CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

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ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. It's properties and applications to solve systems of integral equations are considered.

1. INTRODUCTION

Let F_s be the Fourier sine transform [2]

$$(F_s f)(c) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin xy f(y) dy,$$

and F_c be the Fourier cosine transform [2]

$$(F_c f)(c) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos xy f(y) dy.$$

Convolution theory has been studied in 20^{th}. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated.
The convolution of two functions f and g for the Fourier cosine transform is introduced in [7]

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(y)[g(|x-y|) + g(x+y)]dy, \quad x > 0, \tag{1.1}$$

which satisfied the following factorization equality

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y), \qquad \forall y > 0. \tag{1.2}$$

In 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions f and g with the weight function $\gamma_1(y) = \sin y$ for the Fourier

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Convolution, Fourier sine transform, Fourier cosine transform, integral equation.

sine transform which is of the form [4] and [10]

$$(1.3)\quad (f \overset{\gamma_1}{\underset{F_s}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\mathrm{sign}(x+y-1)g(|x+y-1|) + \mathrm{sign}(x-y+1)g(|x-y+1|) - g(x+y+1) - \mathrm{sign}(x-y-1)g(|x-y-1|)]dy, \quad x > 0,$$

and proved the following factorization identity [4], [10]

$$(1.4)\qquad F_s(f \overset{\gamma_1}{\underset{F_s}{*}} g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$

The convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine transform of two functions f and g is introduced in [11]

$$(1.5)\quad (f \overset{\gamma}{\underset{F_c}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(|y+x-1|)+g(|y-x-1|)+g(y+x+1)+g(|y-x+1|)]dy, x > 0$$

and satisfy the factorization equality [11]

$$(1.6)\qquad F_c(f \overset{\gamma}{\underset{F_c}{*}} g)(y) = \cos y (F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$

In 1941, Churchill R.V introduced the first generalized convolution of two functions f and g for the Fourier sine and Fourier cosine transforms [7]

$$(1.7)\qquad (f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \quad x > 0,$$

and proved the following factorization identity [7]

$$(1.8)\qquad F_s(f \underset{1}{*} g)(y) = (F_s f)(y).(F_c g)(y), \quad \forall y > 0.$$

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, G-transform and H-transform. In 1998, Kakichev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions f and g for the Fourier cosine and sine transforms is defined by [6]

$$(1.9)\qquad (f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[\mathrm{sign}(u-x)g(|u-x|) + g(u+x)]du, \ x > 0.$$

For this generalized convolution the following factorization equality holds [6]

$$(1.10)\qquad F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \ \forall y > 0.$$

Another generalized convolution with the weight function $\gamma_1(y) = \sin y$ for the Fourier cosine and sine has been studied in [15]

(1.11)

$$(f \overset{\gamma_1}{\underset{2}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|)+g(|x-u+1|)-g(x+u+1)-g(|x-u-1|)]du, x > 0.$$

It satisfies the factorization property [15]

$$F_c(f \overset{\gamma_1}{\underset{2}{*}} g)(y) = \sin y (F_s f)(y)(F_c g)(y), \quad \forall y > 0. \tag{1.12}$$

The generalized convolution of two functions f and g with the weight function $\gamma_1(y) = \sin y$ for the Fourier sine and cosine transforms has the form

(1.13)

$$(f \overset{\gamma_1}{\underset{1}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|)+g(|x-u-1|)-g(x+u+1)-g(|x-u+1|)]du, \ x > 0,$$

and satisfy the following factorization identity [14]

$$F_s(f \overset{\gamma_1}{\underset{1}{*}} g)(y) = \sin y (F_c f)(y)(F_c g)(y), \quad \forall y > 0. \tag{1.14}$$

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Is properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

2. The generalized convolution

Definition 1. *A generalized convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine and sine transforms of functions f and g is defined by*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(y+x+1) + \operatorname{sign}(y-x-1)g(|y-x-1|) + \operatorname{sign}(y+x-1)g(|y+x-1|) + \operatorname{sign}(y-x+1)g(|y-x+1|)]dy, \quad x > 0. \tag{2.1}$$

We denote by $L(\mathbb{R}_+)$ the space of all functions f defined on $\mathbb{R}_+$ such that $\int_{+} |f(x)|dx < \infty$.

Theorem 1. *Let f and g be functions in $L(\mathbb{R}_+)$ then the generalized convolution $(f \overset{\gamma}{*} g)(x)$ defined by* (2.1) *also be a $L(\mathbb{R}_+)$ function. Moreover, the following factorization equality holds*

$$F_c(f \overset{\gamma}{*} g)(y) = \cos y (F_s f)(y)(F_s g)(y), \qquad \forall y > 0. \tag{2.2}$$

Proof. From the defining formula of the generalized convolution and the fact that $f, g \in L(\mathbb{R}_+)$ we have

$$\begin{aligned}(2.3)\quad \int_0^{+\infty} |(f \overset{\gamma}{*} g)|(x)dx &= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} |f(y)|.|[g(y+x+1) + \operatorname{sign}(y-x-1)g(|y-x-1|)\\ &\quad + \operatorname{sign}(y+x-1)g(|y+x-1|) + \operatorname{sign}(y-x+1)g(|y-x+1|)]|dydx\\ &\leqslant \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(y)|.\Big[\int_0^{+\infty} |g(y+x+1)|dx + \int_0^{+\infty} |g(|y-x-1|)|dx\\ &\quad + \int_0^{+\infty} |g(|y+x-1|)|dx + \int_0^{+\infty} |g(|y-x+1|)|dx\Big]dy.\end{aligned}$$

On the other hand,

$$\begin{aligned}(2.4)\quad \int_0^{+\infty} |g(y+x+1)|dx + \int_0^{+\infty} |g(|y-x-1|)|dx &= \int_{y+1}^{+\infty} |g(t)|dt + \int_{-y-1}^{+\infty} |g(|t|)|dt\\ &= 2\int_0^{+\infty} |g(t)|dt.\end{aligned}$$

Similary,

$$(2.5)\quad \int_0^{+\infty} |g(|y+x-1|)|dx + \int_0^{+\infty} |g(|y-x+1|)|dx = 2\int_0^{+\infty} |g(t)|dt.$$

From (2.3), (2.4) and (2.5) one holds

$$\int_0^{+\infty} |(f \overset{\gamma}{*} g)|(x)dx \leqslant \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |g(t)|dt \int_0^{+\infty} |g(t)|dt \leqslant +\infty.$$

So $(f \overset{\gamma}{*} g)|(x)$ belong to $L(\mathbb{R}_+)$.

Now we prove the factorization equality (2.2). We have

$$\cos y(F_s f)(y)(F_s g)(y) = \frac{2}{\pi}\int_0^{+\infty}\int_0^{+\infty} \cos y \sin uy \sin vy f(u)g(v)dudv$$

$$= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty} f(u)g(v)[\cos y(u-v+1) + \cos y(u-v-1) - \cos y(u+v+1) - \cos y(u+v-1)]dudv.$$

Changing the variables gives

$$\cos y(F_s f)(y)(F_s g)(y) = \frac{1}{2\pi}\int_0^{+\infty} f(u)\Big[\int_{-u-1}^{+\infty} g(t+u+1)\cos yt dt + \int_{-u+1}^{+\infty} g(t+u-1)\cos yt dt$$
$$-\int_{u+1}^{+\infty} g(t-u-1)\cos yt dt - \int_{u-1}^{+\infty} g(t-u+1)\cos yt dt\Big]du$$
$$=\frac{1}{2\pi}\int_0^{+\infty} f(u)\Big[\int_0^{+\infty} g(u+t+1)\cos yt dt + \int_0^{+\infty} \operatorname{sign}(u-t+1)g(|u-t+1|)\cos yt dt$$
$$+\int_0^{+\infty} \operatorname{sign}(u+t-1)g(|u+t-1|)\cos yt dt + \int_0^{+\infty} \operatorname{sign}(u-t-1)g(|u-t-1|)\cos yt dt\Big]du$$
$$=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty} f(u)[g(u+t+1) + \operatorname{sign}(u-t+1)g(|u-t+1|) + \operatorname{sign}(u+t-1)g(|u+t-1|)$$
$$+\operatorname{sign}(u-t-1)g(|u-t-1|)]du \cos yt dt$$
$$=\sqrt{\frac{2}{\pi}}\int_0^{+\infty} (f \overset{\gamma}{*} g)(t)\cos yt dt$$
$$=F_c(f \overset{\gamma}{*} g)(y).$$

It shows that

$$F_c(f \overset{\gamma}{*} g)(y) = \cos y(F_s f)(y)(F_s g)(y).$$

The proof of Theorem is completes. $\square$

Theorem 2. *In the space $L(\mathbb{R}_+)$ the generalized convolution (2.1) is not associative and the relation with knowns convolutions and generalized convolutions as follows*

a) $f \underset{1}{*} (g \overset{\gamma}{*} h) = g \underset{1}{*} (f \overset{\gamma}{*} h) = h \underset{1}{*} (g \overset{\gamma}{*} f)$, *where* $f \underset{1}{*} g$ *is defined by* (1.7).

b) $f \overset{\gamma}{*} (g \underset{1}{*} h) = (f \overset{\gamma}{*} g) \underset{F_c}{*} h = (f \underset{1}{*} h) \overset{\gamma}{*} g$, *where* $f \underset{F_c}{*} g$ *is the Fourier cosine convolution* (1.1).

c) $f \overset{\gamma_1}{\underset{2}{*}} (g \overset{\gamma}{*} h) = (f \overset{\gamma_1}{\underset{F_s}{*}} h) \overset{\gamma}{*} g = h \overset{\gamma_1}{\underset{2}{*}} (f \overset{\gamma}{*} g)$, *where* $f \overset{\gamma_1}{\underset{F_s}{*}} g$ *and* $f \overset{\gamma_1}{\underset{2}{*}} g$ *are defined by* (1.3) *and* (1.11).

d) $f \overset{\gamma}{*} (g \overset{\gamma_1}{\underset{1}{*}} h) = (f \overset{\gamma_1}{\underset{2}{*}} g) \overset{\gamma}{\underset{F_c}{*}} h = (f \overset{\gamma_1}{\underset{2}{*}} h) \overset{\gamma}{\underset{F_c}{*}} g$, *where* $f \overset{\gamma}{\underset{F_c}{*}} g$ *is the Fourier cosine convolution with a weight function* (1.5).

Proof. a) From the factorization equality, we have

$$\begin{aligned}F_s(f \underset{1}{*} (g \overset{\gamma}{*} h))(y) =&(F_s f)(y)(F_c(g \overset{\gamma}{*} h))(y)\\ =&(F_s f)(y)\cos y(F_s g)(y)(F_s h)(y) = (F_s g)(y)\cos y(F_s f)(y)(F_s h)(y)\\ =&(F_s g)(y)(F_c(f \overset{\gamma}{*} h))(y)\\ =&F_s(g \underset{1}{*} (f \overset{\gamma}{*} h))(y).\end{aligned}$$

Which implies that $f \underset{1}{*} (g \overset{\gamma}{*} h) = g \underset{1}{*} (f \overset{\gamma}{*} h)$

On the other hand,

$$\begin{aligned}F_s(f \underset{1}{*} (g \overset{\gamma}{*} h))(y) =&(F_s f)(y)\cos y(F_s g)(y)(F_s h)(y)\\ =&(F_s h)(y)\cos y(F_s g)(y)(F_s f)(y)\\ =&F_s(g \underset{1}{*} (f \overset{\gamma}{*} h))(y).\end{aligned}$$

Then we obtain the part a). The parts b), c), d) can be obtain in similar way. The Theorem is proved. ☐

Theorem 3. *In the space $L(R_+)$ the generalized convolution* (2.1) *does not have a unit element.*

Proof. Suppose that there exists a unit element e of the generalized convolution (2.1) in $L(\mathbb{R}_+)$. It means that $f \overset{\gamma}{*} e = e \overset{\gamma}{*} f = f$ for any function $f \in L(\mathbb{R}_+)$. It follows that $F_c(f \overset{\gamma}{*} e)(y) = (F_c f)(y)$, $\forall y > 0$.
Hence, $\cos y(F_s e)(y)(F_s f)(y) = (F_c f)(y), \forall y > 0$.
Choosing $f(x) = e^{-x} \in L(\mathbb{R}_+)$. From the fact that

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}}\frac{1}{1+y^2}, \qquad (F_s f)(y) = \sqrt{\frac{2}{\pi}}\frac{y}{1+y^2},$$

we obtain

$$(F_s e)(y) = \frac{1}{y\cos y}.$$

It is contradition from the fact that $\dfrac{1}{y\cos y} \notin L_\infty(\mathbb{R}_+)$ while $(F_s e)(y) \in L_\infty(\mathbb{R}_+)$ since $e \in L(\mathbb{R}_+)$. The Theorem is proved. ☐

Let $L(\mathbb{R}_+, e^x) = \{h, \text{ for all } e^x h(x) \in L(\mathbb{R}_+)\}$.

Theorem 4 (Titchmarch-type Theorem). *Let $f, g \in L(\mathbb{R}_+, e^x)$. If $(f \overset{\gamma}{*} g)(x) \equiv 0$ then either $f \equiv 0$ or $g \equiv 0$.*

Proof. Suppose that $(f \underset{c;s}{*} g)(x) = 0, \forall x > 0$, in view of Theorem 1

$$F_c(f \overset{\gamma}{*} g)(y) = \cos y(F_s f)(y)(F_s g)(y) = 0, \forall y > 0. \tag{2.6}$$

We have

$$\begin{aligned}\left|\frac{d^n}{dy^n}[\sin(xy)f(x)]\right| &= \left|f(x)x^n\sin(xy+n\frac{\pi}{2})\right|\\ &\leqslant \left|f(x)x^n\right|\\ &= \left|e^{-x}.x^n\right|.\left|f_1(x)\right| \leqslant n!\left|f_1(x)\right|,\end{aligned}$$

where $f_1(x) = e^x f(x) \in L(\mathbb{R}_+)$.

Due to Weierstrass criterion, the integral $\int\limits_0^{+\infty} \frac{d^n}{dy^n}[\sin(xy)f(x)]dx$ uniformly converges on $\mathbb{R}_+$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_s f)(y)$ is analytic. Similarly, $(F_s g)(y)$ is analytic. So from (2.6) we have $(F_s f)(y) \equiv 0$ or $(F_s g)(y) \equiv 0$. It completes the proof. $\square$

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations

$$(3.1)\quad \begin{aligned} &f(x) + \lambda_1\int\limits_0^{+\infty} g(t)\varphi_1(x,t)dt + \lambda_2\int\limits_0^{+\infty} g(t)\eta_1(x,t)dt + \lambda_3\int\limits_0^{+\infty} g(t)\mu_1(x,t)dt = h(x)\\ &\lambda_4\frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty}\xi(t)[f(x+t) - f(|x-t|)]dt + \lambda_5\int\limits_0^{+\infty} g(t)\psi_1(x,t)dt + g(x) = k(x),\end{aligned}$$

here $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are complex numbers, φ, η, μ, $\xi \in L(\mathbb{R}_+)$, and

$$\begin{aligned}\varphi_1(x,t) =& \frac{1}{2\sqrt{2\pi}}[\varphi(t+x+1) + \operatorname{sign}(t-x-1)\varphi(|t-x-1|) + \operatorname{sign}(t+x-1)\varphi(|t+x-1|)\\ &+ \operatorname{sign}(t-x+1)\varphi(|t-x+1|)],\\ \eta_1(x,t) =& \frac{1}{2\sqrt{2\pi}}[\eta(|x+t-1|) + \eta(|x-t+1|) - \eta(x+t+1) - \eta(|x-t-1|)],\\ \mu_1(x,t) =& \frac{1}{\sqrt{2\pi}}[\operatorname{sign}(t-x)\mu(|t-x|) + \mu(t+x)],\\ \psi_1(x,t) =& \frac{1}{2\sqrt{2\pi}}[\psi(|x+t-1|) + \psi(|x-t-1|) - \psi(x+t+1) - \psi(|x-t+1|)],\end{aligned}$$

and $\psi(x) = (\nu_1 \underset{2}{*} \nu_2)(x)$.

It shows that φ_1, η_1, μ_1, ψ and ψ_1 are also $L(\mathbb{R}_+)$ - functions

Theorem 5. *With the condition*

$$F_c\big(\lambda_1\lambda_4\varphi \overset{\gamma}{*} \xi + \lambda_2\lambda_4\xi \underset{2}{\overset{\gamma_1}{*}} \eta + \lambda_3\lambda_4\mu \underset{2}{*} \xi + \lambda_1\lambda_5\varphi \overset{\gamma}{*} (\nu_1 \underset{1}{\overset{\gamma_1}{*}} \nu_2) + \lambda_2\lambda_5\nu_1 \underset{2}{\overset{\gamma_1}{*}} (\nu_2 \underset{2}{\overset{\gamma_1}{*}} \eta) + \lambda_3\lambda_5\mu \underset{2}{\overset{\gamma_1}{*}} \psi\big)(y) \neq 1,$$

there exists a unique solution in $L(\mathbb{R}_+)$ of system (3.1) which is of the form

$$f = h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \underset{2}{\overset{\gamma_1}{*}} \eta - \lambda_3 \mu \underset{2}{*} k + h \underset{F_c}{*} l - \lambda_1(\varphi \overset{\gamma}{*} k) \underset{F_c}{*} l - \lambda_2(k \underset{2}{\overset{\gamma_1}{*}} \eta) \underset{F_c}{*} l - \lambda_3(\mu \underset{2}{*} k) \underset{F_c}{*} l$$

$$g = k - \lambda_4 \xi \underset{1}{*} h - \lambda_5 \psi \underset{1}{\overset{\gamma_1}{*}} h + k \underset{1}{*} l - \lambda_4(\xi \underset{1}{*} h) \underset{1}{*} l - \lambda_5(\psi \underset{1}{\overset{\gamma_1}{*}} h) \underset{1}{*} l$$

Here $l \in L(\mathbb{R}_+)$ and is defined by

$$F_c l = \frac{F_c\big(\lambda_1\lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2\lambda_4 \xi \underset{2}{\overset{\gamma_1}{*}} \eta + \lambda_3\lambda_4 \mu \underset{2}{*} \xi + \lambda_1\lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \underset{1}{\overset{\gamma_1}{*}} \nu_2) + \lambda_2\lambda_5 \nu_1 \underset{2}{\overset{\gamma_1}{*}} (\nu_2 \underset{2}{\overset{\gamma_1}{*}} \eta) + \lambda_3\lambda_5 \mu \underset{2}{\overset{\gamma_1}{*}} \psi\big)}{1 - F_c\big(\lambda_1\lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2\lambda_4 \xi \underset{2}{\overset{\gamma_1}{*}} \eta + \lambda_3\lambda_4 \mu \underset{2}{*} \xi + \lambda_1\lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \underset{1}{\overset{\gamma_1}{*}} \nu_2) + \lambda_2\lambda_5 \nu_1 \underset{2}{\overset{\gamma_1}{*}} (\nu_2 \underset{2}{\overset{\gamma_1}{*}} \eta) + \lambda_3\lambda_5 \mu \underset{2}{\overset{\gamma_1}{*}} \psi\big)}$$

Proof. System (3.1) can be rewriten in the form

$$\begin{aligned} f(x) + \lambda_1(g \overset{\gamma}{*} \varphi)(x) + \lambda_2(g \underset{2}{\overset{\gamma_1}{*}} \eta)(x) + \lambda_3(g \underset{2}{*} \mu)(x) &= h(x) \\ \lambda_4(\xi \underset{1}{*} f)(x) + \lambda_5(f \underset{1}{\overset{\gamma_1}{*}} \psi)(x) + g(x) &= k(x). \end{aligned} \tag{3.2}$$

Using the respectively factorization equalities of above generalized convolutions we have

$$\begin{aligned} (F_c f)(y) + \lambda_1 \cos y (F_s g)(y)(F_s\varphi)(y) + \lambda_2 \sin y (F_s g)(y)(F_c\eta)(y) + \lambda_3 (F_s g)(y)(F_s\mu)(y) &= (F_c h)(y) \\ \lambda_4 (F_s\xi)(y)(F_c f)(y) + \lambda_5 \sin y (F_c f)(y)(F_c\psi)(y) + (F_s g)(y) &= (F_s k)(y). \end{aligned} \tag{3.3}$$

We have

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & \lambda_1 \cos y (F_s\varphi)(y) + \lambda_2 \sin y (F_c\eta)(y) + \lambda_3 (F_s\mu)(y) \\ \lambda_4 (F_s\xi)(y) + \lambda_5 \sin y (F_c\psi)(y) & 1 \end{vmatrix} \\ &= 1 - \lambda_1\lambda_4 \cos y (F_s\varphi)(y)(F_s\xi)(y) - \lambda_2\lambda_4 \sin y (F_c\eta)(y)(F_s\xi)(y) - \lambda_3\lambda_4 (F_s\mu)(y)\lambda_4(F_s\xi)(y) - \\ &\quad \lambda_1\lambda_5 \sin y \cos y (F_s\varphi)(y)(F_c\psi)(y) - \lambda_2\lambda_5 \sin^2 y (F_c\eta)(y)(F_c\psi)(y) - \lambda_3\lambda_5 \sin y (F_s\mu)(y)(F_c\psi)(y) \\ &= 1 - F_c\big(\lambda_1\lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2\lambda_4 \xi \underset{2}{\overset{\gamma_1}{*}} \eta + \lambda_3\lambda_4 \mu \underset{2}{*} \xi + \lambda_1\lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \underset{1}{\overset{\gamma_1}{*}} \nu_2) + \lambda_2\lambda_5 \nu_1 \underset{2}{\overset{\gamma_1}{*}} (\nu_2 \underset{2}{\overset{\gamma_1}{*}} \eta) \\ &\qquad + \lambda_3\lambda_5 \mu \underset{2}{\overset{\gamma_1}{*}} \psi\big)(y) \neq 0. \end{aligned}$$

Then due to Wiener-Levi's Theorem [10] there exists a function $l \in L(\mathbb{R}_+)$ such that

$$\begin{aligned} &\frac{1}{\Delta} - 1 = \\ &\frac{F_c\big(\lambda_1\lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2\lambda_4 \xi \underset{2}{\overset{\gamma_1}{*}} \eta + \lambda_3\lambda_4 \mu \underset{2}{*} \xi + \lambda_1\lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \underset{1}{\overset{\gamma_1}{*}} \nu_2) + \lambda_2\lambda_5 \nu_1 \underset{2}{\overset{\gamma_1}{*}} (\nu_2 \underset{2}{\overset{\gamma_1}{*}} \eta) + \lambda_3\lambda_5 \mu \underset{2}{\overset{\gamma_1}{*}} \psi\big)(y)}{1 - F_c\big(\lambda_1\lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2\lambda_4 \xi \underset{2}{\overset{\gamma_1}{*}} \eta + \lambda_3\lambda_4 \mu \underset{2}{*} \xi + \lambda_1\lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \underset{1}{\overset{\gamma_1}{*}} \nu_2) + \lambda_2\lambda_5 \nu_1 \underset{2}{\overset{\gamma_1}{*}} (\nu_2 \underset{2}{\overset{\gamma_1}{*}} \eta) + \lambda_3\lambda_5 \mu \underset{2}{\overset{\gamma_1}{*}} \psi\big)(y)} \\ &= (F_c l)(y), \end{aligned}$$

furthermore

$$\begin{aligned}\Delta_1 =& \begin{vmatrix} (F_c h)(y) & \lambda_1 \cos y (F_s\varphi)(y) + \lambda_2 \sin y (F_c\eta)(y) + \lambda_3 (F_s\mu)(y) \\ (F_s k)(y) & 1 \end{vmatrix} \\ =&(F_c h)(y) - \lambda_1 \cos y (F_s\varphi)(y)(F_s k)(y) - \lambda_2 \sin y (F_c\eta)(y)(F_s k)(y) - \lambda_3 (F_s\mu)(y)(F_s k)(y) \\ =&F_c\big(h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{\underset{2}{*}} \eta - \lambda_3 \mu \underset{2}{*} k\big)(y).\end{aligned}$$

Therefore

$$\begin{aligned}(F_s f)(y) = \frac{\Delta_1}{\Delta} =& F_c\big(h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{\underset{2}{*}} \eta - \lambda_3 \mu \underset{2}{*} k\big)(y)[1 + (F_c l)(y)] \\ =F_c\big(h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{\underset{2}{*}} \eta - \lambda_3 \mu \underset{2}{*} k + h \underset{F_c}{*} l - \lambda_1 (\varphi \overset{\gamma}{*} k) \underset{F_c}{*} l - \lambda_2 (k \overset{\gamma_1}{\underset{2}{*}} \eta) \underset{F_c}{*} l - \lambda_3 (\mu \underset{2}{*} k) \underset{F_c}{*} l\big)(y)\end{aligned}$$

It shows that

$$f = h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{\underset{2}{*}} \eta - \lambda_3 \mu \underset{2}{*} k + h \underset{F_c}{*} l - \lambda_1 (\varphi \overset{\gamma}{*} k) \underset{F_c}{*} l - \lambda_2 (k \overset{\gamma_1}{\underset{2}{*}} \eta) \underset{F_c}{*} l - \lambda_3 (\mu \underset{2}{*} k) \underset{F_c}{*} l$$

Similarlly

$$\begin{aligned}\Delta =& \begin{vmatrix} 1 & (F_c h)(y) \\ \lambda_4 (F_s\xi)(y) + \lambda_5 \sin y (F_c\psi)(y) & (F_s k)(y) \end{vmatrix} \\ =&(F_s k)(y) - \lambda_4 (F_s\xi)(y)(F_c h)(y) - \lambda_5 \sin y (F_c\psi)(y)(F_c h)(y) \\ =&F_s\big(k - \lambda_4 \xi \underset{1}{*} h - \lambda_5 \psi \overset{\gamma_1}{\underset{1}{*}} h\big)(y).\end{aligned}$$

Then

$$\begin{aligned}(F_s g)(y) =&\frac{\Delta_2}{\Delta} = F_s\big(k - \lambda_4 \xi \underset{1}{*} h - \lambda_5 \psi \overset{\gamma_1}{\underset{1}{*}} h\big)(y)[1 + (F_c l)(y)] \\ =&F_s\big(k - \lambda_4 \xi \underset{1}{*} h - \lambda_5 \psi \overset{\gamma_1}{\underset{1}{*}} h\big)(y) + F_s\big(k \underset{1}{*} l - \lambda_4 (\xi \underset{1}{*} h) \underset{1}{*} l - \lambda_5 (\psi \overset{\gamma_1}{\underset{1}{*}} h) \underset{1}{*} l\big)(y).\end{aligned}$$

Hence

$$g = k - \lambda_4 \xi \underset{1}{*} h - \lambda_5 \psi \overset{\gamma_1}{\underset{1}{*}} h + k \underset{1}{*} l - \lambda_4 (\xi \underset{1}{*} h) \underset{1}{*} l - \lambda_5 (\psi \overset{\gamma_1}{\underset{1}{*}} h) \underset{1}{*} l.$$

The proof is complete. □

References

[1] Bateman H. and Erdelyi A. (1954), , Newyork - Toronto - London - McGraw-Hill Book company, INC.

[2] Bochner S. and Chandrasekharan K. (1949), Princeton Univ. Press, Princeton.

[3] Churchill R. V. (1941) New York, 58p.

[4] Kakichev V. A.(1967), "On the Convolution for Integral Transforms." No. 2, 53 − 62 (In Russian).

[5] Kakichev V. A and Nguyen Xuan Thao (1998), "On the design method for the Generalized Integral Convolutions." No. 1, 31−40 (In Russian).

[6] Kakichev V.A, Nguyen Xuan Thao and Vu Kim Tuan (1998), "On the Generalized Convolutions for Fourier Cosine and Sine Transforms." Vol. 1, No. 1 pp. 85−90.

[7] I.N. Sneddon (1972), McGray-Hill, New York.

[8] Stein E.M. and Weiss G (1971), Princeton Univ. Press, Princeton.
[9] Nguyen Xuan Thao (2001), "On the Generalized Convolution for the Stieltjes, Hilbert, Fourier Cosine and Sine Transforms." Vol. 53, p. 560−567 (in Russian).
[10] Nguyen Xuan Thao, Nguyen Thanh Hai (1997), "Convolutions for Integral Transform and their Application." Moscow, 44pages (Russian).
[11] Nguyen Xuan Thao and Nguyen Minh Khoa (2004), "On the Convolution with a weight-function for the Cosine-Fourier integral transform." 29, p.149−162.
[12] Nguyen Xuan Thao and Nguyen Minh Khoa (2004), "Generalized Convolution for Integral Transforms." Methods of complex and clifford Analysis. Proc. Inter. Conf. Appl. Math. SAS Inter. Public. p. 161−180.
[13] Nguyen Xuan Thao and Nguyen Minh Khoa (2005), "On the Generalized Convolution with a Weight-Function for Fourier, Fourier Cosine and Sine Transforms." Vol. 33, N. 4, 421−436.
[14] Nguyen Xuan Thao, Nguyen Minh Khoa (2006), "On the Generalized Convolution with a Weight-Function for the Fourier Sine and Cosine Transforms", Vol 17, No 9, 673−685.
[15] Nguyen Xuan Thao, Vu Kim Tuan, Nguyen Minh Khoa (2004), "On the Generalized Convolution with a Weight-Function for the Fourier Cosine and Sine Transforms." Vol 7, no. 3, 323 −337.
[16] Titchmarsh H. M.(1967), 2nd Ed. Clarendon Press, Oxford.

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