'_1, \qquad\qquad x'_0, x'_1 \in \ker D, \end{aligned} \tag{2.52}$$

where $u \in U$, $U \subset X$, $B \in L_0(U, X)$. Write

$$Q_1 := RAD + B + RC, \quad Q := RQ_1, \quad Q' := RA + BR + RCR. \tag{2.53}$$

The system (2.52) is equivalent to the equation

$$(I - Q)x = R^2Bu + x^0, \quad x^0 := x_0 + Rx_1. \tag{2.54}$$

It is easy to see that $I - Q'$ is the resolving operator for the system (2.52) and $I - Q' = I - Q_1R$. By easy calculations, we find

$$RA\{x_n\} = \{0, x_2, x_3, x_3, \ldots\}, \quad BR\{x_n\} = \{x_1, -x_1, -x_1, 0, 0, \ldots\},$$

$$RCR\{x_n\} = \{0, x_1, x_1, x_1, 0, 0, \ldots\},$$

so that

$$\begin{aligned} (I - Q')\{x_n\} &= \{0, 0, 0, y_4, y_5, \ldots\}, \\ y_k &:= x_k - x_1 - x_3 \ \ (k = 4, 5, \ldots), \end{aligned} \tag{2.55}$$

$$\ker(I - Q') = \{z = x_1, x_2, x_3, x_1 + x_3, x_1 + x_3, \ldots\}, \tag{2.56}$$

$$\Im(I - Q') \neq X. \tag{2.57}$$

The formulae (2.55)-(2.57) imply that the resolving operator $I - Q'$ is not one-sided invertible. However, since $(I - Q')(I - Q') = I - Q'$, we conclude that $I - Q'$ is generalized almost invertible and I is its generalized almost inverse.

By straightforward calculations, we find

$$(I - RQ_1)\{x_n\} = (I - Q)\{x_n\} = \{x_1, 0, 0, x_1, y_5, y_6, \ldots\}, \tag{2.58}$$

where $y_k := x_k - (k-3)x_{k-1} + (k-4)(x_3 - x_2 + x_1) \ \ (k \geqslant 5)$.

Let $x'_0 := 0$, $x'_1 := 0$, i.e. let the initial conditions of the problem $(LS)_0$ be $Fx = 0$, $FDx = 0$. Let $U = X$ and

$$B\{x_n\} = \{0, 0, 0, 0, x_1, x_2, x_3, \ldots\}. \tag{2.59}$$

It is easy to check that

$$BU \oplus \{x_0\} = BX \subset (I - Q)X_2 = (I - Q)X.$$

Hence, the system (2.52) is solvable for every $u \in U$. From (2.54) we find

$$x = (I + RQ_1)R^2Bu = (I + Q)R^2Bu.$$

Therefore, every state $x \in (I + Q)R^2BU$ is reachable from zero.

3 Controllability of linear systems described by generalized almost invertible operators

Let X, Y, U be linear spaces over the same field $\mathcal{F}$ (where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$). Suppose that $V \in W(X)$, $W \in \mathcal{W}_V^\infty$ and $F^{(r)}$, $F^{(l)}$ are right and left initial operators for V corresponding to W; $A \in L_0(X)$, $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$.

By a linear system (LS) we now mean the following system:

$$Vx = Ax + Bu, \quad u \in U, \quad BU \subset (V - A)(\mathrm{dom\,V}), \tag{3.1}$$

$$F^{(r)}x = x_0, \quad x_0 \in \ker V, \tag{3.2}$$

$$y = A_1 x + B_1 u. \tag{3.3}$$

If $A_1 = I$, $B_1 = 0$, i.e. $Y = X$ and $y = x$, then we denote the system (3.1)-(3.3) by $(LS)_0$.

Note that the properties of linear systems depend on the properties of the resolving operators $I - WA$ and $I - AW$. There are eight cases to deal with:

(i) $I - WA \in R(X)$, (ii) $I - WA \in \Lambda(X)$, (iii) $I - WA \in R(X) \cap \Lambda(X)$, (iv) $I - WA \in W(X)$, (v) $I - AW \in R(X)$, (vi) $I - AW \in \Lambda(X)$, (vii) $I - AW \in R(X) \cap \Lambda(X)$, (viii) $I - AW \in W(X)$.

It is sufficient to consider the first four cases (i)-(iv). Since both one-sided invertible and invertible operators are generalized almost invertible, we can reduce those cases to the case of $I - WA$ being generalized almost invertible.

Suppose that we are given a linear system $(LS)_0$. The initial value problem (3.1)-(3.2) has solutions if and only if

$$WBu + x_0 \in (I - WA)X_u \subset (I - WA)(\mathrm{dom\,V}), \tag{3.4}$$

where

$$X_u = \{x \in \mathrm{dom\,V} : \mathrm{F^{(l)}(Ax + Bu) = 0}\}, \quad \mathrm{u \in U},$$

and $x_0 = 0$ if $\dim\ker V = 0$.

So the condition

$$WBU + \{x_0\} \subset (I - WA)X_u \tag{3.5'}$$

is a necessary and sufficient condition for the initial value problem (3.1)-(3.2) to have solutions for every $u \in U$.

It is easy to check that the condition (3.5') is equivalent to the following: $BU \subset (V - A)\mathrm{dom\,V}$.

Suppose that $I - WA$ is generalized almost invertible.
Write
$G(x_0, u) =$

$$= \{x = (I + WW_A A)(WBu + x_0) + z : W_A \in \mathcal{W}_{\mathcal{I}-\mathcal{A}\mathcal{W}}, \ddagger \in \ker(\mathcal{I} - \mathcal{W}\mathcal{A})\}. \quad (3.6)$$

Note that G is the set of all solutions of the problem (3.1)-(3.2). Therefore, to every fixed input (x_0, u) there corresponds an output $x = G(x_0, u)$.
Write

$$\text{Rang}_{U,x_0} G = \bigcup_{x \in U} G(x_0, u), \quad x_0 \in \ker V. \quad (3.7)$$

Definition 3.1. Suppose that we are given a linear system $(LS)_0$ and the set $G(x_0, u)$ of the form (3.6). A state $x \in X$ is said to be reachable from the initial state $x_0 \in \ker V$ if for every $W_A \in \mathcal{W}_{\mathcal{I}-\mathcal{A}\mathcal{W}}$ there exists a control $u \in U$ such that $x \in G(x_0, u)$.

It is easy to see that the set is reachable from the initial state $x_0 \in \ker V$ by means of controls $u \in U$ and this set is contained in dom V.

Lemma 3.1. Write

$$T = I + WW_A A, \quad W_A \in \mathcal{W}_{\mathcal{I}-\mathcal{A}\mathcal{W}}, \quad \mathcal{W} \in \mathcal{W}'_{\mathcal{V}}. \quad (3.8)$$

Then the following equality holds:

$$T(WBU + \{x_0\}) + \ker(I - WA) = TWBU \oplus \{Tx_0\} \oplus \ker(I - WA). \quad (3.9)$$

Theorem 3.1. Suppose that

$$B \in L_0(U \to X, X' \to U'), V \in L(X, X') \cap W(X), W \in L_0(X, X') \cap \mathcal{W}_{\mathcal{V}}^{\infty}$$

and $T \in L_0(X, X')$, where T is defined by (3.8). Then the generalized Kalman condition

$$\ker B^* W^* T^* = \{0\} \quad (3.12)$$

holds if and only if for every initial state $x_0 \in \ker V$, every state

$$x \in WV(\text{dom V}) + \{\text{x}_0\} + \ker(\text{I} - \text{WA})$$

is reachable from x_0.

Now we give another condition for every state $x \in WX + \{Tx_0\} + \ker(I - WA)$ to be reachable from any initial state $x_0 \in \ker V$.

Lemma 3.2. Let $V \in W(X)$, $W \in L_0(X) \cap \mathcal{W}_{\mathcal{V}}^{\infty}$ and let T be given by (3.8). Then

$$T \in \mathcal{W}_{\mathcal{I}-\mathcal{W}\mathcal{A}}, \quad \mathcal{T}\mathcal{W}\mathcal{X} \subset \mathcal{W}\mathcal{X}. \quad (3.14)$$

Lemma 3.2 implies that $F_1^{(r)}TWB$ maps U into WX. Corollary 3.1 yields

Theorem 3.2. Consider a linear system $(LS)_0$ described by a generalized almost invertible operator V. Suppose that $W \in L_0(X) \cap \mathcal{W}_V$ and T is defined by (3.8). Then a necessary and sufficient condition for every element $x \in WX + \{Tx_0\} + \ker(I - WA)$ to be reachable from any initial state $x_0 \in \ker V$ is that

$$TWBU = WX. \tag{3.15}$$

Definition 3.2. Let there be given a linear system $(LS)_0$ of the form (3.1)-(3.2). Let $F_1^{(r)}$ be any right initial operator for V corresponding to $W_1 \in \mathcal{W}_V$.

(i) A state $x_1 \in \ker V$ is said to be $F_1^{(r)}$- reachable from an initial state $x_0 \in \ker V$ if there exists a control $u \in U$ such that $x_1 \in F_1^{(r)}G(x_0, u)$. The state x_1 is then called a finite state.

(ii) The system $(LS)_0$ is said to be $F_1^{(r)}$- controllable if for every initial state $x_0 \in \ker V$, we have

$$F_1^{(r)}(\mathrm{Rang}_{U,x_0}G) = \ker V. \tag{3.16}$$

(iii) The system $(LS)_0$ is said to be $F_1^{(r)}$- controllable to $x_1 \in \ker V$ if

$$x_1 \in F_1^{(r)}(\mathrm{Rang}_{U,x_0}G) \tag{3.17}$$

for every initial state $x_0 \in \ker V$.

Lemma 3.3. Suppose that the system $(LS)_0$ is $F_1^{(r)}$- controllable to zero and that

$$F_1^{(r)}[T(\ker V) + \ker(I - WA)] = \ker V. \tag{3.18}$$

Then every final state $x_1 \in \ker V$ is $F_1^{(r)}$- reachable from zero.

Theorem 3.3. Suppose that all assumptions of Lemma (3.3) are satisfied. Then the linear system $(LS)_0$ is $F_1^{(r)}$- controllable.

Proof. By our assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I - WA)$ such that

$$F_1^{(r)}[T(WBu_0 + x_0) + z_0] = 0 \tag{3.21}$$

By Lemma 3.3, for every $x_1 \in \ker V$ there exist $u_0' \in U$ and $z_1 \in \ker(I - WA)$ such that

$$F_1^{(r)}(TWBu_0' + z_1)] = x_1. \tag{3.22}$$

Now (3.21) and (3.22) imply $F_1^{(r)}[T(WB(u_0 + u_0' + x_0) + (z_0 + z_1) = x_1$, i.e. x_1 is $F_1^{(r)}$- reachable from x_0, which was to be proved.

Corollary 3.4. If the system $(LS)_0$ is $F_1^{(r)}$- controllable to zero and $F_1^{(r)}T(\ker V) = \ker V$, then it is $F_1^{(r)}$- controllable.

Theorem 3.4. Let a linear system $(LS)_0$ of the form (3.1)-(3.2) and an initial operator $F_1^{(r)}$ for V be given. Let T be defined by (3.8) and let $B \in L_0(U \to X, X' \to U')$, $V \in L(X, X')$, A, $W \in L_0(X, X')$. Then $(LS)_0$ is $F_1^{(r)}$- controllable if and only if

$$\ker B^* W^* T^* (F_1^{(r)})^* = \{0\}. \tag{3.23}$$

Theorem 3.5. Let there be given a linear system $(LS)_0$ and an initial operator for $V \in W(X)$. Then the system $(LS)_0$ is $F_1^{(r)}$ - controllable if and only if it is $F_1^{(r)}$ - controllable to every $x' \in F_1^{(r)}TWV(\mathrm{dom\,V})$.

Theorem 3.6. Suppose that the system $(LS)_0$ is $F_1^{(r)}$ -controllable. Then for an arbitrary right initial operator $F_2^{(r)}$ for V, this system is $F_2^{(r)}$ -controllable.

Example 3.1. Let $X := \mathcal{C}[-1,1]$, $D := d/dt$, $R := \int\limits_0^t$, $(Fx)(t) := x(0)$. Define $(Px)(t) := \frac{1}{2}[x(t) + x(-t)]$, $Q := I - P$, $X^+ := PX$, $X^- := QX$, i.e. $X = X^+ \oplus X^-$. Consider the linear system

$$P(D + \beta I)x = Au, \;\; u \in U = X^+, \tag{3.29}$$

$$(I - RPD)x = x_0, \;\; x_0 = RQy_0 + z_0 \in \ker PD, \tag{3.30}$$
$$x_0 \in \ker D, \;\; y_0 \in X,$$

where $A \in L_0(X^+)$, $\beta \in \mathbb{R}$.

Putting $V = PD$, $W = RP$ we find $VWV = V$, $WVW = W$. The right initial operator $F^{(r)}$ for V corresponding to W is $F^{(r)} = I - RPD$. Hence, we can write the system (3.29)-(3.30) in the form

$$(V + \beta P)x = Au, \;\; F^{(r)}x = x_0. \tag{3.31}$$

This system is equivalent to the equation

$$(I + \beta RP)x = RPAu + x_0. \tag{3.32}$$

Since $(I + \beta RP)(I - \beta RP) = I - \beta^2 RPRP = I - \beta^2 R^2 QP = I$, we conclude that every state $x \in \mathrm{dom\,D}$ is reachable from the initial state x_0, i.e. there exists $u \in U$ such that

$$x = (I - \beta RP)(RPAu + x_0).$$

Hence

$$G(x_0, u) = \{x = (I - \beta RP)(RPAu + x_0)\}, \tag{3.33}$$

and since $RPRP = 0$ we get

$$(I - \beta RP)(RPAU + x_0) = RPAU \oplus \{(I - \beta RP)x_0\}. \tag{3.34}$$

From (3.33)-(3.34) we obtain

$$\mathrm{Rang}_{U,x_0} G = RPAU \oplus \{(I - \beta RP)x_0\}.$$

Thus the system (3.29)-(3.30) is $F_1^{(r)}$ -controllable for a right initial operator $F_1^{(r)}$ of V if and only if

$$F_1^{(r)}(\mathrm{Rang}_{U,x_0} G) = \ker(PD).$$

It is easy to check that $\ker(PD)$ consists all even differentiable functions defined on [-1,1].

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Nguyen Van Mau
Department of Analysis, Faculty of Math. Mech. and Informatics
University of Hanoi
334, Nguyen Trai Str., Hanoi, Vietnam
E-mail address: maunv@vnu.edu.vn.