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MATHEMATICAL STRUCTURE FOR FOREST DYNAMICS AND ITS APPLICATIONS

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ABSTRACT

We are concerned with a forest kinematic model presented by Kuznetsov et al. [4]. In this report, we will survey how to construct global solutions and a dynamical system for this model equation. Moreover, we study the structure (including stability and instability) of homogeneous stationary solutions and the existence of inhomogeneous stationary solutions. Especially it shall be shown that in some cases, one can construct an infinite number of discontinuous stationary solutions.

KEYWORDS

Forest ecosystem, Dynamical system, Stationary solutions, Ecotone of forest.

INTRODUCTION

We study the initial-boundary values problem for a parabolic-ordinary system

$$
(1) \qquad \begin{cases}
\dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in} \quad \Omega.
\end{cases}
$$

This system has been introduced by Kuznetsov et al. [4] in order to describe the kinetics of forest from the viewpoint of the age structure. For simplicity they consider a prototype ecosystem of a mono-species and with only two age classes in a two-dimensional domain Ω.

The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and at time $t \in [0,\infty)$. The third unknown function $w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates respectively. While the first and second equations describe the growth of young and old trees respectively; $0 < \delta \leq 1$ is a seed establishment rate, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density v, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees.

For w, the Neumann boundary conditions are imposed on the boundary $\partial\Omega$. The initial value (u_0, v_0, w_0) is taken from the space

$$K = \{(u_0, v_0, w_0); 0 \leq u_0,\ v_0 \in L^\infty(\Omega) \text{ and } 0 \leq w_0 \in L^2(\Omega)\}.$$

In this report, we intend to construct a global solution to (1) for each initial function $U_0 \in K$ and to construct a dynamical system determined from the problem. Furthermore, we are concerned with investigating the stability and instability of homogeneous stationary solutions and seeking inhomogeneous stationary solutions.

Throughout the report, Ω is a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$. We assume as in [4] that the mortality of young trees is given by a square function of the form

$$\gamma(v) = a(v-b)^2 + c, \tag{2}$$

where a, b, $c > 0$ are positive constants. This means that the mortality takes its minimum when the old-age tree density is a specific value b. As mentioned, d, f, h, α, $\beta > 0$ are all positive constants and $0 < \delta \leq 1$.

MATERIALS AND METHODS

We shall formulate the initial boundary value problem (1) as the Cauchy problem for an abstract semilinear equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0 \end{cases}$$

in the underlying product space $X = L^\infty(\Omega) \times L^\infty(\Omega) \times L^2(\Omega)$. Here, the linear operator A and the nonlinear operator F are defined by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \beta\delta w - \gamma(v)u \\ fu \\ \alpha v \end{pmatrix},$$

where Λ is a realization of the operator $-d\Delta + \beta$ in $L^2(\Omega)$ under the homogeneous Neumann boundary condition $\frac{\partial w}{\partial n} = 0$ on the boundary $\partial\Omega$. It is known that Λ is a positive definite self-adjoint operator of $L^2(\Omega)$ with $\mathcal{D}(\Lambda) = H^2_N(\Omega)$, where $H^2_N(\Omega)$ is a closed subspace of $H^2(\Omega)$ consisting of functions w's satisfying the homogeneous Neumann boundary conditions on $\partial\Omega$.

Then for each initial value $U_0 \in K$, we can apply the general results in [5] to construct local solutions. Nonnegativity of local solutions and a priori estimates for local solutions will be established in usual manners. As an immediate consequence of a priori estimates, we can prove the existence and uniqueness of global solutions. Moreover, from the Lipschitz continuity of solution in initial data, we can construct a dynamical system determined from (1).

RESULTS AND DISCUSSION

Theorem 1. *For any $U_0 \in K$, (1) possesses a unique global solution such that*

$$\begin{cases} 0 \le u, v \in \mathcal{C}([0,\infty); L^\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L^\infty(\Omega)), \\ 0 \le w \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H_N^2(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)). \end{cases}$$

For each $U_0 \in K$, let $U = U(t; U_0)$ be the global solution to (1) with the initial value U_0. Put $S(t)U_0 = U(t; U_0)$, $0 < t < \infty$.

Theorem 2. *The nonlinear semigroup $\{S(t)\}_{t\ge0}$ defines a dynamical system $(S(t), K, X)$.*

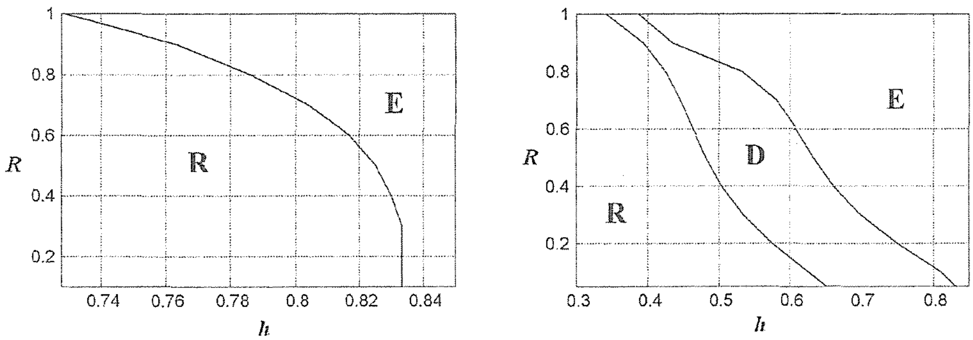
The structure of homogeneous stationary solutions depends on the parameter h drastically. When $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$, it is shown that there exist two homogeneous stationary solutions P_+ (which is non zero solution) and the zero solution $O = (0,0,0)$ and that P_+ is stable and O is unstable. This means that in this case any forest starting from a non zero initial state holds alive. In the meantime, when $\frac{f\alpha\delta}{c+f} < h < \infty$, the zero solution O is a unique stationary solution and is globally stable, that is, every forest is going to extinct. When $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$, there exist three homogeneous stationary solutions $P_\pm$ (which are non zero) and the zero solution O; here, P_+ and O are stable meanwhile P_- is unstable. This means that some forests can hold alive and others are going to extinct. What is more interesting is that, in this case, there exist many inhomogeneous stationary solutions. Especially when a and b are sufficiently large, one can construct an infinite number of discontinuous stationary solutions $(\overline{u}, \overline{v}, \overline{w})$'s, $\overline{u}$, $\overline{v} \in L^\infty(\Omega)$ being discontinuous and $\overline{w} \in H^2(\Omega)$ being continuous.

Such a discontinuous stationary solution is very important in the view point of forest also (see [4]). The interface of discontinuity of a stationary solution is considered as an internal and proper forest boundary, which is called an ecotone. So we can re-create the ecotone of forest by using the prototype model (1). Many interesting problems concerning discontinuous stationary solutions, however, remain to be solved.

We shall present some numerical example. Let Ω be square domain $[0,1] \times [0,1]$. The coefficients are taken as $\alpha = \beta = 1.0$, $\delta = 0.1$, $f = 1.0$, $a = 1.0$, $c = 0.2$ and $d = 0.05$. For each β and h, let $P^+ = (u^+, v^+, w^+)$ be the non zero homogeneous stationary solution. Initial functions u_0, v_0 and w_0 are given by $u_0(x,y) = v_0(x,y) = 0$ if $\sqrt{x^2+y^2} \le R$; $u_0(x,y) = u^+$, $v_0(x,y) = v^+$ if $\sqrt{x^2+y^2} > R$, on the other hand, $w_0 \equiv w^+$ in Ω. Here R is paramater such that $0 < R \le 1$. Our goal is to find the values of R so that forest starting from this initial state is going to extinct, or recover to homogeneous stationary solution P^+, or tend to a inhomogeneous stationary solution. We performed numerical computations for sufficiently large time until the graph of solution and the values of Lyapunov function are stabilized numerically.

In the case $\beta = 1$. As show in Fig. 1(a) below, there are only two regions $\mathbf{R}$ and $\mathbf{E}$. Where, forest is going to extinct if $(h, R) \in \mathbf{E}$ and going to recover the damages if $(h, R) \in \mathbf{R}$.

In the case $\beta = 3$. There are three regions $\mathbf{R}$, $\mathbf{D}$ and $\mathbf{E}$, where forest is going to tend to discontinuous stationary solutions if $(h, R) \in \mathbf{D}$ (see Fig. 1(b)).



(a) $\beta = 1$

(b) $\beta = 3$

Figure 1: Relation between h and R

CONCLUSIONS

We constructed a global solution to (1) for each triplet of initial functions and constructed a dynamical system determined from the problem. Furthermore, we obtained some results about the structure of homogeneous stationary solutions and the existence of inhomogeneous stationary solutions.

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