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CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

NGUYEN XUAN THAO AND NGUYEN THANH HONG

ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. It’s properties and applications to solve systems of integral equations are considered.

1. Introduction

Let $F_s$ be the Fourier sine transform [2]

$$(F_s f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y) dy,$$

and $F_c$ be the Fourier cosine transform [2]

$$(F_c f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y) dy.$$

Convolution theory has been studied in 20th. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated.

The convolution of two functions $f$ and $g$ for the Fourier cosine transform is introduced in [7]

$$(f *_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) [g(|x - y|) + g(x + y)] dy, \quad x > 0,$$

which satisfied the following factorization equality

$$(1.2) \quad F_c(f *_{F_c} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$

Convin 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier
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sine transform which is of the form [4] and [10]

\[(f \ast_{F_s} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\text{sign}(x + y - 1)g(|x + y - 1|) + \text{sign}(x - y + 1)g(|x - y + 1|) - g(x + y + 1) - \text{sign}(x - y - 1)g(|x - y - 1|)]dy, \ x > 0,\]

and proved the following factorization identity [4], [10]

\[(1.4) \quad F_s(f \ast_{F_s} g)(y) = \sin y(Fsf)(y)(Fsg)(y), \ \forall y > 0.\]

The convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine transform of two functions $f$ and $g$ is introduced in [11]

\[(1.5) \quad (f \ast_{F_c} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(|y+x-1|)+g(|y-x-1|)+g(y+x+1)+g(|y-x+1|)]dy, \ x > 0\]

and satisfy the factorization equality [11]

\[(1.6) \quad F_c(f \ast_{F_c} g)(y) = \cos y(Fcf)(y)(Fcg)(y), \ \forall y > 0.\]

In 1941, Churchill R.V introduced the first generalized convolution of two functions $f$ and $g$ for the Fourier sine and Fourier cosine transforms [7]

\[(1.7) \quad (f \ast_{1} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \ x > 0,\]

and proved the following factorization identity [7]

\[(1.8) \quad F_s(f \ast_{1} g)(y) = (Fsf)(y)(Fsg)(y), \ \forall y > 0.\]

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, $G$-transform and $H$-transform. In 1998, Kakiichev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions $f$ and $g$ for the Fourier cosine and sine transforms is defined by [6]

\[(1.9) \quad (f \ast_{2} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[\text{sign}(u - x)g(|u-x|) + g(u+x)]du, \ x > 0.\]

For this generalized convolution the following factorization equality holds [6]

\[(1.10) \quad F_c(f \ast_{2} g)(y) = (Fsf)(y)(Fsg)(y), \ \forall y > 0.\]
Another generalized convolution with the weight function $\gamma_1(y) = \sin y$ for the Fourier cosine and sine has been studied in [15]

\begin{equation}
(f \ast \frac{\gamma_1}{2} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)]du, \ x > 0.
\end{equation}

It satisfies the factorization property [15]

\begin{equation}
F_c(f \ast \frac{\gamma_1}{2} g)(y) = \sin y(F_s f)(y)(F_c g)(y), \ \forall y > 0.
\end{equation}

The generalized convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier sine and cosine transforms has the form

\begin{equation}
(f \ast \frac{\gamma_1}{1} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)]du, \ x > 0,
\end{equation}

and satisfy the following factorization identity [14]

\begin{equation}
F_s(f \ast \frac{\gamma_1}{1} g)(y) = \sin y(F_c f)(y)(F_s g)(y), \ \forall y > 0.
\end{equation}

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Its properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

2. THE GENERALIZED CONVOLUTION

Definition 1. A generalized convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine and sine transforms of functions $f$ and $g$ is defined by

\begin{equation}
(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(y+x+1) + \text{sign}(y-x-1)g(|y-x-1|)
+ \text{sign}(y+x-1)g(|y+x-1|) + \text{sign}(y-x+1)g(|y-x+1|)]dy, \ x > 0.
\end{equation}

We denote by $L(\mathbb{R}_+)$ the space of all functions $f$ defined on $\mathbb{R}_+$ such that $\int_0^+ |f(x)|dx < \infty$.

Theorem 1. Let $f$ and $g$ be functions in $L(\mathbb{R}_+)$ then the generalized convolution $(f \ast g)(x)$ defined by (2.1) also be a $L(\mathbb{R}_+)$ function. Moreover, the following factorization equality holds

\begin{equation}
F_c(f \ast g)(y) = \cos y(F_s f)(y)(F_c g)(y), \ \forall y > 0.
\end{equation}
Proof. From the defining formula of the generalized convolution and the fact that \( f, g \in L(\mathbb{R}_+) \)
we have

\[
\int_0^{+\infty} |(f \otimes g)(x)| \, dx = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} |f(y)| \cdot |g(y + x + 1) + \text{sign}(y - x - 1)g(|y - x - 1|) + \text{sign}(y + x - 1)g(|y + x - 1|) + \text{sign}(y - x + 1)g(|y - x + 1|)| \, dy \, dx
\]

\[
\leq \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(y)| \cdot \left[ \int_0^{+\infty} |g(y + x + 1)| \, dx + \int_0^{+\infty} |g(|y - x - 1|)| \, dx + \int_0^{+\infty} |g(|y - x + 1|)| \, dx \right] \, dy.
\]

On the other hand,

\[
\int_0^{+\infty} |g(y + x + 1)| \, dx + \int_0^{+\infty} |g(|y - x - 1|)| \, dx = \int_{y+1}^{+\infty} |g(t)| \, dt + \int_{-y-1}^{+\infty} |g(|t|)| \, dt
\]

\[= 2 \int_0^{+\infty} |g(t)| \, dt.
\]

Similarly,

\[
\int_0^{+\infty} |g(|y + x - 1|)| \, dx + \int_0^{+\infty} |g(|y - x + 1|)| \, dx = 2 \int_0^{+\infty} |g(t)| \, dt.
\]

From (2.3), (2.4) and (2.5) one holds

\[
\int_0^{+\infty} |(f \otimes g)(x)| \, dx \leq \frac{2}{\pi} \int_0^{+\infty} |g(t)| \, dt \int_0^{+\infty} \frac{1}{\pi} \int_0^{+\infty} \cos y \sin uy \sin vy f(u) g(v) \, dudv
\]

\[\leq +\infty.
\]

So \((f \otimes g)(x)\) belong to \(L(\mathbb{R}_+)\).

Now we prove the factorization equality (2.2). We have

\[
\cos y (F_s f)(y) (F_s g)(y) = \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} \cos y \sin uy \sin vy f(u) g(v) \, dudv
\]

\[= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u) g(v) [\cos y (u - v + 1) + \cos y (u - v - 1) - \cos y (u + v + 1) - \cos y (u + v - 1)] \, dudv.
\]

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Changing the variables gives

\[
\cos y(F_s f)(y)(F_s g)(y) = \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_{-u-1}^{+\infty} g(t + u + 1) \cos y\,dt + \int_{-u+1}^{+\infty} g(t + u - 1) \cos y\,dt 
- \int_{u+1}^{+\infty} g(t - u - 1) \cos y\,dt - \int_{u-1}^{+\infty} g(t - u + 1) \cos y\,dt \right] \,du
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_0^{+\infty} g(u + t + 1) \cos y\,dt + \int_0^{+\infty} \text{sign}(u - t + 1) g(|u - t + 1|) \cos y\,dt 
+ \int_0^{+\infty} \text{sign}(u + t - 1) g(|u + t - 1|) \cos y\,dt + \int_0^{+\infty} \text{sign}(u - t - 1) g(|u - t - 1|) \cos y\,dt \right] \,du
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u)[g(u + t + 1) + \text{sign}(u - t + 1) g(|u - t + 1|) + \text{sign}(u + t - 1) g(|u + t - 1|)
+ \text{sign}(u - t - 1) g(|u - t - 1|)] \,du \cos y\,dt
\]

\[
= \sqrt{2\pi} \int_0^{+\infty} (f \ast g)(t) \cos y\,dt
\]

\[
= F_c(f \ast g)(y).
\]

It shows that

\[
F_c(f \ast g)(y) = \cos y(F_s f)(y)(F_s g)(y).
\]

The proof of Theorem is complete. ∎

**Theorem 2.** In the space \(L(\mathbb{R}_+^+)\) the generalized convolution (2.1) is not associative and the relation with knowns convolutions and generalized convolutions as follows

a) \( f \ast (g \ast h) = g \ast (f \ast h) = h \ast (g \ast f) \), where \( f \ast g \) is defined by (1.7).

b) \( f \ast (g \ast h) = (f \ast g) \ast h = h \ast (f \ast g) \), where \( f \ast g \) is the Fourier cosine convolution (1.1).

c) \( f \ast (g \ast h) = (f \ast g) \ast h = h \ast (f \ast g) \), where \( f \ast g \) and \( f \ast g \) are defined by (1.3) and (1.11).

d) \( f \ast (g \ast h) = (f \ast g) \ast h = h \ast (f \ast g) \), where \( f \ast g \) is the Fourier cosine convolution with a weight function (1.5).
Proof. a) From the factorization equality, we have
\[ F_s(f \ast (g \overset{\gamma}{\ast} h))(y) = (F_s f)(y) (F_c (g \overset{\gamma}{\ast} h))(y) \]
\[ = (F_s f)(y) \cos y (F_s g)(y) (F_s h)(y) = (F_s g)(y) \cos y (F_s f)(y) (F_s h)(y) \]
\[ = (F_s g)(y) (F_c (f \overset{\gamma}{\ast} h))(y) \]
\[ = F_s (g \overset{\gamma}{\ast} h))(y). \]

Which implies that \( f \ast (g \overset{\gamma}{\ast} h) = g \overset{\gamma}{\ast} h \)

On the other hand,
\[ F_s(f \overset{\gamma}{\ast} h))(y) = (F_s f)(y) \cos y (F_s g)(y) (F_s h)(y) \]
\[ = (F_s h)(y) \cos y (F_s g)(y) (F_s f)(y) \]
\[ = F_s (g \overset{\gamma}{\ast} h))(y). \]

Then we obtain the part a). The parts b), c), d) can be obtain in similar way. The Theorem is proved.

Theorem 3. In the space \( L(\mathbb{R}^+) \) the generalized convolution (2.1) does not have a unit element.

Proof. Suppose that there exists a unit element \( e \) of the generalized convolution (2.1) in \( L(\mathbb{R}^+) \).

It means that \( f \overset{\gamma}{\ast} e = e \overset{\gamma}{\ast} f = f \) for any function \( f \in L(\mathbb{R}^+) \). It follows that \( F_c (f \overset{\gamma}{\ast} e)(y) = (F_c f)(y), \forall y > 0. \)

Hence, \( \cos y (F_s e)(y) (F_s f)(y) = (F_s f)(y), \forall y > 0. \)

Choosing \( f(x) = e^{-x} \in L(\mathbb{R}^+) \). From the fact that
\[ (F_c f)(y) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + y^2}, \quad (F_s f)(y) = \sqrt{\frac{2}{\pi}} \frac{y}{1 + y^2}, \]
we obtain
\[ (F_s e)(y) = \frac{1}{y \cos y}. \]

It is contradiction from the fact that \( \frac{1}{y \cos y} \notin L_{\infty}(\mathbb{R}^+) \) while \( (F_s e)(y) \in L_{\infty}(\mathbb{R}^+) \) since \( e \in L(\mathbb{R}^+) \).

The Theorem is proved.

Let \( L(\mathbb{R}^+, e^x) = \{ h, \text{ for all } e^x h(x) \in L(\mathbb{R}^+) \} \).

Theorem 4 (Titchmarsh-type Theorem). Let \( f, g \in L(\mathbb{R}^+, e^x) \). If \( (f \overset{\gamma}{\ast} g)(x) \equiv 0 \) then either \( f \equiv 0 \) or \( g \equiv 0 \).

Proof. Suppose that \( (f \overset{c,s}{\ast} g)(x) = 0, \forall x > 0, \) in view of Theorem 1

\[ F_c (f \overset{\gamma}{\ast} g)(y) = \cos y (F_s f)(y) (F_s g)(y) = 0, \forall y > 0. \]
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We have
\[
\left| \frac{d^n}{dy^n} [\sin(xy)f(x)] \right| = \left| f(x) x^n \sin(xy + n\frac{\pi}{2}) \right| \\
\leq \left| f(x) x^n \right| \\
= e^{-\pi x^n} \cdot |f_1(x)| \leq n!|f_1(x)|,
\]
where $f_1(x) = e^{\pi} f(x) \in L(\mathbb{R}_+)$. Due to Weierstrass criterion, the integral \( \int_0^{+\infty} \frac{d^n}{dy^n} [\sin(xy)f(x)] \, dx \) uniformly converges on $\mathbb{R}_+$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_s f)(y)$ is analytic. Similarly, $(F_s g)(y)$ is analytic. So from (2.6) we have $(F_s f)(y) \equiv 0$ or $(F_s g)(y) \equiv 0$. It completes the proof. 

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations
\[
f(x) + \lambda_1 \int_0^{+\infty} g(t)\varphi_1(x,t) \, dt + \lambda_2 \int_0^{+\infty} g(t)\eta_1(x,t) \, dt + \lambda_3 \int_0^{+\infty} g(t)\mu_1(x,t) \, dt = h(x) \tag{3.1}
\]
\[
\lambda_4 \int_0^{+\infty} \frac{\xi(t)(f(x+t) - f(|x-t|))}{\sqrt{2\pi}} \, dt + \lambda_5 \int_0^{+\infty} g(t)\psi_1(x,t) \, dt + g(x) = k(x),
\]
here $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are complex numbers, $\varphi, \eta, \mu, \xi \in L(\mathbb{R}_+)$, and
\[
\varphi_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \varphi(t+x+1) + \operatorname{sign}(t-x-1)\varphi(|t-x-1|) + \operatorname{sign}(t+x-1)\varphi(|t+x-1|) \right. \\
\left. + \operatorname{sign}(t-x+1)\varphi(|t-x+1|) \right],
\]
\[
\eta_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \eta(|x+t-1|) + \eta(|x-t+1|) - \eta(x+t+1) - \eta(|x-t-1|) \right],
\]
\[
\mu_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \operatorname{sign}(t-x)\mu(|t-x|) + \mu(t+x) \right],
\]
\[
\psi_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \psi(|x+t-1|) + \psi(|x-t+1|) - \psi(x+t+1) - \psi(|x+t+1|) \right],
\]
and $\psi(x) = (\nu_1 \ast \nu_2)(x)$. It shows that $\varphi_1, \eta_1, \mu_1, \psi$ and $\psi_1$ are also $L(\mathbb{R}_+)$ - functions.

Theorem 5. With the condition
\[
F_e(\lambda_1 \lambda_4 \varphi \ast \xi + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \psi + \nu_2 \ast \eta) + \lambda_2 \lambda_5 \nu_1 \ast \psi + \lambda_3 \lambda_5 \mu \ast \psi)(y) \neq 1,
\]
there exists a unique solution in $L(\mathbb{R}^+)$ of system (3.1) which is of the form

$$
\begin{align*}
  f &= h - \lambda_1 \varphi \ast k - \lambda_2 k \ast \eta - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \ast \eta) \ast l - \lambda_3 (\mu \ast k) \ast l \\
  g &= k - \lambda_4 \xi \ast h - \lambda_5 \psi \ast h + k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \ast h) \ast l
\end{align*}
$$

Here $l \in L(\mathbb{R}^+)$ and is defined by

$$
F_c l = \frac{F_c(\lambda_1 \lambda_4 \varphi \ast \gamma \xi + \lambda_2 \lambda_4 \xi \ast \gamma \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \gamma (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \gamma \psi)}{1 - F_c(\lambda_1 \lambda_4 \varphi \ast \gamma \xi + \lambda_2 \lambda_4 \xi \ast \gamma \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \gamma (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \gamma \psi)}
$$

**Proof.** System (3.1) can be rewritten in the form

$$
\begin{align*}
  f(x) + \lambda_1 (g \ast \varphi)(x) + \lambda_2 (g \ast \eta)(x) + \lambda_3 (g \ast \mu)(x) &= h(x) \\
  \lambda_4 (\xi \ast f)(x) + \lambda_5 (f \ast \psi)(x) + g(x) &= k(x)
\end{align*}
$$

Using the respective factorization equalities of above generalized convolutions we have

$$
\begin{align*}
  (F_c f)(y) + \lambda_1 \cos y (F_e g)(y)(F_s \varphi)(y) + \lambda_2 \sin y (F_e g)(y)(F_s \eta)(y) + \lambda_3 (F_e g)(y)(F_s \mu)(y) &= (F_c h)(y) \\
  \lambda_4 (F_e \xi)(y)(F_e f)(y) + \lambda_5 \sin y (F_e f)(y)(F_e \psi)(y) + (F_s g)(y) &= (F_e k)(y).
\end{align*}
$$

We have

$$
\begin{align*}
  \Delta &= \begin{vmatrix}
  1 & \lambda_1 \cos y (F_s \varphi)(y) + \lambda_2 \sin y (F_s \eta)(y) + \lambda_3 (F_s \mu)(y) \\
  \lambda_4 (F_e \xi)(y) + \lambda_5 \sin y (F_e \psi)(y) & 1
\end{vmatrix}
\end{align*}
$$

$$
\begin{align*}
  &= 1 - \lambda_1 \lambda_4 \cos y (F_s \varphi)(y)(F_e \xi)(y) - \lambda_2 \lambda_4 \sin y (F_s \eta)(y)(F_e \xi)(y) - \lambda_3 \lambda_4 (F_s \mu)(y)(F_e \xi)(y) \\
  &\quad - \lambda_1 \lambda_5 \sin y \cos y (F_s \varphi)(y)(F_e \psi)(y) - \lambda_2 \lambda_5 \sin y \sin y (F_s \eta)(y)(F_e \psi)(y) - \lambda_3 \lambda_5 \sin y (F_s \mu)(y)(F_e \psi)(y)
\end{align*}
$$

$$
\begin{align*}
  &= 1 - F_c(\lambda_1 \lambda_4 \varphi \ast \gamma \xi + \lambda_2 \lambda_4 \xi \ast \gamma \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \gamma (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \gamma \psi)(y) \\
  &\quad + \lambda_3 \lambda_5 \mu \ast \gamma \psi)(y) \neq 0.
\end{align*}
$$

Then due to Wiener-Levi's Theorem [10] there exists a function $l \in L(\mathbb{R}^+)$ such that

$$
\begin{align*}
  \frac{1}{\Delta} - 1 &= \frac{F_c(\lambda_1 \lambda_4 \varphi \ast \gamma \xi + \lambda_2 \lambda_4 \xi \ast \gamma \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \gamma (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \gamma \psi)(y)}{1 - F_c(\lambda_1 \lambda_4 \varphi \ast \gamma \xi + \lambda_2 \lambda_4 \xi \ast \gamma \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \gamma (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \gamma \psi)(y)}
\end{align*}
$$

$$
= (F_c l)(y),
$$
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Furthermore

\[
\Delta_1 = \begin{vmatrix} (F_c k)(y) & \lambda_1 \cos y(F_c \varphi)(y) + \lambda_2 \sin y(F_c \eta)(y) + \lambda_3(F_c \mu)(y) \\ (F_s k)(y) & 1 \end{vmatrix}
\]

\[
= (F_c h)(y) - \lambda_1 \cos y(F_c \varphi)(y)(F_s k)(y) - \lambda_2 \sin y(F_c \eta)(y)(F_s k)(y) - \lambda_3(F_c \mu)(y)(F_s k)(y)
\]

\[
= F_c \left( h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu \ast k \right)(y).
\]

Therefore

\[
(F_s f)(y) = \frac{\Delta_1}{\Delta} = F_c \left( h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu \ast k \right)(y) [1 + (F_c l)(y)]
\]

\[
= F_c \left( h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \frac{\gamma}{2} \eta) \ast l - \lambda_3 (\mu \ast k) \ast l \right)(y)
\]

It shows that

\[
f = h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \frac{\gamma}{2} \eta) \ast l - \lambda_3 (\mu \ast k) \ast l
\]

Similarly

\[
\Delta = \begin{vmatrix} \lambda_4(F_s \xi)(y) + \lambda_5 \sin y(F_c \psi)(y) & (F_c h)(y) \\ (F_s k)(y) & (F_s k)(y) \end{vmatrix}
\]

\[
= (F_s k)(y) - \lambda_4(F_s \xi)(y)(F_c h)(y) - \lambda_5 \sin y(F_c \psi)(y)(F_c h)(y)
\]

\[
= F_s \left( k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\gamma}{2} \eta \right)(y).
\]

Then

\[
(F_s g)(y) = \frac{\Delta_2}{\Delta} = F_s \left( k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\gamma}{2} \eta \right)(y) [1 + (F_c l)(y)]
\]

\[
= F_s \left( k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \frac{\gamma}{2} \eta) \ast l \right)(y).
\]

Hence

\[
g = k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\gamma}{2} \eta \ast h + k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \frac{\gamma}{2} \eta) \ast l.
\]

The proof is complete.

\[\square\]

REFERENCES

THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE $F_c,F_s$ TRANSFORMS


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