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CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED
CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER
COSINE AND SINE TRANSFORMS

NGUYEN XUAN THAO AND NGUYEN THANH HONG

ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. It’s properties and applications to solve systems of integral equations are considered.

1. INTRODUCTION

Let \( F_s \) be the Fourier sine transform [2]

\[
(F_s f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y) dy,
\]

and \( F_c \) be the Fourier cosine transform [2]

\[
(F_c f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y) dy.
\]

Convolution theory has been studied in 20th. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated.

The convolution of two functions \( f \) and \( g \) for the Fourier cosine transform is introduced in [7]

\[
(f \ast_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(|x - y|) + g(x + y)]dy, \quad x > 0,
\]

which satisfied the following factorization equality

\[
F_c(f \ast_{F_c} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0.
\]

In 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions \( f \) and \( g \) with the weight function \( \gamma_1(y) = \sin y \) for the Fourier
sine transform which is of the form [4] and [10]

\[(f \ast_{F_s} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(y)[\text{sign}(x + y - 1)g(|x + y - 1|) + \text{sign}(x - y + 1)g(|x - y + 1|) \nonumber \\
- g(x + y + 1) - \text{sign}(x - y - 1)g(|x - y - 1|)]dy, \quad x > 0,\]

and proved the following factorization identity [4], [10]

\[F_s(f \ast_{F_s} g)(y) = \sin y(F_sf)(y)(F_sg)(y), \quad \forall y > 0.\]

The convolution with the weight function \(\gamma(y) = \cos y\) for the Fourier cosine transform of two functions \(f\) and \(g\) is introduced in [11]

\[(f \ast_{F_c} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(y)[g(|y+x-1|)+g(|y-x-1|)+g(y+x+1)+g(|y-x+1|)]dy, x > 0\]

and satisfy the factorization equality [11]

\[F_c(f \ast_{F_c} g)(y) = \cos y(F_c f)(y)(F_c g)(y), \quad \forall y > 0.\]

In 1941, Churchill R.V introduced the first generalized convolution of two functions \(f\) and \(g\) for the Fourier sine and Fourier cosine transforms [7]

\[(f \ast_{1} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \quad x > 0,\]

and proved the following factorization identity [7]

\[F_s(f \ast_{1} g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0.\]

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, \(G\)-transform and \(H\)-transform. In 1998, Kachihev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions \(f\) and \(g\) for the Fourier cosine and sine transforms is defined by [6]

\[(f \ast_{2} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} f(u)[\text{sign}(u-x)g(|u-x|) + g(u+x)]du, \quad x > 0.\]

For this generalized convolution the following factorization equality holds [6]

\[F_c(f \ast_{2} g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0.\]
Another generalized convolution with the weight function $\gamma_1(y) = \sin y$ for the Fourier cosine and sine has been studied in [15]

\[(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u+1|)]du, \ x > 0.\]

It satisfies the factorization property [15]

\[(1.12) \quad F_c(f \ast g)(y) = \sin y(F_s f)(y)(F_c g)(y), \ \forall y > 0.\]

The generalized convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier sine and cosine transforms has the form

\[(1.13) \quad (f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u+1|)]du, \ x > 0,\]

and satisfy the following factorization identity [14]

\[(1.14) \quad F_s(f \ast g)(y) = \sin y(F_c f)(y)(F_s g)(y), \ \forall y > 0.\]

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Its properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

2. The Generalized Convolution

**Definition 1.** A generalized convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine and sine transforms of functions $f$ and $g$ is defined by

\[(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(y)[g(y + x + 1) + \text{sign}(y - x - 1)g(|y - x - 1|)]
+ \text{sign}(y + x - 1)g(|y + x - 1|) + \text{sign}(y - x + 1)g(|y - x + 1|)]dy, \ x > 0.\]

We denote by $L(\mathbb{R}_+)$ the space of all functions $f$ defined on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} |f(x)|dx < \infty$.

**Theorem 1.** Let $f$ and $g$ be functions in $L(\mathbb{R}_+)$ then the generalized convolution $(f \ast g)(x)$ defined by (2.1) also be a $L(\mathbb{R}_+)$ function. Moreover, the following factorization equality holds

\[(2.2) \quad F_c(f \ast g)(y) = \cos y(F_s f)(y)(F_c g)(y), \ \forall y > 0.\]
Proof. From the defining formula of the generalized convolution and the fact that \( f, g \in L(\mathbb{R}_+) \) we have

\[
(2.3) \quad \int_0^{+\infty} |(f \ast g)(x)| \, dx = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} |f(y)| |g(y + x + 1) + \text{sign}(y - x - 1) g(|y - x - 1|) + \text{sign}(y + x - 1) g(|y + x - 1|) + \text{sign}(y - x + 1) g(|y - x + 1|)| \, dy \, dx
\]

\[
+ \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(y)| \left[ \int_0^{+\infty} |g(y + x + 1)| \, dx + \int_0^{+\infty} |g(|y - x - 1|)| \, dx \right] \, dy.
\]

On the other hand,

\[
(2.4) \quad \int_0^{+\infty} |g(y + x + 1)| \, dx + \int_0^{+\infty} |g(|y - x - 1|)| \, dx = \int_0^{+\infty} |g(t)| \, dt + \int_{-y-1}^{y+1} |g(|t|)| \, dt
\]

\[
= 2 \int_0^{+\infty} |g(t)| \, dt.
\]

Similarly,

\[
(2.5) \quad \int_0^{+\infty} |g(|y + x - 1|)| \, dx + \int_0^{+\infty} |g(|y - x + 1|)| \, dx = 2 \int_0^{+\infty} |g(t)| \, dt.
\]

From (2.3), (2.4) and (2.5) one holds

\[
\int_0^{+\infty} |(f \ast g)(x)| \, dx \leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |g(t)| \, dt \int_0^{+\infty} |g(t)| \, dt \leq +\infty.
\]

So \((f \ast g)(x)\) belong to \(L(\mathbb{R}_+)\).

Now we prove the factorization equality (2.2). We have

\[
\cos y (F_s f)(y)(F_s g)(y) = \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} \cos y \, \sin u \, v \, f(u) g(v) \, du \, dv
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u) g(v) (\cos y(u - v + 1) + \cos y(u - v - 1) - \cos y(u + v + 1) - \cos y(u + v - 1)) \, du \, dv.
\]
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Changing the variables gives

$$
\cos y(F_s f)(y)(F_s g)(y) = \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_{-u-1}^{+\infty} g(t+u+1) \cos ytdt + \int_{-u+1}^{+\infty} g(t+u-1) \cos ytdt \right. \\
\left. - \int_{u+1}^{+\infty} g(t-u-1) \cos ytdt - \int_{u-1}^{+\infty} g(t-u+1) \cos ytdt \right] du
$$

$$
= \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_0^{+\infty} g(u+t+1) \cos ytdt + \int_0^{+\infty} \text{sign}(u-t+1)g(|u-t+1|) \cos ytdt \\
+ \int_0^{+\infty} \text{sign}(u-t-1)g(|u-t-1|) \cos ytdt + \int_0^{+\infty} \text{sign}(u-t-1)g(|u-t-1|) \cos ytdt \right] du
$$

$$
= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (f \hat{\gamma} g)(t) \cos ytdt
$$

$$
= F_c(f \hat{\gamma} g)(y).
$$

It shows that

$$
F_c(f \hat{\gamma} g)(y) = \cos y(F_s f)(y)(F_s g)(y).
$$

The proof of Theorem is completes. \qed

**Theorem 2.** In the space $L(\mathbb{R}_+)$ the generalized convolution (2.1) is not associative and the relation with knowns convolutions and generalized convolutions as follows

a) $f \ast (g \hat{\gamma} h) = g \ast (f \hat{\gamma} h) = h \ast (g \hat{\gamma} f)$, where $f \ast g$ is defined by (1.7).

b) $f \hat{\gamma} (g \ast h) = (f \hat{\gamma} g) \ast h = (f \hat{\gamma} h) \ast g$, where $f \ast g$ is the Fourier cosine convolution (1.1).

c) $f \hat{\gamma} (g \hat{\gamma} h) = (f \hat{\gamma} g) \hat{\gamma} h = h \hat{\gamma} (f \hat{\gamma} g)$, where $f \hat{\gamma} g$ and $f \hat{\gamma} g$ are defined by (1.3) and (1.11).

d) $f \hat{\gamma} (g \hat{\gamma} h) = (f \hat{\gamma} g) \hat{\gamma} h = (f \hat{\gamma} h) \hat{\gamma} g$, where $f \hat{\gamma} g$ is the Fourier cosine convolution with a weight function (1.5).
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**Proof.** a) From the factorization equality, we have

\[
F_s(f \ast (g \ast h))(y) = (F_s f)(y) (F_c (g \ast h))(y)
\]
\[
= (F_s f)(y) \cos y (F_s g)(y) (F_s h)(y)
\]
\[
= (F_s g)(y) (F_c (f \ast h))(y)
\]
\[
= F_s (g \ast (f \ast h))(y).
\]

Which implies that $f \ast (g \ast h) = g \ast (f \ast h)$

On the other hand,

\[
F_s(f \ast (g \ast h))(y) = (F_s f)(y) \cos y (F_s g)(y) (F_s h)(y)
\]
\[
= (F_s h)(y) \cos y (F_s g)(y) (F_s f)(y)
\]
\[
= F_s (g \ast (f \ast h))(y).
\]

Then we obtain the part a). The parts b), c), d) can be obtain in similar way. The Theorem is proved.

**Theorem 3.** In the space $L(R_+)$ the generalized convolution (2.1) does not have a unit element.

**Proof.** Suppose that there exists a unit element $e$ of the generalized convolution (2.1) in $L(R_+)$. It means that $f \ast e = e \ast f = f$ for any function $f \in L(R_+)$. It follows that $F_c(f \ast e)(y) = (F_c f)(y), \forall y > 0$.

Hence, $\cos y (F_s e)(y) (F_s f)(y) = (F_s f)(y), \forall y > 0$.

Choosing $f(x) = e^{-x} \in L(R_+)$. From the fact that

\[
(F_c f)(y) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + y^2}, \quad (F_s f)(y) = \sqrt{\frac{2}{\pi}} \frac{y}{1 + y^2},
\]

we obtain

\[
(F_s e)(y) = \frac{1}{y \cos y}.
\]

It is contradiction from the fact that $\frac{1}{y \cos y} \notin L_\infty(R_+)$ while $(F_s e)(y) \in L_\infty(R_+)$ since $e \in L(R_+)$. The Theorem is proved.

Let $L(R_+, e^x) = \{ h, \text{ for all } e^x h(x) \in L(R_+) \}$.

**Theorem 4** (Titchmarsh-type Theorem). Let $f, g \in L(R_+, e^x)$. If $(f \ast g)(x) \equiv 0$ then either $f \equiv 0$ or $g \equiv 0$.

**Proof.** Suppose that $(f \ast g)(x) = 0, \forall x > 0$, in view of Theorem 1

\[
F_c(f \ast g)(y) = \cos y (F_s f)(y) (F_s g)(y) = 0, \forall y > 0.
\]
We have
\[
\left| \frac{d^n}{dy^n}[\sin(xy)f(x)] \right| = \left| f(x)x^n \sin(x + n\frac{\pi}{2}) \right| \\
\leq \left| f(x)x^n \right| \\
= e^{-x.x^n}.|f_1(x)| \leq n!|f_1(x)|,
\]
where \( f_1(x) = e^x f(x) \in L(\mathbb{R}_+) \).

Due to Weierstrass criterion, the integral \( \int_0^{+\infty} \frac{d^n}{dy^n}[\sin(xy)f(x)]dx \) uniformly converges on \( \mathbb{R}_+ \).

Therefore, based on the differentiability of integrals depending on parameter, we conclude that \( (F_{s,f})(y) \) is analytic. Similarly, \( (F_{s,g})(y) \) is analytic. So from (2.6) we have \( (F_{s,f})(y) \equiv 0 \) or \( (F_{s,g})(y) \equiv 0 \). It completes the proof. \( \square \)

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations
\[
f(x) + \lambda_1 \int_0^{+\infty} g(t)\varphi_1(x,t)dt + \lambda_2 \int_0^{+\infty} g(t)\eta_1(x,t)dt + \lambda_3 \int_0^{+\infty} g(t)\mu_1(x,t)dt = h(x)
\]
(3.1)
\[
\lambda_4 \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \xi(t)[f(x + t) - f(|x - t|)]dt + \lambda_5 \int_0^{+\infty} g(t)\psi_1(x,t)dt + g(x) = k(x),
\]
where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) are complex numbers, \( \varphi, \eta, \mu, \xi \in L(\mathbb{R}_+) \), and
\[
\varphi_1(x,t) = \frac{1}{2\sqrt{2\pi}}[\varphi(t + x + 1) + \text{sign}(t - x - 1)\varphi(|t - x - 1|) + \text{sign}(t + x - 1)\varphi(|t + x - 1|)
+ \text{sign}(t - x + 1)\varphi(|t - x + 1|)],
\]
\[
\eta_1(x,t) = \frac{1}{2\sqrt{2\pi}}[\eta(|x + t - 1|) + \eta(|x - t + 1|) - \eta(x + t + 1) - \eta(|x - t - 1|)],
\]
\[
\mu_1(x,t) = \frac{1}{2\sqrt{2\pi}}[\text{sign}(t - x)\mu(|t - x|) + \mu(t + x)],
\]
\[
\psi_1(x,t) = \frac{1}{2\sqrt{2\pi}}[\psi(|x + t - 1|) + \psi(|x - t + 1|) - \psi(x + t + 1) - \psi(|x - t + 1|)],
\]
and \( \psi(x) = (\nu_1 \ast \nu_2)(x) \).

It shows that \( \varphi_1, \eta_1, \mu_1, \psi \) and \( \psi_1 \) are also \( L(\mathbb{R}_+) \) - functions

**Theorem 5. With the condition**

\[
F_c(\lambda_1\lambda_4\varphi \ast \xi + \lambda_2\lambda_4\xi \ast \eta + \lambda_3\lambda_4\mu \ast \xi + \lambda_1\lambda_5\varphi \ast (\nu_1 \ast \nu_2) + \lambda_2\lambda_5\nu_1 \ast (\nu_2 \ast \eta) + \lambda_3\lambda_5\mu \ast \psi)(y) \neq 1,
\]
there exists a unique solution in \( L(\mathbb{R}^+) \) of system (3.1) which is of the form
\[
f = h - \lambda_1 \varphi \ast k - \lambda_2 \eta \ast k + h \ast l - \lambda_1 \varphi \ast k \ast f_{\mathcal{C}} - \lambda_2 \eta \ast k \ast f_{\mathcal{C}} - \lambda_3 (\mu \ast k) \ast f_{\mathcal{C}}
\]
\[
g = k - \lambda_4 \xi \ast h - \lambda_5 \psi \ast h + k \ast l - \lambda_4 (\xi \ast h) \ast f_{\mathcal{C}} - \lambda_5 (\psi \ast h) \ast f_{\mathcal{C}}
\]
Here \( l \in L(\mathbb{R}^+) \) and is defined by
\[
F_{\mathcal{C}} l = \\
1 - F_{\mathcal{C}} (\lambda_1 \lambda_4 \varphi \ast (\gamma \ast \xi) + \lambda_2 \lambda_4 \xi \ast \gamma \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma \ast \eta + \lambda_3 \lambda_5 \mu \ast \gamma \ast \psi)
\]

**Proof.** System (3.1) can be rewritten in the form
\[
f(x) + \lambda_1 (g \ast \varphi)(x) + \lambda_2 (g \ast \eta)(x) + \lambda_3 (g \ast \mu)(x) = h(x)
\]
(3.2)
\[
\lambda_4 (\xi \ast f)(x) + \lambda_5 (f \ast \psi)(x) + g(x) = k(x).
\]
Using the respectively factorization equalities of above generalized convolutions we have
(3.3)
\[
(F_c f)(y) + \lambda_1 \cos (F_s g)(y)(F_s \varphi)(y) + \lambda_2 \sin (F_s g)(y)(F_c \eta)(y) + \lambda_3 (F_s g)(y)(F_s \mu)(y) = (F_c h)(y)
\]
\[
\lambda_4 (F_s \xi)(y)(F_c f)(y) + \lambda_5 \sin (F_c f)(y)(F_c \psi)(y) + (F_s g)(y) = (F_s k)(y).
\]
We have
\[
\Delta = \\
1 + \lambda_1 \cos (F_s \varphi)(y) + \lambda_2 \sin (F_c \eta)(y) + \lambda_3 (F_s \mu)(y)
\]
\[
1 - \lambda_4 \cos (F_s \xi)(y) + \lambda_5 \sin (F_c \psi)(y)
\]
\[
= 1 - \lambda_1 \lambda_4 \cos y(F_s \varphi)(y)(F_s \xi)(y) - \lambda_2 \lambda_4 \sin y(F_s \varphi)(y)(F_c \xi)(y) - \lambda_3 (F_s \mu)(y) \lambda_4 (F_s \xi)(y) - \lambda_1 \lambda_5 \sin y(F_s \varphi)(y)(F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 y(F_c \eta)(y)(F_c \psi)(y) - \lambda_3 \lambda_5 \sin y(F_s \mu)(y)(F_c \psi)(y)
\]
\[
= 1 - F_c (\lambda_1 \lambda_4 \varphi \ast (\gamma \ast \xi) + \lambda_2 \lambda_4 \xi \ast \gamma \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma \ast \eta + \lambda_3 \lambda_5 \mu \ast \gamma \ast \psi)(y)
\]
\[
+ \lambda_3 \lambda_5 \mu \ast \gamma \ast \psi)(y) \neq 0.
\]
Then due to Wiener-Levi’s Theorem [10] there exists a function \( l \in L(\mathbb{R}^+) \) such that
\[
\frac{1}{\Delta} = 1 - F_c (\lambda_1 \lambda_4 \varphi \ast (\gamma \ast \xi) + \lambda_2 \lambda_4 \xi \ast \gamma \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma \ast \eta + \lambda_3 \lambda_5 \mu \ast \gamma \ast \psi)(y)
\]
\[
1 - F_c (\lambda_1 \lambda_4 \varphi \ast (\gamma \ast \xi) + \lambda_2 \lambda_4 \xi \ast \gamma \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \gamma \ast \eta + \lambda_3 \lambda_5 \mu \ast \gamma \ast \psi)(y)
\]
\[
= (F_c l)(y),
\]
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furthermore
\[
\Delta_1 = \begin{vmatrix} (F_c k)(y) & \lambda_1 \cos y(F_s \varphi)(y) + \lambda_2 \sin y(F_c \eta)(y) + \lambda_3 (F_s \mu)(y) \\ (F_s k)(y) & 1 \end{vmatrix}
\]
\[
= (F_c h)(y) - \lambda_1 \cos y(F_s \varphi)(y)(F_s k)(y) - \lambda_2 \sin y(F_c \eta)(y)(F_s k)(y) - \lambda_3 (F_s \mu)(y)(F_s k)(y)
\]
\[
= F_c \left( h - \lambda_1 \varphi * k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu * k \right)(y).
\]

Therefore
\[
(F_s f)(y) = \frac{\Delta_1}{\Delta} = F_c \left( h - \lambda_1 \varphi * k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu * k \right)(y)[1 + (F_c l)(y)]
\]
\[
= F_c \left( h - \lambda_1 \varphi * k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu * k + h * l - \lambda_1 \varphi * k \right) * l - \lambda_2 (k \frac{\gamma_1}{2} \eta) * l - \lambda_3 (\mu * k) * l \right)(y)
\]
It shows that
\[
f = h - \lambda_1 \varphi * k - \lambda_2 k \frac{\gamma_1}{2} \eta - \lambda_3 \mu * k + h * l - \lambda_1 \varphi * k \right) * l - \lambda_2 (k \frac{\gamma_1}{2} \eta) * l - \lambda_3 (\mu * k) * l
\]
Similarly
\[
\Delta = \begin{vmatrix} 1 & (F_c h)(y) \\ \lambda_4 (F_s \xi)(y) + \lambda_5 \sin y(F_c \psi)(y) & (F_s k)(y) \end{vmatrix}
\]
\[
= (F_s k)(y) - \lambda_4 (F_s \xi)(y)(F_c h)(y) - \lambda_5 \sin y(F_c \psi)(y)(F_c h)(y)
\]
\[
= F_s \left( k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma_1}{1} \right)(y).
\]

Then
\[
(F_s g)(y) = \frac{\Delta_2}{\Delta} = F_s \left( k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma_1}{1} \right)(y)[1 + (F_c l)(y)]
\]
\[
= F_s \left( k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma_1}{1} \right)(y) + F_s \left( k * l - \lambda_4 (\xi * h) * l - \lambda_5 (\psi \frac{\gamma_1}{1} * l) \right)(y).
\]
Hence
\[
g = k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma_1}{1} \right) * h + k * l - \lambda_4 (\xi * h) * l - \lambda_5 (\psi \frac{\gamma_1}{1} * l).
\]
The proof is complete.

REFERENCES

THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE \( F_c, F_s \) TRANSFORMS


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