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CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

NGUYEN XUAN THAO AND NGUYEN THANH HONG

ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. Its properties and applications to solve systems of integral equations are considered.

1. INTRODUCTION

Let $F_s$ be the Fourier sine transform [2]

$$(F_s f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y) dy,$$

and $F_c$ be the Fourier cosine transform [2]

$$(F_c f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y) dy.$$

Convolution theory has been studied in 20th. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated. The convolution of two functions $f$ and $g$ for the Fourier cosine transform is introduced in [7]

$$(f \ast_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(|x - y|) + g(x + y)] dy, \quad x > 0,$$

which satisfied the following factorization equality

$$(1.2) \quad F_c(f \ast_{F_c} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$

In 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier

Convolution, Fourier sine transform, Fourier cosine transform, integral equation.
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sine transform which is of the form [4] and [10]

$$(1.3) \quad (f \ast_{F_s} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} f(y)[\text{sign}(x + y - 1)g(|x + y - 1|) + \text{sign}(x - y + 1)g(|x - y + 1|)$$

$$- g(x + y + 1) - \text{sign}(x - y - 1)g(|x - y - 1|)]dy, \quad x > 0,$$

and proved the following factorization identity [4], [10]

$$(1.4) \quad F_s(f \ast_{F_s} g)(y) = \sin y(F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$ 

The convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine transform of two functions $f$ and $g$ is introduced in [11]

$$(1.5) \quad (f \ast_{F_c} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} f(y)[g(|y+x-1|)+g(|y-x-1|)+g(y+x+1)+g(|y-x+1|)]dy, \quad x > 0$$

and satisfy the factorization equality [11]

$$(1.6) \quad F_c(f \ast_{F_c} g)(y) = \cos y(F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$ 

In 1941, Churchill R.V introduced the first generalized convolution of two functions $f$ and $g$ for the Fourier sine and Fourier cosine transforms [7]

$$(1.7) \quad (f \ast_{F} \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \quad x > 0,$$

and proved the following factorization identity [7]

$$(1.8) \quad F_s(f \ast_{F} \ast g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$ 

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, $G$-transform and $H$-transform. In 1998, Kakiachev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions $f$ and $g$ for the Fourier cosine and sine transforms is defined by [6]

$$(1.9) \quad (f \ast_{F} \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(u)[\text{sign}(u-x)g(|u-x|) + g(u+x)]du, \quad x > 0.$$ 

For this generalized convolution the following factorization equality holds [6]

$$(1.10) \quad F_c(f \ast_{F} \ast g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$
Another generalized convolution with the weight function \( \gamma_1(y) = \sin y \) for the Fourier cosine and sine has been studied in [15]

\[
(f \ast_2 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)\left[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)\right]du, \ x > 0.
\]

It satisfies the factorization property [15]

\[
F_c(f \ast_2 g)(y) = \sin y(F_s f)(y)(F_c g)(y), \ \forall y > 0.
\]

The generalized convolution of two functions \( f \) and \( g \) with the weight function \( \gamma_1(y) = \sin y \) for the Fourier sine and cosine transforms has the form

\[
(f \ast_1 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)\left[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)\right]du, \ x > 0,
\]

and satisfy the following factorization identity [14]

\[
F_s(f \ast_1 g)(y) = \sin y(F_c f)(y)(F_s g)(y), \ \forall y > 0.
\]

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Its properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

## 2. The Generalized Convolution

**Definition 1.** A generalized convolution with the weight function \( \gamma(y) = \cos y \) for the Fourier cosine and sine transforms of functions \( f \) and \( g \) is defined by

\[
(f \ast_\gamma g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(y)\left[g(y+x+1) + \text{sign}(y-x-1)g(|y-x-1|)
+ \text{sign}(y+x-1)g(|y+x-1|) + \text{sign}(y-x+1)g(|y-x+1|)\right]dy, \ x > 0.
\]

We denote by \( L(\mathbb{R}_+) \) the space of all functions \( f \) defined on \( \mathbb{R}_+ \) such that \( \int_{\mathbb{R}_+} |f(x)|dx < \infty \).

**Theorem 1.** Let \( f \) and \( g \) be functions in \( L(\mathbb{R}_+) \) then the generalized convolution \( (f \ast_\gamma g)(x) \) defined by (2.1) also be a \( L(\mathbb{R}_+) \) function. Moreover, the following factorization equality holds

\[
F_c(f \ast_\gamma g)(y) = \cos y(F_s f)(y)(F_s g)(y), \ \forall y > 0.
\]
Proof. From the defining formula of the generalized convolution and the fact that \( f, g \in L(\mathbb{R}_+) \) we have

\[
(2.3) \quad \int_0^{+\infty} |(f \ast g)(x)| \, dx = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} |f(y)| \, |g(y + x + 1) + \text{sign}(y - x - 1)g(|y - x - 1|) + \text{sign}(y + x - 1)g(|y + x - 1|) + \text{sign}(y - x + 1)g(|y - x + 1|)| \, dy \, dx \\
\leq \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(y)| \left[ \int_0^{+\infty} |g(y + x + 1)| \, dx + \int_0^{+\infty} |g(|y - x - 1|)| \, dx \right] \, dy.
\]

On the other hand,

\[
(2.4) \quad \int_0^{+\infty} |g(y + x + 1)| \, dx + \int_0^{+\infty} |g(|y - x - 1|)| \, dx = \int_0^{+\infty} |g(t)| \, dt + \int_0^{+\infty} |g(|t|)| \, dt \\
= 2 \int_0^{+\infty} |g(t)| \, dt.
\]

Similarly,

\[
(2.5) \quad \int_0^{+\infty} |g(|y + x - 1|)| \, dx + \int_0^{+\infty} |g(|y - x + 1|)| \, dx = 2 \int_0^{+\infty} |g(t)| \, dt.
\]

From (2.3), (2.4) and (2.5) one holds

\[
\int_0^{+\infty} |(f \ast g)(x)| \, dx \leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |g(t)| \, dt \int_0^{+\infty} |g(t)| \, dt \leq +\infty.
\]

So \((f \ast g)(x)\) belong to \(L(\mathbb{R}_+)\).

Now we prove the factorization equality (2.2). We have

\[
\cos y(F_s f)(y)(F_s g)(y) = \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} \cos y \sin uy \sin vy f(u)g(v) \, dudv \\
= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u)g(v)[\cos y(u - v + 1) + \cos y(u - v - 1) - \cos y(u + v + 1) - \cos y(u + v - 1)] \, dudv.
\]
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Changing the variables gives

$$ \cos y(F_s f)(y)(F_s g)(y) = \frac{1}{2\pi} \int_0^\infty f(u) \left[ \int_{-u-1}^{+\infty} g(t + u + 1) \cos y t \, dt + \int_{-u+1}^{+\infty} g(t + u - 1) \cos y t \, dt \right] du $$

$$ - \int_{u+1}^{+\infty} g(t - u - 1) \cos y t \, dt - \int_{u-1}^{+\infty} g(t - u + 1) \cos y t \, dt \right] du $$

$$ = \frac{1}{2\pi} \int_0^\infty f(u) \left[ \int_0^{+\infty} g(u + t + 1) \cos y t \, dt + \int_0^{+\infty} \text{sign}(u - t + 1) g(|u - t + 1|) \cos y t \, dt \right] du $$

$$ + \int_0^{+\infty} \text{sign}(u + t - 1) g(|u + t - 1|) \cos y t \, dt + \int_0^{+\infty} \text{sign}(u - t - 1) g(|u - t - 1|) \cos y t \, dt \right] du $$

$$ = \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ g(u + t + 1) + \text{sign}(u - t + 1) g(|u - t + 1|) + \text{sign}(u + t - 1) g(|u + t - 1|) + \text{sign}(u - t - 1) g(|u - t - 1|) \right] du \cos y t \, dt $$

$$ = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_s^\gamma g)(t) \cos y t \, dt $$

$$ = F_c(f^\gamma g)(y). $$

It shows that

$$ F_c(f^\gamma g)(y) = \cos y(F_s f)(y)(F_s g)(y). $$

The proof of Theorem is completes. $\square$

**Theorem 2.** In the space $L(\mathbb{R}_+)$ the generalized convolution (2.1) is not associative and the relation with knowns convolutions and generalized convolutions as follows

a) $f \ast (g \ast h) = g \ast (f \ast h) = h \ast (g \ast f)$, where $f \ast g$ is defined by (1.7).

b) $f^\gamma_1 \ast (g \ast h) = (f^\gamma_1 \ast g) \ast h = (f \ast h)^\gamma_1 \ast g$, where $f \ast g$ is the Fourier cosine convolution (1.1).

c) $f^{\gamma_1}_2 \ast (g \ast h) = (f^{\gamma_1}_2 \ast h) \ast g = h^{\gamma_1}_2 \ast (f \ast g)$, where $f^{\gamma_1}_2 \ast g$ and $f^{\gamma_1}_2 \ast g$ are defined by (1.3) and (1.11).

d) $f^\gamma_1 \ast (g \ast h) = (f^{\gamma_1}_2 \ast g) \ast h = (f^{\gamma_1}_2 \ast h) \ast g$, where $f^\gamma_1 \ast g$ is the Fourier cosine convolution with a weight function (1.5).
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**Proof.** a) From the factorization equality, we have

\[
F_s(f \ast (g \ast h))(y) = (F_s f)(y)(F_c(g \ast h))(y)
\]

\[
= (F_s f)(y) \cos y(F_s g)(y)(F_s h)(y) = (F_s g)(y) \cos y(F_s f)(y)(F_s h)(y)
\]

\[
= (F_s g)(y)(F_c(f \ast h))(y)
\]

\[
= F_s(g \ast (f \ast h))(y).
\]

Which implies that \( f \ast (g \ast h) = g \ast (f \ast h) \)

On the other hand,

\[
F_s(f \ast (g \ast h))(y) = (F_s f)(y) \cos y(F_s g)(y)(F_s h)(y)
\]

\[
= (F_s h)(y) \cos y(F_s g)(y)(F_s f)(y)
\]

\[
= F_s(g \ast (f \ast h))(y).
\]

Then we obtain the part a). The parts b), c), d) can be obtain in similar way. The Theorem is proved.

**Theorem 3.** In the space \( L(R^+) \) the generalized convolution (2.1) does not have a unit element.

**Proof.** Suppose that there exists a unit element \( e \) of the generalized convolution (2.1) in \( L(R^+) \). It means that \( f \ast e = e \ast f = f \) for any function \( f \in L(R^+) \). It follows that \( F_c(f \ast e)(y) = (F_c f)(y), \forall y > 0. \)

Hence, \( \cos y(F_s e)(y)(F_s f)(y) = (F_c f)(y), \forall y > 0. \)

Choosing \( f(x) = e^{-x} \in L(R^+) \). From the fact that

\[
(F_c f)(y) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + y^2}, \quad (F_s f)(y) = \sqrt{\frac{2}{\pi}} \frac{y}{1 + y^2},
\]

we obtain

\[
(F_s e)(y) = \frac{1}{y \cos y}.
\]

It is contradiction from the fact that \( \frac{1}{y \cos y} \notin L_\infty(R^+) \) while \( (F_s e)(y) \in L_\infty(R^+) \) since \( e \in L(R^+) \).

The Theorem is proved.

Let \( L(R^+, e^x) = \{ h, \text{ for all } e^x h(x) \in L(R^+) \}. \)

**Theorem 4** (Titchmarsh-type Theorem). Let \( f, g \in L(R^+, e^x) \). If \( f \ast g \equiv 0 \) then either \( f \equiv 0 \) or \( g \equiv 0 \).

**Proof.** Suppose that \( (f \ast g)(x) = 0, \forall x > 0, \) in view of Theorem 1

\[
F_c(f \ast g)(y) = \cos y(F_s f)(y)(F_s g)(y) = 0, \forall y > 0.
\]
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We have

\[
\left| \frac{d^n}{dy^n} [\sin(xy)f(x)] \right| = \left| f(x)x^n \sin(xy + n \frac{\pi}{2}) \right|
\]

\[
\leq \left| f(x)x^n \right|
\]

\[
= e^{-\pi x_n} \left| f_1(x) \right| \leq n! \left| f_1(x) \right|
\]

where $f_1(x) = e^x f(x) \in L(\mathbb{R}_+)$. Due to Weierstrass criterion, the integral \[ \int_0^{+\infty} \frac{d^n}{dy^n} [\sin(xy)f(x)] \, dx \] uniformly converges on $\mathbb{R}_+$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_s f)(y)$ is analytic. Similarly, $(F_s g)(y)$ is analytic. So from (2.6) we have $(F_s f)(y) \equiv 0$ or $(F_s g)(y) \equiv 0$. It completes the proof. \( \square \)

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations

\[
f(x) + \lambda_1 \int_0^{+\infty} g(t) \varphi_1(x, t) \, dt + \lambda_2 \int_0^{+\infty} g(t) \eta_1(x, t) \, dt + \lambda_3 \int_0^{+\infty} g(t) \mu_1(x, t) \, dt = h(x)
\]

(3.1)

\[
\lambda_4 \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \xi(t) \left[ f(x + t) - f(|x - t|) \right] \, dt + \lambda_5 \int_0^{+\infty} g(t) \psi_1(x, t) \, dt + g(x) = k(x),
\]

here $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are complex numbers, $\varphi, \eta, \mu, \xi \in L(\mathbb{R}_+)$, and

\[
\varphi_1(x, t) = \frac{1}{2\sqrt{2\pi}} \left[ \varphi(t + x + 1) + \text{sign}(t - x - 1) \varphi(|t - x - 1|) + \text{sign}(t + x + 1) \varphi(|t + x - 1|) \right. \\
+ \text{sign}(t - x + 1) \varphi(|t - x + 1|),
\]

\[
\eta_1(x, t) = \frac{1}{2\sqrt{2\pi}} \left[ \eta(|x - t + 1|) + \eta(|x - t + 1|) - \eta(x + t + 1) - \eta(|x - t - 1|) \right],
\]

\[
\mu_1(x, t) = \frac{1}{2\sqrt{2\pi}} \left[ \text{sign}(t - x) \mu(|t - x|) + \mu(t + x) \right],
\]

\[
\psi_1(x, t) = \frac{1}{2\sqrt{2\pi}} \left[ \psi(|x + t - 1|) + \psi(|x - t + 1|) - \psi(x + t + 1) - \psi(|x - t + 1|) \right],
\]

and $\psi(x) = (\nu_1 * \nu_2)(x)$.

It shows that $\varphi_1, \eta_1, \mu_1, \psi$ and $\psi_1$ are also $L(\mathbb{R}_+)$ - functions.

Theorem 5. With the condition

\[
F_c(\lambda_1 \lambda_4 \varphi_1 \xi + \lambda_2 \lambda_4 \xi \gamma_2 \eta + \lambda_3 \lambda_4 \mu \xi + \lambda_1 \lambda_5 \varphi_2 (\nu_1 \gamma_1 \nu_2) + \lambda_2 \lambda_5 \nu_1 \gamma_2 (\nu_2 \gamma \eta) + \lambda_3 \lambda_5 \mu \gamma \psi)(y) \neq 1,
\]

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there exists a unique solution in \( L(\mathbb{R}_+) \) of system (3.1) which is of the form

\[
\begin{align*}
f = h - \lambda_1 \varphi * k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu * k + h * l - \lambda_1 (\varphi * k) * \hat{F}_c - \lambda_2 (k \frac{\gamma}{2} \eta) * l - \lambda_3 (\mu * k) * l \\
g = k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma}{2} \eta + h * l - \lambda_4 (\xi * h) * l - \lambda_5 (\psi \frac{\gamma}{2} \eta) * l
\end{align*}
\]

Here \( l \in L(\mathbb{R}_+) \) and is defined by

\[
F_c l = \frac{F_c (\lambda_1 \lambda_4 \varphi \frac{\gamma}{2} \eta + \lambda_2 \lambda_4 \xi \frac{\gamma}{2} \eta + \lambda_3 \lambda_4 \mu \frac{\gamma}{2} \eta + \lambda_1 \lambda_5 \varphi (\nu_1 \frac{\gamma}{2} \eta + \nu_2 \frac{\gamma}{2} \eta) + \lambda_2 \lambda_5 \nu_1 \frac{\gamma}{2} \eta + \lambda_3 \lambda_5 \mu \frac{\gamma}{2} \eta)}{1 - F_c (\lambda_1 \lambda_4 \varphi \frac{\gamma}{2} \eta + \lambda_2 \lambda_4 \xi \frac{\gamma}{2} \eta + \lambda_3 \lambda_4 \mu \frac{\gamma}{2} \eta + \lambda_1 \lambda_5 \varphi (\nu_1 \frac{\gamma}{2} \eta + \nu_2 \frac{\gamma}{2} \eta) + \lambda_2 \lambda_5 \nu_1 \frac{\gamma}{2} \eta + \lambda_3 \lambda_5 \mu \frac{\gamma}{2} \eta)}
\]

**Proof.** System (3.1) can be rewritten in the form

\[
f(x) + \lambda_1 (g * \varphi)(x) + \lambda_2 (g \frac{\gamma}{2} \eta)(x) + \lambda_3 (g * \mu)(x) = h(x)
\]

(3.2)

\[
\lambda_4 (\xi * f)(x) + \lambda_5 (f \frac{\gamma}{2} \psi)(x) + g(x) = k(x).
\]

Using the respectively factorization equalities of above generalized convolutions we have

\[
(F_c f)(y) + \lambda_1 \cos y (F_s g)(y) (F_s \varphi)(y) + \lambda_2 \sin y (F_s g)(y) (F_s \xi)(y) + \lambda_3 (F_s g)(y) (F_s \mu)(y) = (F_c h)(y)
\]

\[
\lambda_4 (F_s \xi)(y) (F_c f)(y) + \lambda_5 \sin y (F_c f)(y) (F_c \psi)(y) + (F_s g)(y) = (F_s k)(y).
\]

We have

\[
\Delta = \begin{vmatrix}
1 & \lambda_1 \cos y (F_s \varphi)(y) + \lambda_2 \sin y (F_s \xi)(y) + \lambda_3 (F_s \mu)(y) \\
\lambda_4 (F_s \xi)(y) + \lambda_5 \sin y (F_c \psi)(y) & 1
\end{vmatrix} = 1 - \lambda_1 \cos y (F_s \varphi)(y) (F_c \xi)(y) - \lambda_2 \lambda_4 \sin y (F_s \xi)(y) (F_c \xi)(y) - \lambda_3 \lambda_4 \sin y (F_s \xi)(y) (F_c \xi)(y) - \lambda_1 \lambda_5 \sin y (F_s \varphi)(y) (F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 y (F_s \xi)(y) (F_c \psi)(y) - \lambda_3 \lambda_5 \sin y (F_s \xi)(y) (F_c \psi)(y) - \lambda_1 \lambda_5 \sin y (F_s \varphi)(y) (F_c \psi)(y) - \lambda_2 \lambda_5 \sin y (F_s \xi)(y) (F_c \psi)(y) - \lambda_3 \lambda_5 \sin y (F_s \xi)(y) (F_c \psi)(y)
\]

Then due to Wiener-Levi's Theorem [10] there exists a function \( l \in L(\mathbb{R}_+) \) such that

\[
\frac{1}{\Delta} - 1 = F_c (\lambda_1 \lambda_4 \varphi \frac{\gamma}{2} \eta + \lambda_2 \lambda_4 \xi \frac{\gamma}{2} \eta + \lambda_3 \lambda_4 \mu \frac{\gamma}{2} \eta + \lambda_1 \lambda_5 \varphi (\nu_1 \frac{\gamma}{2} \eta + \nu_2 \frac{\gamma}{2} \eta) + \lambda_2 \lambda_5 \nu_1 \frac{\gamma}{2} \eta + \lambda_3 \lambda_5 \mu \frac{\gamma}{2} \eta) (y)
\]

\[
1 - F_c (\lambda_1 \lambda_4 \varphi \frac{\gamma}{2} \eta + \lambda_2 \lambda_4 \xi \frac{\gamma}{2} \eta + \lambda_3 \lambda_4 \mu \frac{\gamma}{2} \eta + \lambda_1 \lambda_5 \varphi (\nu_1 \frac{\gamma}{2} \eta + \nu_2 \frac{\gamma}{2} \eta) + \lambda_2 \lambda_5 \nu_1 \frac{\gamma}{2} \eta + \lambda_3 \lambda_5 \mu \frac{\gamma}{2} \eta) (y) = (F_c l)(y),
\]
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furthermore

$$\Delta_1 = \frac{(F_c k)(y) \quad \lambda_1 \cos y (F_s \varphi)(y) + \lambda_2 \sin y (F_c \eta)(y) + \lambda_3 (F_s \mu)(y) }{(F_s k)(y) \quad 1}$$

$$= (F_c h)(y) - \lambda_1 \cos y (F_s \varphi)(y) (F_s k)(y) - \lambda_2 \sin y (F_c \eta)(y) (F_s k)(y) - \lambda_3 (F_s \mu)(y) (F_s k)(y)$$

$$= F_c (h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu \ast k)(y).$$

Therefore

$$\frac{(F_s f)(y)}{\Delta_1} = F_c (h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu \ast k)(y)[1 + (F_c l)(y)]$$

$$= F_c (h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \frac{\gamma_1}{2} \eta) \ast l - \lambda_3 (\mu \ast k) \ast l)(y)$$

It shows that

$$f = h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \frac{\gamma_1}{2} \eta) \ast l - \lambda_3 (\mu \ast k) \ast l$$

Similarly

$$\Delta = \left| \begin{array}{c} \lambda_2 (F_s \xi)(y) + \lambda_5 \sin y (F_c \psi)(y) \\ \lambda_2 (F_s \xi)(y) \end{array} \right|$$

$$= (F_c k)(y) - \lambda_4 (F_s \xi)(y) (F_c h)(y) - \lambda_5 \sin y (F_c \psi)(y) (F_c h)(y)$$

$$= F_c (k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\gamma_1}{2} \eta)(y).$$

Then

$$\frac{(F_s g)(y)}{\Delta} = \frac{\Delta_2}{\Delta} = F_c (k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\gamma_1}{2} \eta)(y)[1 + (F_c l)(y)]$$

$$= F_c (k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \frac{\gamma_1}{2} \eta) \ast l)(y).$$

Hence

$$g = k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\gamma_1}{2} \eta + k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \frac{\gamma_1}{2} \eta) \ast l.$$

The proof is complete.

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