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<thead>
<tr>
<th>Title</th>
<th>Controllability of Linear Systems with Generalized Invertible Operators</th>
</tr>
</thead>
<tbody>
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Osaka University
Controllability of Linear Systems with Generalized Invertible Operators

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1 Controllability of first order linear systems with right invertible operators

Let $X$, $Y$ and $U$ be linear spaces (all over the same field $\mathcal{F}$, where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$). Suppose that $D \in \mathcal{R}(X)$, $\dim \ker D \neq 0$, $F \in \mathcal{F}_D$ corresponds to an $R \in \mathcal{R}_D$, $A \in \mathcal{L}_0(X)$, $A_1 \in \mathcal{L}_0(X \to Y)$, $B \in \mathcal{L}_0(U \to X)$, $B_1 \in \mathcal{L}_0(U \to Y)$ (cf. Section 1). By a first order linear system (shortly: $(LS)$) we mean the system

$$\begin{align*}
Dx &= Ax + Bu, \quad RBU \oplus \{x_0\} \subset (I - RA)(\text{dom} D), \\
Fx &= x_0, \quad x_0 \in \ker D, \\
y &= A_1x + B_1u.
\end{align*}$$

The spaces $X$ and $U$ are called the space of states and the space of controls, respectively. The element $x_0 \in \ker D$ is called an initial state. A pair $(x_0, u) \in (\ker D) \times U$ is called an input. The space $(\ker D) \times U$ is called the input space, and the corresponding set of $y$'s in $Y$ the output space. Very often there are considered linear systems with $A_1 = I$ and $B_1 = 0$, i.e. with $Y = X$ and the output $y = x$. We shall denote such systems by $(LS)_0$.

The properties of linear systems depend on the properties of the resolving operators $I - RA$ and $I - AR$, respectively. In a series of papers (cf. [54-56]) Nguyen Dinh Quyêt studied some properties of linear systems in the case $I - RA$ invertible. His results concerning controllability were generalized by Pogorzelec [84-85] in the case $I - RA$ and $I - AR$ either left or right invertible, and in the case $I - AR$ invertible.

Hence, there are six cases to deal with:
(i) \( I - RA \in R(X) \), (ii) \( I - RA \in \mathbb{L}(X) \), (iii) \( I - RA \) is invertible,
(iv) \( I - AR \in R(X) \), (v) \( I - AR \in \mathbb{L}(X) \), (vi) \( I - AR \) is invertible.

We show that \( I - RA \) is right invertible (left invertible, invertible) if and only if so is \( I - AR \), i.e. it is sufficient to consider the first three cases. On the other hand, since every one-sided invertible operator and every invertible operator are generalized almost invertible, we can reduce those cases to the case of \( I - RA \) being generalized almost invertible.

Suppose that we are given a linear system \((LS)_0\). The initial value problem (1.1)-(1.2) is equivalent to the equation

\[
(I - RA)x = RBu + x_0. \tag{1.4}
\]

Hence, the inclusion

\[
RBu \oplus \{x_0\} \subseteq (I - RA)(\text{dom } D) \tag{1.5}
\]

is a necessary and sufficient condition for the problem (1.1)-(1.2) to have solutions for every \(u \in U\).

Denote by \(G_i\) \((i = 1, 2, 3, 4)\) following sets defined for every \(x_0 \in \ker D\), \(u \in U\):

(i) If \( I - RA \in R(X) \) and \( T_1 \in \mathcal{R}_{I-RA} \), then

\[
G_1(x_0, u) := \{x = R_1(RBu + x_0) + z : z \in \ker(I - RA)\}. \tag{1.6}
\]

(ii) If \( I - RA \in \mathbb{L}(X) \) and \( T_2 \in \mathcal{L}_{I-RA} \), then

\[
G_2(x_0, u) := \{x = T_2(RBu + x_0)\}. \tag{1.7}
\]

(iii) If \( I - RA \) is invertible, then

\[
G_3(x_0, u) := \{x = T_3(RBu + x_0)\}, \quad T_3 = (I - RA)^{-1}. \tag{1.8}
\]

(iv) If \( I - RA \in \mathbb{W}(X) \) and \( T_4 \in \mathcal{W}_{I-RA} \), then

\[
G_4(x_0, u) := \{x = T_4(RBu + x_0) + z : z \in \ker(I - RA)\}. \tag{1.9}
\]

Note that the \(G_i\) are the sets of all solutions of the problem (1.1)-(1.2) in the corresponding cases. Therefore, to every fixed input \((x_0, u)\) there corresponds an output \(x \in G_i(x_0, u)\) for each case.

**Definition 1.1.** Suppose that we are given a system \((LS)_0\) and the sets \(G_i(x_0, u)\) of the forms (1.6)-(1.9). A state \(x \in X\) is said to be \((i)\)-reachable \((i = 1, 2, 3, 4)\) from an initial state \(x_0 \in \ker D\) if for every \(T_i\) \((T_1 \in \mathcal{R}_{I-RA}, T_2 \in \mathcal{L}_{I-RA}, T_3 = (I - RA)^{-1}, T_4 \in \mathcal{W}_{I-RA})\) there exists a control \(u \in U\) such that \(x \in G_i(x_0, u)\).
Write
\[ \text{Rang}_{U,x_0} G_i = \bigcup_{u \in U} G_i(x_0, u), \quad x_0 \in \ker D \quad (i = 1, 2, 3, 4). \]

It is easy to see that \( \text{Rang}_{U,x_0} G_i \) is \((i)\)-reachable from \( x_0 \in \ker D \) by means of controls \( u \in U \) and it is contained in \( \text{dom} \ D \).

**Lemma 1.1.** Suppose that \( T_i \ (i = 1,2,3,4) \) are defined as in (1.6)- (1.9). Then
\[ T_i(RBU \oplus \{x_0\}) + \ker(I - RA) = T_iRBU \oplus \{T_ix_0\} \oplus \ker(I - RA). \quad (1.10) \]

**Remark 1.1.** If either \( I - RA \in \mathbb{L}(X) \) or \( I - RA \) is invertible then \( \ker(I - RA) = \{0\} \), and (1.10) takes the form \( T_i(RBU \oplus \{x_0\}) = T_iRBU \oplus \{T_ix_0\} \).

The formulae (1.5)-(1.9) imply

**Corollary 1.1.**
\[ \text{Rang}_{U,x_0} G_i = T_iRBU \oplus \{T_ix_0\} \oplus \ker(I - RA). \quad (1.11) \]

**Corollary 1.2.** A state \( x \) is \((i)\)-reachable from a given initial state \( x_0 \in \ker D \) if and only if
\[ x \in T_iRBU \oplus \{T_ix_0\} \oplus \ker(I - RA), \quad i = 1, 2, 3, 4. \quad (1.12) \]

**Lemma 1.2.** Write
\[ E_i := T_iRB, \quad X_i := T_i(I - RA)(\text{dom} \ D) - \{x_0\}. \]

Then the operator \( E_i \) maps \( U \) into \( X_i \).

**Proof.** By our assumption, \( RBU \oplus \{x_0\} \subset (I - RA)(\text{dom} \ D) \), thus for every \( u \in U \) there exist \( v \in X \) and \( x_1 \in \ker D \) such that
\[ R Bu + x_0 = (I - RA)(Rv + x_1), \]
i.e. \( T_iRBu = T_i[(I - RA)(Rv + x_1) - x_0] \).

**Theorem 1.1.** Suppose that \( B \in L_0(U \to X, X' \to U') \), \( D \in L(X, X') \), \( R \in L_0(X, X') \) and \( T_i \in L_0(X, X') \ (i = 1,2,3,4) \). Then the generalized Kalman condition
\[ \ker B^*R^*T_i^* = \{0\} \quad (1.13) \]
holds if and only if for every initial state \( x_0 \in \ker D \), every state \( x \in RX \oplus \{ x_0 \} + \ker(I - RA) \) is (i)-reachable from \( x_0 \).

**Proof.** By Lemma 1.2, the condition (1.13) holds if and only if for every \( x_1 \in \ker D \) and \( v \in X \) there exists \( u \in U \) such that \( RBu + x_0 = (I - RA)(Rv + x_1) \). This means that for every \( x_1 \in \ker D \), \( v \in X \) and \( z \in \ker(I - RA) \) there exists \( u \in U \) such that

\[
T_i(RBu + x_0) + z = T_i(I - RA)(Rv + x_1) + z.
\] (1.14)

It is sufficient to consider \( i = 4 \), i.e. the case when \( (I - RA) \) is generalized almost invertible. Write \( F' := I - T_4(I - RA) \). It is easy to check that \( (I - RA)F' = 0, F'_2 = F' \) and \( F'X = \ker(I - RA) \). Choosing \( x_1 := x_0 \), \( z := F'(Rv + x_1) \in \ker(I - RA) \), we get from (1.14) the equalities

\[
T_4(RBu + x_0) + z = (I - F')(Rv + x_0) + F'(Rv + x_0) = Rv + x_0.
\]

This means that for every \( v \in X \), \( z_1 \in \ker(I - RA) \) there exist \( z' = z_1 + F'(Rv + x_0) \in \ker(I - RA) \) and \( u \in U \) such that

\[
T_4(RBu + x_0) + z' \in RX \oplus \{ x_0 \} + \ker(I - RA),
\]

i.e.

\[
\text{Rang}_{0; x_0} G_4 = RX \oplus \{ x_0 \} + \ker(I - RA).
\]

Note that the generalized Kalman condition (1.13) in the case of \( (I - RA) \) invertible was introduced and applied by Nguyen Dinh Quyet [54-56]. Theorem 1.1 in the case of \( I - RA \) one-sided invertible was obtained by Pogorzelec [84].

Now we give another condition for every state \( x \in RX \oplus \{ TiX_0 \} + \ker(I - RA) \) to be (i)-reachable from any initial state \( x_0 \in \ker D \). To begin with, note that

\[
T_iRX \subset RX \quad (i = 1, 2, 3, 4).
\] (1.15)

Indeed, there exist \( T'_i \) (\( i = 1, 2, 3, 4 \)) such that \( T_i = I + RT'_iA \). Thus

\[
T_iRX = (I + RT'_iA)RX = R(I + T'_iAR)X \subset RX.
\]

Therefore, \( T_iRB \) map \( U \) into \( RX \). Corollary 6.1 gives the following

**Theorem 1.2.** A necessary and sufficient condition for every element

\[
x \in RX \oplus \{ TiX_0 \} + \ker(I - RA)
\]

to be (i)-reachable from any initial state \( x_0 \in \ker D \) is that \( T_iRBU = RX \).
Definition 1.2. Let there be given a linear system $(LS)_0$ of the form (1.1)-(1.2). Let $F_i \in \mathcal{F}_D$ ($i = 1, 2, 3, 4$) be arbitrary initial operators (not necessarily different).

(i) A state $x_1 \in \ker D$ is said to be $F_i$-reachable from an initial state $x_0 \in \ker D$ if there exists a control $u \in U$ such that $x_1 \in F_i G_i(x_0, u)$. The state $x_1$ is then called a final state.

(ii) The system $(LS)_0$ is said to be $F_i$-controllable if for every initial state $x_0 \in \ker D$,

$$F_i(\text{Rang}_{U, x_0} G_i) = \ker D. \quad (1.16)$$

(iii) The system $(LS)_0$ is said to be $F_i$-controllable to $x_1 \in \ker D$ if

$$x_i \in F_i(\text{Rang}_{U, x_0} G_i) \quad (1.17)$$

for every initial state $x_0 \in \ker D$.

Lemma 1.3. Let there be given a linear system $(LS)_0$ and an initial operator $F_i \in \mathcal{F}_D$. Suppose that the system $(LS)_0$ is $F_i$-controllable to zero and that

$$F_i(T_i \ker D + \ker(I - RA)) = \ker D. \quad (1.18)$$

Then every final state $x_1 \in \ker D$ is $F_i$-reachable from zero.

Theorem 1.3. Suppose that all assumptions of Lemma 1.3 are satisfied. Then the system $(LS)_0$ is $F_i$-controllable.

Proof. Suppose that $I - RA \in W(X)$. By our assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I - RA)$ such that

$$F_4[T_4(RBu_0 + x_0) + z_0] = 0. \quad (1.21)$$

By Lemma 1.3, for every $x_1 \in \ker D$ there exist $u'_0 \in U$ and $z_1 \in \ker(I - RA)$ such that

$$F_4(T_4 RBu'_0 + z_1) = x_1. \quad (1.22)$$

Add (1.21) and (1.22) to find

$$F_4[T_4[RBU_0 + u'_0] + x_0 + (z_0 + z_1)] = x_1,$$

i.e. $x_1$ is $F_4$-reachable from $x_0$, which was to be proved.

Corollary 1.4 (cf. Pogorzelec [84]). Let $T'_1 \in \mathcal{R}_{I - AR}$, $T'_2 \in L_{I - AR}$, $T'_3 = (I - AR)^{-1}$ and $T'_4 \in \mathcal{W}_{I - AR}$ for $I - AR \in R(X)$, $I - AR \in \mathbb{L}(X)$, $I - AR$ invertible and $I - AR \in W(X)$, respectively. If the system $(LS)_0$ is $F_i$-controllable to zero and

$$F_i(I + RT'_i A)(\ker D) = \ker D, \quad (1.23)$$

Then every final state $x_1 \in \ker D$ is $F_i$-reachable from zero.
then \((LS)_0\) is \(F_i\)-controllable.

Indeed, by (6.10)-(6.12), \(I + RT_i'A = T_i\). Therefore (1.23) takes the form \(F_iT_i(ker D) = ker D\) and we get a sufficient condition for \(F_i\)-controllability.

**Corollary 1.5** (cf. Pogorzelec [84-85]). If the system \((LS)_0\) is \(F_i\)-controllable to zero and \(F_iT_i(ker D) = ker D\), then \((LS)_0\) is \(F_i\)-controllable.

So the conditions \(F_iT_i(ker D) = ker D\) and \(F_i(I + RT_i'A)(ker D) = ker D\), found by Pogorzelec for the one-sided invertible resolving operators, are identical.

**Theorem 1.4.** Let a linear system \((LS)_0\) of the form (1.1)-(1.2) and an initial operator \(F_i \in \mathcal{F}_D\) be given. Let \(T_i \in \mathcal{R}_{I-RA}\) if \(I-RA \in R(X)\) is invertible,

\[
T_2 \in \mathcal{L}_{I-RA} \quad \text{if} \quad I-RA \text{ is left invertible,}
\]

\[
T_3 = (I-RA)^{-1} \quad \text{if} \quad I-RA \text{ is invertible and}
\]

\[
T_4 \in \mathcal{W}_{I-RA} \quad \text{if} \quad I-RA \text{ is generalized almost invertible.}
\]

Suppose that \(B \in L_0(U \rightarrow X, X' \rightarrow U')\), \(D \in L(X, X')\), \(A, R \in L_0(X, X')\). Then the system \((LS)_0\) is \(F_i\)-controllable if and only if

\[
ker B'R_i'T_i^*F_i^* = \{0\}. \tag{1.24}
\]

**Theorem 1.5.** Let there be given a linear system \((LS)_0\) and an initial operator \(F_i \in \mathcal{F}_D\). Then the system \((LS)_0\) is \(F_i\)-controllable if and only if it is \(\tilde{F}_i\)-controllable to every element \(v' \in F_iRX\).

**Corollary 1.6.** The system \((LS)_0\) is \(F_i\)-controllable if and only if it is \(\tilde{F}_i\)-controllable to every element \(v_0 \in F_iRX\).

Indeed, it is easy to check that \(T_iRX \subset RX\). Thus \(F_iRX \subset F_iRX\).

**Theorem 1.6.** Suppose that the system \((LS)_0\) is \(F_i\)-controllable. Then it is \(\tilde{F}_i\)-controllable for every initial operator \(\tilde{F}_i \in \mathcal{F}_D\).

**Proof.** Let \(R_i \in \mathcal{R}_D\) be the right inverse of \(D\) corresponding to \(F_i\), i.e. \(F_iR_i = 0\). For every \(x_1 \in ker D\) and \(v \in X\) there exists \(x_2 \in ker D\) such that \(x_1 = x_2 + F_i'R_i v\). By the assumption, the system \((LS)_0\) is \(F_i\)-controllable.

Hence for every \(x_0, x_2 \in ker D\) there exist \(u \in U\) and \(z \in ker(I-RA)\) such that \(F_i[T_i(RBu + x_0) + z] = x_2\), or equivalently

\[
T_i(RBu + x_0) + z = x_2 + R_i v
\]

for some \(v \in X\). Thus

\[
F_i'[T_i(RBu + x_0) + z] = x_2 + F_i'R_i v = x_1.
\]

The arbitrariness of \(x_0, x_1 \in ker D\) implies the assertion.
Example 1.1. Let $X = (s)$ be the space of all real sequences. Write

$$\{e_n\} = \{1, 1, 1, \ldots\}, \quad \{0_n\} = \{0, 0, 0, \ldots\},$$

$$D\{x_n\} := \{x_{n+1} - x_n\}, \quad F\{x_n\} := x_1\{e_n\},$$

$$R\{x_n\} := \{y_n\}, \quad y_1 := 0, \quad y_n = \sum_{j=1}^{n-1} x_j \quad (n = 2, 3, \ldots),$$

$$A\{x_n\} := \{z_n\}, \quad z_1 := 2x_2 - x_1, \quad z_n := x_{n+1} - x_n \quad (n = 2, 3, \ldots),$$

$$B := \beta I, \quad \text{where} \quad \beta \in \mathbb{R},$$

$$U := \{\{u_n\} : u_n = 0 \text{ for } n = 2, 3, \ldots\}.$$ 

It is easy to check that $D \in R(X)$, dom $D = X$, $R \in \mathcal{R}_D$ and $F$ is an initial operator for $D$ corresponding to $R$. Moreover, ker $D = \{\{ce_n\} : c \in \mathbb{R}\}$.

Consider the following linear system $(LS)_a$

$$Dx = Ax + Bu, \quad Fx = x'_0, \quad x'_0 \in \text{ker } D. \quad (1.30)$$

Since $(I - RA)\{x_n\} = \{x_1 + x_2, x_3, x_3, \ldots\}$, we conclude that ker $(I - RA) \neq \{0\}, (I - RA)X \neq X$. Therefore, $I - RA$ is not one-sided invertible. Write $T_4\{x_n\} := \{x_1, 0, x_3, 0, 0, \ldots\}$. Then

$$T_4(I - RA)\{x_n\} = T_4\{x_1 + x_2, x_3, x_3, \ldots\} = \{x_1 + x_2, 0, x_3, 0, 0, \ldots\},$$

$$(I - RA)T_4(I - RA)\{x_n\} = \{x_1 + x_2, x_3, x_3, \ldots\},$$

i.e. $(I - RA)T_4(I - RA) = I - RA$. Hence, the resolving operator is generalized almost invertible, but it is neither invertible nor one-sided invertible.

Let $x'_0 = \{be_n\} \in \text{ker } D$. Then

$$RBU \oplus \{x'_0\} = \{\{x_n\} : x_1 = b, x_k = b + c \quad (k \geq 2), \quad c \in \mathbb{R}\}. \quad (1.31)$$

Hence $RBU \oplus \{x'_0\} \subset (I - RA)(\text{dom } D)$, i.e. the system $(1.30)$ has solutions for every control $u \in U$.

If $x'_1 = \{se_n\}, v = \{v_1, v_2, \ldots\} \in X$ then

$$(I - RA)(Rv + x'_1) = \{2s, s + v_1, v_2, s + v_1 + v_2, \ldots\}. \quad (1.32)$$

Now $(1.31)$ and $(1.32)$ together imply ker $B^*R^*T_4^* \neq \{0\}$, i.e. not every state $x$ in $(RX \oplus \{x'_0\} + \text{ker } (I - RA)$ is reachable from $x'_0$.

By simple calculation, we also have

$$T_4RBU = \{\{0, 0, c, 0, 0, \ldots\} : c \in \mathbb{R}\},$$
RX + \ker(I - RA) = \{\beta, x_1 - \beta, x_1 + x_2 - \beta, y_4, y_5, \ldots \} : \beta \in \mathbb{R},
\quad x = \{x_n\} \in X, \ y_k = x_1 + \cdots + x_{k-1} (k \geq 4)\}.

Hence \(T_4RBU \neq RX + \ker(I - RA)\). By Theorem 1.2, there is

\[ x \in RX + \{x'_0\} + \ker(I - RA), \]

which is not reachable from \(x'_0\).

Let \(F_4\{x_n\} = x_3\{e_n\}\). Then

\[ F_4T_4(\ker D) = \{\beta, \beta, \ldots\}, \]

i.e. \(F_4T_4(\ker D) = \ker D\). Corollary 1.5 implies that the system (1.30) is \(F_4\)-controllable.

If we put \(F'_4\{x_n\} = x_2\{e_n\}\), then \(F'_4T_4(\ker D) = \{0\}\). Hence \(F'_4T_4(\ker D) \neq \ker D\). However, \(F'_4(\ker(I - RA)) = \ker D\), so that

\[ F'_4T_4(\ker D) + \ker(I - RA) = \ker D. \]

By Theorem 1.3, the system (1.30) is \(F'_4\)-controllable.

**Example 1.2.** Suppose that \(X, D, R, F\) are defined as in Example 1.1 and that

\[ A\{x_n\} := \{0, x_3, x_4 - x_3, x_5 - x_4, \ldots \}, \quad U := X, \quad B := I. \]

It is easy to check that

\[ (I - RA)\{x_n\} = \{x_1, x_2, 0, 0, \ldots\}. \quad (1.33) \]

Hence \(I - RA\) is a projection, and so it is not one-sided invertible, but it is generalized almost invertible. The kernel of \(I - RA\) is

\[ \ker(I - RA) = \{\{0, 0, x_3, x_4, x_5, \ldots \} : x_n \in \mathbb{R} (n \geq 3)\}. \quad (1.34) \]

Fix \(x'_0 = \{be_n\} \in \ker D\). Then

\[ RBU \oplus \{x'_0\} = RX \oplus \{x'_0\}. \quad (1.35) \]

Since \((I - RA)^2 = I - RA\), we get \(T_4 = I \in \mathcal{W}_{\mathbb{Z}, R, A}\), and

\[ T_4RBU = RX. \quad (1.36) \]

Now (1.34) and (1.36) yield

\[ T_4RBU = RX + \ker(I - RA). \]
Theorem 1.2 implies that every state \( x \in RX + \{ T_4 x'_0 \} + \ker(I - RA) \) is (4)-reachable from \( x_0 \in \ker D \).

Let \( F_4 \in \mathcal{F}_D \), \( F_4 \{ e_n \} := x_3 \{ e_n \} \). Then \( F_4 T_4(\ker D) = \ker D \). Hence, by Corollary 1.5, the system (1.30) is \( F_4 \)-controllable.

Suppose now that \( T_4^* = I - RA \). Then \( I - RA \in \mathcal{W}_{I - RA} \) since \( (I - RA)^3 = I - RA \). In this case, we obtain

\[
T_4 RBU = \{0, \beta, 0, 0, \ldots \}, \quad T_4(\ker D) = \{\{\beta, \beta, 0, 0, \ldots \} : \beta \in \mathbb{R}\},
\]

\[
F_4 T_4(\ker D) = \{\{\beta, \beta, 0, 0, \ldots \} : \beta \in \mathbb{R}\}
\]

and \( F_4 (T_4(\ker D) + \ker(I - RA)) = \{\{c e_n \} : c \in \mathbb{R}\} \). Thus \( F_4 T_4(\ker D) \not\subset \ker D \). However,

\[
F_4 (T_4(\ker D) + \ker(I - RA)) = \ker D.
\]

Theorem 1.3 implies that the system (1.30) is \( F_4 \)-controllable for the given generalized almost inverse \( T_4 = I - RA \).

## 2 Controllability of general systems with right invertible operators

Let \( X, Y \) and \( U \) be linear spaces (all over the same field \( \mathcal{F} \), where \( \mathcal{F} = \mathbb{C} \) or \( \mathcal{F} = \mathbb{R} \)). Let \( D \in \mathcal{R}(X) \), \( R \in \mathcal{R}_D \) and let \( F \) be an initial operator corresponding to \( R \). Write

\[
X_k := \text{dom} D^k, \quad Z_k := \ker D^k \quad (k \in \mathbb{N}).
\]

Suppose that we are given \( A_1 \in L_0(X \to Y) \), \( B \in L_0(U \to X) \), \( B_1 \in L_0(U \to Y) \).

**Definition 2.1.** A linear system (shortly (LS)) is any system

\[
Q[D] = Bu, \quad FD^j x = x_j, \quad x_j \in Z_1 \quad (j = 0, \ldots, M + N - 1), \quad (2.1)
\]

\[
y = A_1 x + B_1 u, \quad (2.2)
\]

where

\[
Q[D] := \sum_{m=0}^{M} \sum_{n=0}^{N} D^m A_{mn} D^n, \quad (2.3)
\]

\( A_{mn} \in L(X) \), \( A_{mn} X_{M+N-n} \subset X_m \) \( (m = 0, \ldots, M; n = 0, \ldots, N; m + n < M + N) \), \( A_{MN} = I \).
Herein we assume that
\[ R^{M+N}B \oplus \{x^0\} \subset (I + Q)X_{M+N}, \tag{2.4} \]
where
\[ x^0 := \sum_{j=0}^{M+N-1} R^j x_j \in Z_{M+N}, \tag{2.5} \]
\[ Q := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M+N-m} B_{mn} D^n, \tag{2.6} \]
where
\[ B_{mn} := \begin{cases} A'_{0m} & \text{if } m = 0, \\ A'_{mn} - \sum_{\mu=m}^{M} FD^{\mu-m} A'_{\mu m} & \text{otherwise}, \end{cases} \]
\[ A'_{mn} := \begin{cases} 0 & \text{if } m = M \text{ and } n = N, \\ A_{mn} & \text{otherwise } (m = 0, \ldots, M; n = 0, \ldots, N). \end{cases} \]

The assumption (2.4) is a necessary and sufficient condition for the initial value problem (2.1) to have solutions for every \( u \in U \).

If \( A_1 = I \) and \( B_1 = 0 \) then we shall denote the system (2.1)-(2.2) by \((LS)_0\).

**Definition 2.2.** The linear system (2.1)-(2.2) is said to be well-defined if for every fixed \( u \in U \) the corresponding initial value problem (2.1) is well-posed. If there is \( u \in U \) such that the initial value problem (2.1) is ill-posed, then the linear system is said to be ill-defined.

**Theorem 2.1.** Suppose that the condition (2.4) is satisfied. Then the system (2.1)-(2.2) is well-defined if and only if the corresponding resolving operator \( I + Q' \), where
\[ Q' := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} B_{mn} R^{N-n} \tag{2.7} \]
is either invertible or left invertible.

Indeed, if \( I + Q' \) is either invertible or left invertible, then for every \( u \in U \), the initial value problem (2.1) has a unique solution of the form \( x = G(x^0, u) \), where
\[ G(x^0, u) = E_Q(R^{M+N}Bu + x^0), \tag{2.8} \]
\[ E_Q := \begin{cases} I - R^N E_{Q'} Q_1 & \text{if } I + Q' \text{ is invertible}, \\ I - R^N L_{Q'} Q_1 & \text{if } I + Q' \text{ is left invertible}, \end{cases} \tag{2.9} \]
\[ E_{Q'} := (I + Q')^{-1}, \quad L_{Q'} \in L_{I + Q'}, \]
\[ Q_1 := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} B_{mn} D^n. \]  

(2.10)

So, according to (2.2), the output \( y \) is uniquely determined by any \( u \in U \) and \( x^0 \in Z_{M+N} \), and is of the form \( y = A_1 G(x^0, u) + B_1 u \). If we consider a linear system \((LS)_0\), then \( y = x = G(x^0, u) \).

**Definition 2.3.** Write
\[ G_0 := A_1 E_Q, \quad G_1 := G_0 R^{M+N} B + B_1, \]  

(2.11)

where \( E_Q \) is defined by (2.9). The matrix operator \( G^0 = (G_0, G_1) \) defined on the input space \( Z_{M+N} \times U \) is said to be the transfer operator for the linear system with the resolving operator \( I + Q' \) invertible.

Therefore, to every input \((x^0, u)\) there corresponds a uniquely determined output \( y \), which can be written as
\[ y = G_0(x^0, u) = G_0 x^0 + G_1 u. \]

Consider now the linear system \((LS)_0\), i.e. the system (2.1)-(2.2) with \( A_1 = I, B_1 = 0 \):
\[ Q[D]x = B_u, \quad FD^j x = x_j, \quad x_j \in Z_1 \quad (j = 0, \ldots, M + N - 1), \]  

(2.12)
\[ R^{M+N} BU \oplus \{x^0\} \subset (I + Q)X_{M+N}. \]  

(2.13)

Write this system in an equivalent form
\[ (I + Q)x = R^{M+N} Bu + x^0. \]  

(2.14)

Denote by \( H_1 \) \((i = 1, 2, 3, 4)\) the following sets defined for any \( x^0 \in Z_{M+N}, u \in U \).

(1) If \( I + Q' \in R(X) \), then
\[ H_1(x^0, u) := \{T_1(R^{M+N} Bu + x^0) + z : z \in \ker(I + Q)\}, \]  

(2.15)

where
\[ T_1 := I - R^N R_{Q'} Q_1, \quad R_{Q'} \in \mathcal{R}_{I + Q'}. \]  

(2.16)

\( Q_1 \) is given by (21.10).

(2) If \( I + Q' \in \Lambda(X) \) and \( L_{Q'} \in L_{I + Q'} \), then
\[ H_2(x^0, u) := \{T_2(R^{M+N} Bu + x^0)\}, \]  

(2.17)
where
\[ T_2 := I - R^N L_Q Q_1, \quad Q_1 \text{ is defined by (2.10).} \]  
(2.18)

(3) If \( I + Q' \) is invertible, then
\[ H_3(x^0, u) := \{ T_3(R^M N B u + x^0) \} \]  
(2.19)

where
\[ T_3 := I - R^N (I + Q')^{-1} Q_1. \]  
(2.20)

(4) If \( I + Q' \in W(X) \) and \( W_{Q'} \in W_{I+Q} \), then
\[ H_4(x^0, u) := \{ T_4(R^M N B u + x^0) + z : z \in \ker(I + Q) \} \]  
(2.21)

where
\[ T_4 := I - R^N W_{Q'} Q_1. \]  
(2.22)

Note that \( H_i \) \( (i = 1, 2, 3, 4) \) are the sets of all solutions of the system \((LS)_0\) in the respective cases.

As in Section 33, we need the following

**Definition 2.5.** A state \( x \in X \) is said to be \((i)-reachable \) \( (i = 1, 2, 3, 4) \) from an initial state \( x^0 \in Z_{M+N} \) if for every \( T_i \) \( (T_1 \in R_{I+Q}, \ T_2 \in L_{I+Q}, \ T_3 = (I + Q)^{-1}, \ T_4 \in W_{I+Q}) \) there exists a control \( u \in U \) such that \( x \in H_i(x^0, u) \).

In the following we only deal with the above four cases. Write
\[ \text{Rang}_{U, x^0} H_i = \bigcup_{u \in U} H_i(x^0, u), \quad x^0 \in Z_{M+N}. \]  
(2.23)

It is easy to see that \( \text{Rang}_{U, x^0} H_i \) is \((i)-reachable \) from \( x^0 \) by means of controls \( u \in U \) and it is contained in \( X_{M+N} \).

**Lemma 2.1.** Suppose that \( T_i \) \( (i = 1, 2, 3, 4) \) are given by (2.16), (2.18), (2.20) and (2.22), respectively. Then
\[ T_i(R^M N B U \oplus \{ x^0 \}) + \ker(I + Q) = T_i R^M N B U \oplus \{ T_i x^0 \} + \ker(I + Q). \]  
(2.24)

**Remark 2.1.** If \( I + Q' \) is either invertible or left invertible, the formula (2.24) is of the form
\[ T_i(R^M N B U \oplus \{ x_0 \}) = T_i R^M N B U \oplus \{ T_i x_0 \}. \]

**Corollary 2.1.**
\[ \text{Rang}_{U, x^0} H_i = T_i R^M N B U \oplus \{ T_i x^0 \} + \ker(I + Q). \]  
(2.25)
Corollary 2.2. The state \( x \in X_{M+N} \) is (i)-reachable from \( x_0 \in Z_{M+N} \) if and only if
\[
x \in T_i R^{M+N} BU \oplus \{ T_i x^0 \} \oplus \ker(I + Q).
\]

Lemma 2.2. Write
\[
E_i := T_i R^{M+N} B,
X_{0i} := T_i (R^N (I + Q') R^M X + (I + Q) Z_{M+N} - \{ x^0 \}).
\]

Then the operator \( E_i \) maps the space \( U \) into \( X_{0i} \).

Theorem 2.3. Let there be given a system \((LS)_0\) described by (2.12)-(2.13). Suppose that \( B \in L_0(U \to X, X' \to U') \), \( D \in L(X, X') \), \( T_i \in L_0(X_{M+N}, X'_{M+N}) \), \( i = 1, 2, 3, 4 \); \( R \in L_0(X, X') \). Then the generalized Kalman condition
\[
\ker B^* (R^*)^{M+N} T_i^* = \{ 0 \}
\]
holds if and only if for every initial state \( x^0 \in Z_{M+N} \), every state \( x \in R^{M+N} X + x^0 + \ker(I + Q) \) is reachable from \( x^0 \).

Definition 2.6. Let there be given a linear system \((LS)_0\) of the form (2.12)-(2.13) and let \( F'_i \in F_{D^{M+N}} \).

(i) The state \( x^1 \in Z_{M+N} \) is said to be \( F_i \)-reachable from an initial state \( x^0 \in Z_{M+N} \) if there exists a control \( u \in U \) such that \( x^1 \in F'_i H_i(x^0, u) \). The state \( x^1 \) is then called a final state.

(ii) The system \((LS)_0\) is said to be \( F_i \)-controllable if for every initial state \( x^0 \in Z_{M+N} \),
\[
F'_i (\text{Rang}_{U,x^0} H_i) = Z_{M+N}.
\]

(iii) The system \((LS)_0\) is said to be \( F_i \)-controllable to \( x^1 \in Z_{M+N} \) if
\[
x^1 \in F'_i (\text{Rang}_{U,x^0} H_i)
\]
for every initial state \( x^0 \in Z_{M+N} \).

Lemma 2.3. Let there be given a linear system \((LS)_0\) of the form (2.12)-(2.13) and an initial operator \( F'_i \in F_{D^{M+N}} \). Suppose that \((LS)_0\) is \( F'_i \)-controllable to zero and that
\[
F'_i T_i Z_{M+N} = Z_{M+N}.
\]
Then every final state \( x^1 \in Z_{M+N} \) is \( F'_i \)-reachable from zero.

Proof. It is sufficient to deal with the case \( i = 4 \). Since the system is \( F'_4 \)-controllable to zero, there exists a control \( u' \in U \) such that \( 0 \in F'_4 H_4(x^0, u') \),
i.e. there exists $z_0 \in \ker(I + Q)$ such that $F'_4(T_4(R^{M+N}Bu' + x^0) + z_0) = 0$, or equivalently

$$F'_4(T_4(R^{M+N}Bu' + z_0) = -F'_4T_4x^0.$$  

By the assumption (2.32), for every given state $x^1 \in Z_{M+N}$ we find $x^2 \in Z_{M+N}$ such that $-F'_4T_4x^2 = x^1$. Hence, there are $u \in U$ and $z_0 \in \ker(I + Q)$ such that

$$F'_4(T_4(R^{M+N}Bu) + z_0) = -F'_4T_4x^2 = x^1.$$  

This proves that an arbitrary final state $x^1$ is reachable from the initial state 0.

**Theorem 2.4.** Suppose that all assumptions of Lemma 2.3 are satisfied. Then the linear system $(LS)_0$ is $F'_i$-controllable.

**Proof.** It is sufficient to deal with the case of a generalized almost invertible resolving operator. By the assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I + Q)$ such

$$F'_4[T_4(R^{M+N}Bu_0 + x^0) + z_0] = 0. \quad (2.33)$$

On the other hand, by Lemma 2.3, for every given $x^1 \in Z_{M+N}$ there exist $u_2 \in U$, that $z_2 \in \ker(I + Q)$ such that

$$F'_4[T_4(R^{M+N}Bu_2 + 0) + z_2] = x^1. \quad (2.34)$$

If we add (2.33) and (2.34), we obtain $F'_4[T_4(R^{M+N}Bu) + x^0 + z_1] = x^1$, where $u_1 := u_0 + u_2 \in U$, $z_1 := z_0 + z_2 \in \ker(I + Q)$. Thus every final state $x^1 \in Z_{M+N}$ is $F_4$-reachable from the initial state $x^0 \in Z_{M+N}$.

Note that Theorem 2.4 was given by Nguyen Dinh Quyet [54-56] and Pogorzelski [84] for systems of the first order with invertible and one-sided invertible resolving operators (cf. Section 33). Theorem 2.4 can be generalized as follows:

**Theorem 2.5.** Let there be given a system $(LS)_0$ of the form (2.12)-(2.13) and an initial operator $F'_i \in F_{V^{M+N}}$. Suppose that $(LS)_0$ is $F'_i$-controllable to zero and that

$$F'_i[T_4(Z_{M+N} + \ker(I + Q)] = Z_{M+N}. \quad (2.35)$$

Then $(LS)_0$ is $F'_i$-controllable.

Note that the conditions of Theorem 2.4 and 2.5 are sufficient but not necessary.

**Theorem 2.6.** Let there be given a system $(LS)_0$ of the form (2.12)-(2.13) and an initial operator $F'_i \in F_{V^{M+N}}$. Then $(LS)_0$ is $F'_i$-controllable if and only if it is $F'_i$-controllable to every element $v^0 \in F'_i(T_iR^{M+N}X_{M+N})$. 

---514---
Note that the operator \( F^T_i T_i R^{M+N} B \) maps \( U \) into \( Z_{M+N} \). The following theorem shows that this operator determines the properties of the system \((LS)_0\).

**Theorem 2.7.** Let a linear system \((LS)_0\) of the form (2.12)- (2.13) and an initial operator \( F'_i \in \mathcal{F}_{DM+M} \) be given. Suppose that \( B \in L_0(U \to X, X' \to U') \), \( D \in L(X, X') \), \( R \in L_0(X, X') \) and \( T_i \in L_0(X_{M+N}, X_{M+N}) \). Then \((LS)_0\) is \( F'_i \)-controllable if and only if

\[
\ker B^*(R^*)^{M+N} T_i^*(F'_i)^* = \{0\}. \tag{2.42}
\]

**Theorem 2.7.** Suppose that the system \((LS)_0\) is \( F'_i \)-controllable. Then it is \( F' \)-controllable for every initial operator \( F' \in \mathcal{F}_{DM+M} \).

**Example 2.1.** Let \( X := C[0,1] \) over \( \mathbb{C} \). Let \( D := d/dt \),

\[
R := \int_{t_0}^{t} (F'x)(t) := x(t_0), \quad t_0 \in [0,1].
\]

Consider the system

\[
\begin{align*}
[D^N + P_0(D,I) + P_1(D,I)F' + R^k P_2(D,I)]x &= Bu, \tag{2.46} \\
FD^j x &= x_j, \quad x_j \in \mathbb{C} \quad (j = 0, \ldots, N-1), \tag{2.47}
\end{align*}
\]

where \( F' \in \mathcal{F}_{DN} \), \( U = X \), \( B \in L_0(X) \), \( k \in \mathbb{N}_0 \),

\[
P_{\mu}(t, s) := \sum_{i=0}^{N-1} a_{\mu i} t^i s^{N-1-i}, \quad a_{\mu i} \in \mathbb{C} \quad (\mu = 0, 1, 2). \tag{2.48}
\]

As before, we write

\[
\begin{align*}
Q_1 &:= P_0(D,I) + P_1(D,I)F' + R^k P_2(D,I), \\
Q &:= R^N Q_1, \quad Q' := P_0(I,R) + R^k P_2(I,R).
\end{align*}
\]

Since \( R \in V(X) \), the resolving operator \( I + Q' \) is invertible (Theorem I in Section 6). On the other hand, it is easy to check that \( Q' = Q_1 R^N \), so that by Theorem 2.1, \( I + Q \) is also invertible, and

\[
(I + Q)^{-1} = I - R^N (I + Q') Q_1. \tag{2.49}
\]

Write the system (2.46)-(2.47) in the following equivalent form:

\[
(I + Q)x = R^N Bu + x^0, \quad x^0 = \sum_{j=0}^{N-1} R^j x_j. \tag{2.50}
\]
From (2.49), we conclude that $x = [I - R^N(I + Q')^{-1}Q_1](R^N Bu + x^0) \in X_N$. Hence, (2.50) has solutions for every $u \in X$. This means that the condition (2.13) is satisfied. A unique solution of the system (2.46)-(2.47) is

$$x = [I - R^N(I + Q')^{-1}Q_1](R^N Bu + x^0) \in X_N.$$  \hfill (2.51)

Thus, every state $x \in [I - R^N(I + Q')^{-1}Q_1](R^N Bu \oplus \{x^0\})$ is reachable from $x^0 \in Z_N$.

Let $F'_1, F'_2 \in \mathcal{F}_{DN}$ be initial operators for $D^N$ given by

$$F'_1 := I - R^N, \quad F'_2 := I - R^N - 1 D^N \text{ on } \text{dom}D^N,$$

where $R_1 := \int_t^t$, $t_1 \neq t_0$, $t_0, t_1 \in [0, 1]$. Let $T_3 := (I + Q)^{-1}$. It is easy to check that $F'_1 R^N X = Z_N, F'_2 R^N X \neq Z_N$, so that for every $B \in L_0(X)$, we find

$$F'_2(I - R^N(I + Q')^{-1}Q_1)R^N Bu = F'_2 R^N(I - (I + Q')^{-1}Q_1 R^N)BX \neq Z_N,$$

i.e. ker $B^*(R^*)^N T^*_3 F'^*_1 \neq \{0\}$. This means that the system (2.46)-(2.47) is not $F'_2$-controllable.

Let $B = I$. Since $I - (I + Q')^{-1}Q_1 R^N$ is invertible because $I - R^N(I + Q')^{-1}Q_1$ is invertible, we conclude that

$$[I - (I + Q')^{-1}Q_1 R^N]X = X.$$

This implies

$$F'_1 T_3 R^N Bu = F'_1 T_3 R^N X = F'_1(I - R^N(I + Q')^{-1}Q_1)R^N X = F'_1 R^N X = Z_N.$$

Hence ker $B^*(R^*)^N T^*_3 F'^*_1 \neq \{0\}$. Thus, by Theorem 2.7, the system (2.46)-(2.47) is $F'_1$-controllable.

**Example 2.2.** Let $X = (s)$ be the space of all real sequences. Write $\{e_n\} := \{1, 1, \ldots\}, \{0_n\} := \{0, 0, \ldots\}$. Define the following operators:

$$D\{x_n\} := \{x_{n+1} - x_n\}, \quad F\{x_n\} := x_1\{e_n\},$$

$$R\{x_n\} := \{y_n\}, \quad y_1 := 0, \quad y_n := x_1 + \cdots + x_{n-1} \quad (n \geq 2),$$

$$A\{x_n\} = \{x_2, x_3 - x_2, 0, 0, \ldots\}, \quad B\{x_n\} = \{x_2, -x_2 - x_2, 0, 0, \ldots\},$$

$$C\{x_n\} = \{x_2 - x_1, 0, 0, \ldots\}.$$
Consider the system
\[
(D^2 - AD - DB - C)x = Bu,
Fx = x_0',
FDx = x_1',
x'_0, x'_1 \in \ker D,
\]
where \( u \in U, \; U \subset X, \; B \in L_0(U, X) \). Write
\[
Q_1 := RAD + B + RC, \quad Q := RQ_1, \quad Q' := RA + BR + RCR.
\]
The system (2.52) is equivalent to the equation
\[
(I - Q)x = R^2 Bu + x^0,
\]
\[
x^0 := x_0 + Rx_1.
\]
It is easy to see that \( I - Q' \) is the resolving operator for the system (2.52) and \( I - Q' = I - Q_1 R \). By easy calculations, we find
\[
RA\{x_n\} = \{0, x_2, x_3, x_3, \ldots\},
BR\{x_n\} = \{x_1, -x_1, -x_1, 0, 0, \ldots\},
RCR\{x_n\} = \{0, x_1, x_1, x_1, 0, 0, \ldots\},
\]
so that
\[
(I - Q')\{x_n\} = \{0, 0, 0, y_4, y_5, \ldots\},
\]
\[
y_k := x_k - x_1 - x_3 \quad (k = 4, 5, \ldots),
\]
\[
\ker(I - Q') = \{z = x_1, x_2, x_3, x_1 + x_3, x_1 + x_3, \ldots\},
\]
\[
3(I - Q') \neq X.
\]
The formulae (2.55)-(2.57) imply that the resolving operator \( I - Q' \) is not one-sided invertible. However, since \( (I - Q')(I - Q') = I - Q' \), we conclude that \( I - Q' \) is generalized almost invertible and \( I \) is its generalized almost inverse.

By straightforward calculations, we find
\[
(I - RQ_1)\{x_n\} = (I - Q)\{x_n\} = \{x_1, 0, 0, x_5, y_6, \ldots\},
\]
where \( y_k := x_k - (k - 3)x_{k-1} + (k - 4)(x_3 - x_2 + x_1) \quad (k \geq 5) \).

Let \( x'_0 := 0, \; x'_1 := 0 \), i.e. let the initial conditions of the problem \((LS)_0\) be \( Fx = 0, \; FDx = 0 \). Let \( U = X \) and
\[
B\{x_n\} = \{0, 0, 0, 0, x_1, x_2, x_3, \ldots\}.
\]
It is easy to check that
\[
BU \oplus \{x_0\} = BX \subset (I - Q)X_2 = (I - Q)X.
\]
Hence, the system (2.52) is solvable for every \( u \in U \). From (2.54) we find
\[
x = (I + RQ_1)R^2 Bu = (I + Q)R^2 Bu.
\]
Therefore, every state \( x \in (I + Q)R^2 BU \) is reachable from zero.
3 Controllability of linear systems described by generalized almost invertible operators

Let $X, Y, U$ be linear spaces over the same field $F$ (where $F = \mathbb{C}$ or $F = \mathbb{R}$). Suppose that $V \in W(X)$, $W \in W_\infty$ and $F^{(r)}$, $F^{(l)}$ are right and left initial operators for $V$ corresponding to $W$; $A \in L_0(X)$, $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$.

By a linear system $(LS)$ we now mean the following system:

$$Vx = Ax + Bu, \quad u \in U, \quad BU \subset (V - A)(\text{dom } V), \quad (3.1)$$

$$F^{(r)}x = x_0, \quad x_0 \in \ker V, \quad (3.2)$$

$$y = A_1x + B_1u. \quad (3.3)$$

If $A_1 = I$, $B_1 = 0$, i.e. $Y = X$ and $y = x$, then we denote the system (3.1)-(3.3) by $(LS)_0$.

Note that the properties of linear systems depend on the properties of the resolving operators $I - WA$ and $I - AW$. There are eight cases to deal with:

(i) $I - WA \in R(X)$, (ii) $I - WA \in \Lambda(X)$, (iii) $I - WA \in R(X) \cap \Lambda(X)$,
(iv) $I - WA \in W(X)$, (v) $I - AW \in R(X)$, (vi) $I - AW \in \Lambda(X)$, (vii) $I - AW \in R(X) \cap \Lambda(X)$, (viii) $I - AW \in W(X)$.

It is sufficient to consider the first four cases (i)-(iv). Since both one-sided invertible and invertible operators are generalized almost invertible, we can reduce those cases to the case of $I - WA$ being generalized almost invertible.

Suppose that we are given a linear system $(LS)_0$. The initial value problem (3.1)-(3.2) has solutions if and only if

$$WBu + x_0 \in (I - WA)X_u \subset (I - WA)(\text{dom } V), \quad (3.4)$$

where

$$X_u = \{x \in \text{dom } V : F^{(l)}(Ax + Bu) = 0\}, \quad u \in U,$$

and $x_0 = 0$ if $\dim \ker V = 0$.

So the condition

$$WBu + \{x_0\} \subset (I - WA)X_u \quad (3.5')$$

is a necessary and sufficient condition for the initial value problem (3.1)-(3.2) to have solutions for every $u \in U$.

It is easy to check that the condition (3.5') is equivalent to the following: $BU \subset (V - A)\text{dom } V$. 

—518—
Suppose that $I - WA$ is generalized almost invertible.

Write

\[ G(x_0, u) = \{ x = (I + WW_A A)(WBu + x_0) + z : W_A \in W_{I - AW}, \, z \in \ker(I - WA) \}. \]  

(3.6)

Note that $G$ is the set of all solutions of the problem (3.1)-(3.2). Therefore, to every fixed input $(x_0, u)$ there corresponds an output $x = G(x_0, u)$.

Write

\[ \text{Ran}_{U,x_0} G = \bigcup_{x \in U} G(x_0, u), \, x_0 \in \ker V. \]  

(3.7)

**Definition 3.1.** Suppose that we are given a linear system $(LS)_0$ and the set $G(x_0, u)$ of the form (3.6). A state $x \in X$ is said to be reachable from the initial state $x_0 \in \ker V$ if for every $W_A \in W_{I - AW}$ there exists a control $u \in U$ such that $x \in G(x_0, u)$.

It is easy to see that the set is reachable from the initial state $x_0 \in \ker V$ by means of controls $u \in U$ and this set is contained in $\text{dom } V$.

**Lemma 3.1.** Write

\[ T = I + WW_A A, \, W_A \in W_{I - AW}, \, W \in W_V. \]  

(3.8)

Then the following equality holds:

\[ T(WBU + \{x_0\}) + \ker(I - WA) = TWBU \oplus \{Tx_0\} \oplus \ker(I - WA). \]  

(3.9)

**Theorem 3.1.** Suppose that

\[ B \in L_0(U \to X, \, X' \to U'), \, V \in L(X, X') \cap W(X), \, W \in L_0(X, X') \cap W_V \]  

and $T \in L_0(X, X')$, where $T$ is defined by (3.8). Then the generalized Kalman condition

\[ \ker B^*W^*T^* = \{0\} \]  

(3.12)

holds if and only if for every initial state $x_0 \in \ker V$, every state

\[ x \in WV(\text{dom } V) + \{x_0\} + \ker(I - WA) \]  

is reachable from $x_0$.

Now we give another condition for every state $x \in WX + \{Tx_0\} + \ker(I - WA)$ to be reachable from any initial state $x_0 \in \ker V$.

**Lemma 3.2.** Let $V \in W(X)$, $W \in L_0(X) \cap W_V^\infty$ and let $T$ be given by (3.8). Then

\[ T \in W_{I - W_A}, \, TWX \subset WX. \]  

(3.14)
Lemma 3.2 implies that $F^{(r)}_1 T W B$ maps $U$ into $WX$. Corollary 3.1 yields

**Theorem 3.2.** Consider a linear system $(LS)_0$ described by a generalized almost invertible operator $V$. Suppose that $W \in L_0(X) \cap \mathcal{W}_V$ and $T$ is defined by (3.8). Then a necessary and sufficient condition for every element $x \in WX + \{T x_0\} + \ker(I - WA)$ to be reachable from any initial state $x_0 \in \ker V$ is that

$$T W B U = WX. \tag{3.15}$$

**Definition 3.2.** Let there be given a linear system $(LS)_0$ of the form (3.1)-(2). Let $F_1^{(r)}$ be any right initial operator for $V$ corresponding to $W_1 \in \mathcal{W}_V$.

(i) A state $x_1 \in \ker V$ is said to be $F_1^{(r)}$- reachable from an initial state $x_0 \in \ker V$ if there exists a control $u \in U$ such that $x_1 \in F_1^{(r)} G(x_0, u)$. The state $x_1$ is then called a finite state.

(ii) The system $(LS)_0$ is said to be $F_1^{(r)}$- controllable if for every initial state $x_0 \in \ker V$, we have

$$F_1^{(r)}(\text{Rang}_{U, x_0} G) = \ker V. \tag{3.16}$$

(iii) The system $(LS)_0$ is said to be $F_1^{(r)}$- controllable to $x_1 \in \ker V$ if

$$x_1 \in F_1^{(r)}(\text{Rang}_{U, x_0} G) \tag{3.17}$$

for every initial state $x_0 \in \ker V$.

**Lemma 3.3.** Suppose that the system $(LS)_0$ is $F_1^{(r)}$- controllable to zero and that

$$F_1^{(r)}[T(\ker V) + \ker(I - WA)] = \ker V. \tag{3.18}$$

Then every final state $x_1 \in \ker V$ is $F_1^{(r)}$- reachable from zero.

**Theorem 3.3.** Suppose that all assumptions of Lemma (3.3) are satisfied. Then the linear system $(LS)_0$ is $F_1^{(r)}$- controllable.

**Proof.** By our assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I - WA)$ such that

$$F_1^{(r)}[T(WBu_0 + x_0) + z_0] = 0 \tag{3.21}$$

By Lemma 3.3, for every $x_1 \in \ker V$ there exist $u'_0 \in U$ and $z_1 \in \ker(I - WA)$ such that

$$F_1^{(r)}(T W B u'_0 + z_1) = x_1. \tag{3.22}$$

Now (3.21) and (3.22) imply

$$F_1^{(r)}[T(WBu_0 + u'_0 + x_0) + (z_0 + z_1)] = x_1,$$

i.e. $x_1$ is $F_1^{(r)}$- reachable from $x_0$, which was to be proved.
Corollary 3.4. If the system \((LS)_0\) is \(F_1^{(r)}\)-controllable to zero and \(F_1^{(r)} T(\ker V) = \ker V\), then it is \(F_1^{(r)}\)-controllable.

**Theorem 3.4.** Let a linear system \((LS)_0\) of the form (3.1)-(3.2) and an initial operator \(F_1^{(r)}\) for \(V\) be given. Let \(T\) be defined by (3.8) and let \(B \in L_0(U \to X, X' \to U')\), \(V \in L(X, X')\), \(A, W \in L_0(X, X')\). Then \((LS)_0\) is \(F_1^{(r)}\)-controllable if and only if

\[
\ker B^{*}W^{*}T^{*}(F_1^{(r)})^* = \{0\}. \tag{3.23}
\]

**Theorem 3.5.** Let there be given a linear system \((LS)_0\) and an initial operator for \(V \in W(X)\). Then the system \((LS)_0\) is \(F_1^{(r)}\)-controllable if and only if it is \(F_1^{(r)}\)-controllable to every \(x' \in F_1^{(r)} TW V (\text{dom } V)\).

**Theorem 3.6.** Suppose that the system \((LS)_0\) is \(F_1^{(r)}\)-controllable. Then for an arbitrary right initial operator \(F_2^{(r)}\) for \(V\), this system is \(F_2^{(r)}\)-controllable.

**Example 3.1.** Let \(X := C[-1,1]\), \(D := d/dt\), \(R := \int_0^t, (Fx)(t) := x(0)\). Define \((Px)(t) := \frac{1}{2}[x(t) + x(-t)], Q := I - P, X^+ := PX, X^- := QX\), i.e. \(X = X^+ \oplus X^-\). Consider the linear system

\[
P(D + \beta I)x = Au, \ u \in U = X^+, \tag{3.29}
\]

\[
(I - RP D)x = x_0, \ x_0 = RQy_0 + z_0 \in \ker PD, \tag{3.30}
\]

where \(A \in L_0(X^+)\), \(\beta \in \mathbb{R}\).

Putting \(V = PD, W = RP\) we find \(VWV = V, WVW = W\). The right initial operator \(F^{(r)}\) for \(V\) corresponding to \(W\) is \(F^{(r)} = I - RP D\). Hence, we can write the system (3.29)-(3.30) in the form

\[
(V + \beta P)x = Au, \ F^{(r)}x = x_0. \tag{3.31}
\]

This system is equivalent to the equation

\[
(I + \beta RP)x = RP Au + x_0. \tag{3.32}
\]

Since \((I + \beta RP)(I - \beta RP) = I - \beta^2 RPRP = I - \beta^2 R^2 QP = I\), we conclude that every state \(x \in \text{dom } D\) is reachable from the initial state \(x_0\), i.e. there exists \(u \in U\) such that

\[
x = (I - \beta RP)(RP Au + x_0).
\]
Hence
\[ G(x_0, u) = \{ x = (I - \beta RP)(RPAu + x_0) \}, \tag{3.33} \]
and since \( RPRP = 0 \) we get
\[ (I - \beta RP)(RPAu + x_0) = RPAU \oplus \{(I - \beta RP)x_0\}. \tag{3.34} \]

From (3.33)-(3.34) we obtain
\[ \text{Rang}_{U,x_0} G = RPAU \oplus \{(I - \beta RP)x_0\}. \]

Thus the system (3.29)-(3.30) is \( F^{(r)}_1 \)-controllable for a right initial operator \( F^{(r)}_1 \) of \( V \) if and only if
\[ F^{(r)}_1(\text{Rang}_{U,x_0} G) = \text{ker}(PD). \]

It is easy to check that \( \text{ker}(PD) \) consists all even differentiable functions defined on \([-1,1]\).

**References**


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