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CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

NGUYEN XUAN THAO AND NGUYEN THANH HONG

ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. It's properties and applications to solve systems of integral equations are considered.

1. INTRODUCTION

Let F_s be the Fourier sine transform [2]

$$(F_s f)(c) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin xy f(y) dy,$$

and F_c be the Fourier cosine transform [2]

$$(F_c f)(c) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos xy f(y) dy.$$

Convolution theory has been studied in 20th. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated. The convolution of two functions f and g for the Fourier cosine transform is introduced in [7]

$$(1.1) \quad (f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(y)[g(|x-y|) + g(x+y)] dy, \quad x > 0,$$

which satisfied the following factorization equality

$$(1.2) \quad F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$

In 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions f and g with the weight function $\gamma_1(y) = \sin y$ for the Fourier

Convolution, Fourier sine transform, Fourier cosine transform, integral equation.

sine transform which is of the form [4] and [10]

$$(1.3) \quad (f \underset{F_s}{*}^{\gamma_1} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\text{sign}(x+y-1)g(|x+y-1|) + \text{sign}(x-y+1)g(|x-y+1|) - g(x+y+1) - \text{sign}(x-y-1)g(|x-y-1|)]dy, \quad x > 0,$$

and proved the following factorization identity [4], [10]

$$(1.4) \quad F_s(f \underset{F_s}{*}^{\gamma_1} g)(y) = \sin y(F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$

The convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine transform of two functions f and g is introduced in [11]

$$(1.5) \quad (f \underset{F_c}{*}^{\gamma} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(|y+x-1|)+g(|y-x-1|)+g(y+x+1)+g(|y-x+1|)]dy, \quad x > 0$$

and satisfy the factorization equality [11]

$$(1.6) \quad F_c(f \underset{F_c}{*}^{\gamma} g)(y) = \cos y(F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$

In 1941, Churchill R.V introduced the first generalized convolution of two functions f and g for the Fourier sine and Fourier cosine transforms [7]

$$(1.7) \quad (f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \quad x > 0,$$

and proved the following factorization identity [7]

$$(1.8) \quad F_s(f \underset{1}{*} g)(y) = (F_s f)(y) \cdot (F_c g)(y), \quad \forall y > 0.$$

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, G -transform and H -transform. In 1998, Kakichev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions f and g for the Fourier cosine and sine transforms is defined by [6]

$$(1.9) \quad (f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[\text{sign}(u-x)g(|u-x|) + g(u+x)]du, \quad x > 0.$$

For this generalized convolution the following factorization equality holds [6]

$$(1.10) \quad F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$

Another generalized convolution with the weight function $\gamma_1(y) = \sin y$ for the Fourier cosine and sine has been studied in [15]

(1.11)

$$(f \overset{\gamma_1}{*}_2 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)]du, x > 0.$$

It satisfies the factorization property [15]

$$(1.12) \quad F_c(f \overset{\gamma_1}{*}_2 g)(y) = \sin y (F_s f)(y) (F_c g)(y), \quad \forall y > 0.$$

The generalized convolution of two functions f and g with the weight function $\gamma_1(y) = \sin y$ for the Fourier sine and cosine transforms has the form

(1.13)

$$(f \overset{\gamma_1}{*}_1 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)]du, x > 0,$$

and satisfy the following factorization identity [14]

$$(1.14) \quad F_s(f \overset{\gamma_1}{*}_1 g)(y) = \sin y (F_c f)(y) (F_c g)(y), \quad \forall y > 0.$$

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Its properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

2. THE GENERALIZED CONVOLUTION

Definition 1. A generalized convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine and sine transforms of functions f and g is defined by

$$(2.1) \quad (f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(y+x+1) + \text{sign}(y-x-1)g(|y-x-1|) \\ + \text{sign}(y+x-1)g(|y+x-1|) + \text{sign}(y-x+1)g(|y-x+1|)]dy, \quad x > 0.$$

We denote by $L(\mathbb{R}_+)$ the space of all functions f defined on \mathbb{R}_+ such that $\int_+ |f(x)|dx < \infty$.

Theorem 1. Let f and g be functions in $L(\mathbb{R}_+)$ then the generalized convolution $(f \overset{\gamma}{*} g)(x)$ defined by (2.1) also be a $L(\mathbb{R}_+)$ function. Moreover, the following factorization equality holds

$$(2.2) \quad F_c(f \overset{\gamma}{*} g)(y) = \cos y (F_s f)(y) (F_s g)(y), \quad \forall y > 0.$$

Proof. From the defining formula of the generalized convolution and the fact that $f, g \in L(\mathbb{R}_+)$ we have

$$\begin{aligned}
 (2.3) \quad \int_0^{+\infty} |(f \overset{\gamma}{*} g)|(x) dx &= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} |f(y)| \cdot [|g(y+x+1) + \text{sign}(y-x-1)g(|y-x-1|) \\
 &\quad + \text{sign}(y+x-1)g(|y+x-1|) + \text{sign}(y-x+1)g(|y-x+1|)] dy dx \\
 &\leq \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(y)| \cdot \left[\int_0^{+\infty} |g(y+x+1)| dx + \int_0^{+\infty} |g(|y-x-1|)| dx \right. \\
 &\quad \left. + \int_0^{+\infty} |g(|y+x-1|)| dx + \int_0^{+\infty} |g(|y-x+1|)| dx \right] dy.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (2.4) \quad \int_0^{+\infty} |g(y+x+1)| dx + \int_0^{+\infty} |g(|y-x-1|)| dx &= \int_{y+1}^{+\infty} |g(t)| dt + \int_{-y-1}^{+\infty} |g(|t|)| dt \\
 &= 2 \int_0^{+\infty} |g(t)| dt.
 \end{aligned}$$

Similarly,

$$(2.5) \quad \int_0^{+\infty} |g(|y+x-1|)| dx + \int_0^{+\infty} |g(|y-x+1|)| dx = 2 \int_0^{+\infty} |g(t)| dt.$$

From (2.3), (2.4) and (2.5) one holds

$$\int_0^{+\infty} |(f \overset{\gamma}{*} g)|(x) dx \leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |g(t)| dt \int_0^{+\infty} |g(t)| dt \leq +\infty.$$

So $(f \overset{\gamma}{*} g)|(x)$ belong to $L(\mathbb{R}_+)$.

Now we prove the factorization equality (2.2). We have

$$\begin{aligned}
 \cos y (F_s f)(y) (F_s g)(y) &= \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} \cos y \sin uy \sin vy f(u) g(v) dudv \\
 &= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u) g(v) [\cos y(u-v+1) + \cos y(u-v-1) - \cos y(u+v+1) - \cos y(u+v-1)] dudv.
 \end{aligned}$$

Changing the variables gives

$$\begin{aligned}
 \cos y(F_s f)(y)(F_s g)(y) &= \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[\int_{-u-1}^{+\infty} g(t+u+1) \cos ytdt + \int_{-u+1}^{+\infty} g(t+u-1) \cos ytdt \right. \\
 &\quad \left. - \int_{u+1}^{+\infty} g(t-u-1) \cos ytdt - \int_{u-1}^{+\infty} g(t-u+1) \cos ytdt \right] du \\
 &= \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[\int_0^{+\infty} g(u+t+1) \cos ytdt + \int_0^{+\infty} \text{sign}(u-t+1)g(|u-t+1|) \cos ytdt \right. \\
 &\quad \left. + \int_0^{+\infty} \text{sign}(u+t-1)g(|u+t-1|) \cos ytdt + \int_0^{+\infty} \text{sign}(u-t-1)g(|u-t-1|) \cos ytdt \right] du \\
 &= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u)[g(u+t+1) + \text{sign}(u-t+1)g(|u-t+1|) + \text{sign}(u+t-1)g(|u+t-1|) \\
 &\quad + \text{sign}(u-t-1)g(|u-t-1|)] du \cos ytdt \\
 &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (f \overset{\gamma}{*} g)(t) \cos ytdt \\
 &= F_c(f \overset{\gamma}{*} g)(y).
 \end{aligned}$$

It shows that

$$F_c(f \overset{\gamma}{*} g)(y) = \cos y(F_s f)(y)(F_s g)(y).$$

The proof of Theorem is completes. □

Theorem 2. *In the space $L(\mathbb{R}_+)$ the generalized convolution (2.1) is not associative and the relation with knowns convolutions and generalized convolutions as follows*

- a) $f \overset{\gamma}{*}_1 (g \overset{\gamma}{*}_1 h) = g \overset{\gamma}{*}_1 (f \overset{\gamma}{*}_1 h) = h \overset{\gamma}{*}_1 (f \overset{\gamma}{*}_1 g)$, where $f \overset{\gamma}{*}_1 g$ is defined by (1.7).
- b) $f \overset{\gamma}{*}_1 (g \overset{\gamma}{*}_1 h) = (f \overset{\gamma}{*}_1 g) \overset{\gamma}{*}_{F_c} h = (f \overset{\gamma}{*}_1 h) \overset{\gamma}{*}_1 g$, where $f \overset{\gamma}{*}_{F_c} g$ is the Fourier cosine convolution (1.1).
- c) $f \overset{\gamma_1}{*}_2 (g \overset{\gamma}{*}_1 h) = (f \overset{\gamma_1}{*}_{F_s} h) \overset{\gamma}{*}_1 g = h \overset{\gamma_1}{*}_2 (f \overset{\gamma}{*}_1 g)$, where $f \overset{\gamma_1}{*}_{F_s} g$ and $f \overset{\gamma_1}{*}_2 g$ are defined by (1.3) and (1.11).
- d) $f \overset{\gamma}{*}_1 (g \overset{\gamma_1}{*}_1 h) = (f \overset{\gamma_1}{*}_2 g) \overset{\gamma}{*}_{F_c} h = (f \overset{\gamma_1}{*}_2 h) \overset{\gamma}{*}_{F_c} g$, where $f \overset{\gamma}{*}_{F_c} g$ is the Fourier cosine convolution with a weight function (1.5).

Proof. a) From the factorization equality, we have

$$\begin{aligned} F_s(f \underset{1}{*} (g \overset{\gamma}{*} h))(y) &= (F_s f)(y) (F_c(g \overset{\gamma}{*} h))(y) \\ &= (F_s f)(y) \cos y (F_s g)(y) (F_s h)(y) = (F_s g)(y) \cos y (F_s f)(y) (F_s h)(y) \\ &= (F_s g)(y) (F_c(f \overset{\gamma}{*} h))(y) \\ &= F_s(g \underset{1}{*} (f \overset{\gamma}{*} h))(y). \end{aligned}$$

Which implies that $f \underset{1}{*} (g \overset{\gamma}{*} h) = g \underset{1}{*} (f \overset{\gamma}{*} h)$

On the other hand,

$$\begin{aligned} F_s(f \underset{1}{*} (g \overset{\gamma}{*} h))(y) &= (F_s f)(y) \cos y (F_s g)(y) (F_s h)(y) \\ &= (F_s h)(y) \cos y (F_s g)(y) (F_s f)(y) \\ &= F_s(g \underset{1}{*} (f \overset{\gamma}{*} h))(y). \end{aligned}$$

Then we obtain the part a). The parts b), c), d) can be obtain in similar way. The Theorem is proved. \square

Theorem 3. *In the space $L(\mathbb{R}_+)$ the generalized convolution (2.1) does not have a unit element.*

Proof. Suppose that there exists a unit element e of the generalized convolution (2.1) in $L(\mathbb{R}_+)$. It means that $f \overset{\gamma}{*} e = e \overset{\gamma}{*} f = f$ for any function $f \in L(\mathbb{R}_+)$. It follows that $F_c(f \overset{\gamma}{*} e)(y) = (F_c f)(y)$, $\forall y > 0$.

Hence, $\cos y (F_s e)(y) (F_s f)(y) = (F_c f)(y)$, $\forall y > 0$.

Choosing $f(x) = e^{-x} \in L(\mathbb{R}_+)$. From the fact that

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \frac{1}{1+y^2}, \quad (F_s f)(y) = \sqrt{\frac{2}{\pi}} \frac{y}{1+y^2},$$

we obtain

$$(F_s e)(y) = \frac{1}{y \cos y}.$$

It is contradiction from the fact that $\frac{1}{y \cos y} \notin L_\infty(\mathbb{R}_+)$ while $(F_s e)(y) \in L_\infty(\mathbb{R}_+)$ since $e \in L(\mathbb{R}_+)$.

The Theorem is proved. \square

Let $L(\mathbb{R}_+, e^x) = \{h, \text{ for all } e^x h(x) \in L(\mathbb{R}_+)\}$.

Theorem 4 (Titchmarch-type Theorem). *Let $f, g \in L(\mathbb{R}_+, e^x)$. If $(f \overset{\gamma}{*} g)(x) \equiv 0$ then either $f \equiv 0$ or $g \equiv 0$.*

Proof. Suppose that $(f \underset{c;s}{*} g)(x) = 0, \forall x > 0$, in view of Theorem 1

$$(2.6) \quad F_c(f \overset{\gamma}{*} g)(y) = \cos y (F_s f)(y) (F_s g)(y) = 0, \forall y > 0.$$

We have

$$\begin{aligned} \left| \frac{d^n}{dy^n} [\sin(xy)f(x)] \right| &= \left| f(x)x^n \sin\left(xy + n\frac{\pi}{2}\right) \right| \\ &\leq \left| f(x)x^n \right| \\ &= |e^{-x} \cdot x^n| \cdot |f_1(x)| \leq n! |f_1(x)|, \end{aligned}$$

where $f_1(x) = e^x f(x) \in L(\mathbb{R}_+)$.

Due to Weierstrass criterion, the integral $\int_0^{+\infty} \frac{d^n}{dy^n} [\sin(xy)f(x)] dx$ uniformly converges on \mathbb{R}_+ .

Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_s f)(y)$ is analytic. Similarly, $(F_s g)(y)$ is analytic. So from (2.6) we have $(F_s f)(y) \equiv 0$ or $(F_s g)(y) \equiv 0$. It completes the proof. \square

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations

$$(3.1) \quad \begin{aligned} f(x) + \lambda_1 \int_0^{+\infty} g(t)\varphi_1(x,t)dt + \lambda_2 \int_0^{+\infty} g(t)\eta_1(x,t)dt + \lambda_3 \int_0^{+\infty} g(t)\mu_1(x,t)dt &= h(x) \\ \lambda_4 \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \xi(t)[f(x+t) - f(|x-t|)]dt + \lambda_5 \int_0^{+\infty} g(t)\psi_1(x,t)dt + g(x) &= k(x), \end{aligned}$$

here $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are complex numbers, $\varphi, \eta, \mu, \xi \in L(\mathbb{R}_+)$, and

$$\begin{aligned} \varphi_1(x,t) &= \frac{1}{2\sqrt{2\pi}} [\varphi(t+x+1) + \text{sign}(t-x-1)\varphi(|t-x-1|) + \text{sign}(t+x-1)\varphi(|t+x-1|) \\ &\quad + \text{sign}(t-x+1)\varphi(|t-x+1|)], \end{aligned}$$

$$\eta_1(x,t) = \frac{1}{2\sqrt{2\pi}} [\eta(|x+t-1|) + \eta(|x-t+1|) - \eta(x+t+1) - \eta(|x-t-1|)],$$

$$\mu_1(x,t) = \frac{1}{\sqrt{2\pi}} [\text{sign}(t-x)\mu(|t-x|) + \mu(t+x)],$$

$$\psi_1(x,t) = \frac{1}{2\sqrt{2\pi}} [\psi(|x+t-1|) + \psi(|x-t-1|) - \psi(x+t+1) - \psi(|x-t+1|)],$$

and $\psi(x) = (\nu_1 *_2 \nu_2)(x)$.

It shows that $\varphi_1, \eta_1, \mu_1, \psi$ and ψ_1 are also $L(\mathbb{R}_+)$ - functions

Theorem 5. *With the condition*

$$F_c(\lambda_1 \lambda_4 \varphi *_2^\gamma \xi + \lambda_2 \lambda_4 \xi *_2^{\gamma_1} \eta + \lambda_3 \lambda_4 \mu *_2^\gamma \xi + \lambda_1 \lambda_5 \varphi *_2^\gamma (\nu_1 *_1^{\gamma_1} \nu_2) + \lambda_2 \lambda_5 \nu_1 *_2^{\gamma_1} (\nu_2 *_2^{\gamma_1} \eta) + \lambda_3 \lambda_5 \mu *_2^{\gamma_1} \psi)(y) \neq 1,$$

there exists a unique solution in $L(\mathbb{R}_+)$ of system (3.1) which is of the form

$$\begin{aligned} f &= h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{*}_2 \eta - \lambda_3 \mu \overset{*}{*_2} k + h \overset{*}{*_c} l - \lambda_1 (\varphi \overset{\gamma}{*} k) \overset{*}{*_c} l - \lambda_2 (k \overset{\gamma_1}{*}_2 \eta) \overset{*}{*_c} l - \lambda_3 (\mu \overset{*}{*_2} k) \overset{*}{*_c} l \\ g &= k - \lambda_4 \xi \overset{*}{*_1} h - \lambda_5 \psi \overset{\gamma_1}{*}_1 h + k \overset{*}{*_1} l - \lambda_4 (\xi \overset{*}{*_1} h) \overset{*}{*_1} l - \lambda_5 (\psi \overset{\gamma_1}{*}_1 h) \overset{*}{*_1} l \end{aligned}$$

Here $l \in L(\mathbb{R}_+)$ and is defined by

$$\begin{aligned} F_c l &= \\ \frac{F_c(\lambda_1 \lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2 \lambda_4 \xi \overset{\gamma_1}{*}_2 \eta + \lambda_3 \lambda_4 \mu \overset{*}{*_2} \xi + \lambda_1 \lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \overset{\gamma_1}{*}_1 \nu_2) + \lambda_2 \lambda_5 \nu_1 \overset{\gamma_1}{*}_2 (\nu_2 \overset{\gamma_1}{*}_2 \eta) + \lambda_3 \lambda_5 \mu \overset{\gamma_1}{*}_2 \psi)}{1 - F_c(\lambda_1 \lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2 \lambda_4 \xi \overset{\gamma_1}{*}_2 \eta + \lambda_3 \lambda_4 \mu \overset{*}{*_2} \xi + \lambda_1 \lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \overset{\gamma_1}{*}_1 \nu_2) + \lambda_2 \lambda_5 \nu_1 \overset{\gamma_1}{*}_2 (\nu_2 \overset{\gamma_1}{*}_2 \eta) + \lambda_3 \lambda_5 \mu \overset{\gamma_1}{*}_2 \psi)} \end{aligned}$$

Proof. System (3.1) can be rewritten in the form

$$\begin{aligned} (3.2) \quad f(x) + \lambda_1 (g \overset{\gamma}{*} \varphi)(x) + \lambda_2 (g \overset{\gamma_1}{*}_2 \eta)(x) + \lambda_3 (g \overset{*}{*_2} \mu)(x) &= h(x) \\ \lambda_4 (\xi \overset{*}{*_1} f)(x) + \lambda_5 (f \overset{\gamma_1}{*}_1 \psi)(x) + g(x) &= k(x). \end{aligned}$$

Using the respectively factorization equalities of above generalized convolutions we have

$$\begin{aligned} (3.3) \quad (F_c f)(y) + \lambda_1 \cos y (F_s g)(y) (F_s \varphi)(y) + \lambda_2 \sin y (F_s g)(y) (F_c \eta)(y) + \lambda_3 (F_s g)(y) (F_s \mu)(y) &= (F_c h)(y) \\ \lambda_4 (F_s \xi)(y) (F_c f)(y) + \lambda_5 \sin y (F_c f)(y) (F_c \psi)(y) + (F_s g)(y) &= (F_s k)(y). \end{aligned}$$

We have

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & \lambda_1 \cos y (F_s \varphi)(y) + \lambda_2 \sin y (F_c \eta)(y) + \lambda_3 (F_s \mu)(y) \\ \lambda_4 (F_s \xi)(y) + \lambda_5 \sin y (F_c \psi)(y) & 1 \end{vmatrix} \\ &= 1 - \lambda_1 \lambda_4 \cos y (F_s \varphi)(y) (F_s \xi)(y) - \lambda_2 \lambda_4 \sin y (F_c \eta)(y) (F_s \xi)(y) - \lambda_3 \lambda_4 (F_s \mu)(y) \lambda_4 (F_s \xi)(y) - \\ &\quad \lambda_1 \lambda_5 \sin y \cos y (F_s \varphi)(y) (F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 y (F_c \eta)(y) (F_c \psi)(y) - \lambda_3 \lambda_5 \sin y (F_s \mu)(y) (F_c \psi)(y) \\ &= 1 - F_c(\lambda_1 \lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2 \lambda_4 \xi \overset{\gamma_1}{*}_2 \eta + \lambda_3 \lambda_4 \mu \overset{*}{*_2} \xi + \lambda_1 \lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \overset{\gamma_1}{*}_1 \nu_2) + \lambda_2 \lambda_5 \nu_1 \overset{\gamma_1}{*}_2 (\nu_2 \overset{\gamma_1}{*}_2 \eta) \\ &\quad + \lambda_3 \lambda_5 \mu \overset{\gamma_1}{*}_2 \psi)(y) \neq 0. \end{aligned}$$

Then due to Wiener-Levi's Theorem [10] there exists a function $l \in L(\mathbb{R}_+)$ such that

$$\begin{aligned} \frac{1}{\Delta} - 1 &= \\ \frac{F_c(\lambda_1 \lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2 \lambda_4 \xi \overset{\gamma_1}{*}_2 \eta + \lambda_3 \lambda_4 \mu \overset{*}{*_2} \xi + \lambda_1 \lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \overset{\gamma_1}{*}_1 \nu_2) + \lambda_2 \lambda_5 \nu_1 \overset{\gamma_1}{*}_2 (\nu_2 \overset{\gamma_1}{*}_2 \eta) + \lambda_3 \lambda_5 \mu \overset{\gamma_1}{*}_2 \psi)(y)}{1 - F_c(\lambda_1 \lambda_4 \varphi \overset{\gamma}{*} \xi + \lambda_2 \lambda_4 \xi \overset{\gamma_1}{*}_2 \eta + \lambda_3 \lambda_4 \mu \overset{*}{*_2} \xi + \lambda_1 \lambda_5 \varphi \overset{\gamma}{*} (\nu_1 \overset{\gamma_1}{*}_1 \nu_2) + \lambda_2 \lambda_5 \nu_1 \overset{\gamma_1}{*}_2 (\nu_2 \overset{\gamma_1}{*}_2 \eta) + \lambda_3 \lambda_5 \mu \overset{\gamma_1}{*}_2 \psi)(y)} & \\ = (F_c l)(y), \end{aligned}$$

furthermore

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} (F_c h)(y) & \lambda_1 \cos y (F_s \varphi)(y) + \lambda_2 \sin y (F_c \eta)(y) + \lambda_3 (F_s \mu)(y) \\ (F_s k)(y) & 1 \end{vmatrix} \\ &= (F_c h)(y) - \lambda_1 \cos y (F_s \varphi)(y) (F_s k)(y) - \lambda_2 \sin y (F_c \eta)(y) (F_s k)(y) - \lambda_3 (F_s \mu)(y) (F_s k)(y) \\ &= F_c (h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{*}_2 \eta - \lambda_3 \mu \overset{*}{*}_2 k)(y). \end{aligned}$$

Therefore

$$\begin{aligned} (F_s f)(y) &= \frac{\Delta_1}{\Delta} = F_c (h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{*}_2 \eta - \lambda_3 \mu \overset{*}{*}_2 k)(y) [1 + (F_c l)(y)] \\ &= F_c (h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{*}_2 \eta - \lambda_3 \mu \overset{*}{*}_2 k + h \overset{*}{*}_{F_c} l - \lambda_1 (\varphi \overset{\gamma}{*} k) \overset{*}{*}_{F_c} l - \lambda_2 (k \overset{\gamma_1}{*}_2 \eta) \overset{*}{*}_{F_c} l - \lambda_3 (\mu \overset{*}{*}_2 k) \overset{*}{*}_{F_c} l)(y) \end{aligned}$$

It shows that

$$f = h - \lambda_1 \varphi \overset{\gamma}{*} k - \lambda_2 k \overset{\gamma_1}{*}_2 \eta - \lambda_3 \mu \overset{*}{*}_2 k + h \overset{*}{*}_{F_c} l - \lambda_1 (\varphi \overset{\gamma}{*} k) \overset{*}{*}_{F_c} l - \lambda_2 (k \overset{\gamma_1}{*}_2 \eta) \overset{*}{*}_{F_c} l - \lambda_3 (\mu \overset{*}{*}_2 k) \overset{*}{*}_{F_c} l$$

Similarly

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & (F_c h)(y) \\ \lambda_4 (F_s \xi)(y) + \lambda_5 \sin y (F_c \psi)(y) & (F_s k)(y) \end{vmatrix} \\ &= (F_s k)(y) - \lambda_4 (F_s \xi)(y) (F_c h)(y) - \lambda_5 \sin y (F_c \psi)(y) (F_c h)(y) \\ &= F_s (k - \lambda_4 \xi \overset{*}{*}_1 h - \lambda_5 \psi \overset{\gamma_1}{*}_1 h)(y). \end{aligned}$$

Then

$$\begin{aligned} (F_s g)(y) &= \frac{\Delta_2}{\Delta} = F_s (k - \lambda_4 \xi \overset{*}{*}_1 h - \lambda_5 \psi \overset{\gamma_1}{*}_1 h)(y) [1 + (F_c l)(y)] \\ &= F_s (k - \lambda_4 \xi \overset{*}{*}_1 h - \lambda_5 \psi \overset{\gamma_1}{*}_1 h)(y) + F_s (k \overset{*}{*}_1 l - \lambda_4 (\xi \overset{*}{*}_1 h) \overset{*}{*}_1 l - \lambda_5 (\psi \overset{\gamma_1}{*}_1 h) \overset{*}{*}_1 l)(y). \end{aligned}$$

Hence

$$g = k - \lambda_4 \xi \overset{*}{*}_1 h - \lambda_5 \psi \overset{\gamma_1}{*}_1 h + k \overset{*}{*}_1 l - \lambda_4 (\xi \overset{*}{*}_1 h) \overset{*}{*}_1 l - \lambda_5 (\psi \overset{\gamma_1}{*}_1 h) \overset{*}{*}_1 l.$$

The proof is complete. □

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