<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Nguyen, Xuan Thao; Nguyen, Thanh Hong</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

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Osaka University
CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED
CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER
COSINE AND SINE TRANSFORMS

NGUYEN XUAN THAO AND NGUYEN THANH HONG

ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. Its properties and applications to solve systems of integral equations are considered.

1. Introduction

Let $F_s$ be the Fourier sine transform \[ (F_s f)(c) = \frac{2}{\pi} \int_0^\infty \sin xy f(y) dy, \]

and $F_c$ be the Fourier cosine transform \[ (F_c f)(c) = \frac{2}{\pi} \int_0^\infty \cos xy f(y) dy. \]

Convolution theory has been studied in 20th. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated. The convolution of two functions $f$ and $g$ for the Fourier cosine transform is introduced in [7]

\[ (f \ast g)_c(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(|x - y|) + g(x + y)]dy, \quad x > 0, \]

which satisfied the following factorization equality

\[ F_c(f \ast g)_c(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0. \]

In 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier

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Convolution, Fourier sine transform, Fourier cosine transform, integral equation.
THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE $F_c, F_s$ TRANSFORMS

sine transform which is of the form [4] and [10]

\begin{equation}
(f_{\gamma}^{F_s}g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} f(y)[\text{sign}(x+y-1)g(|x+y-1|) + \text{sign}(x-y+1)g(|x-y+1|) - g(x+y+1) - \text{sign}(x-y-1)g(|x-y-1|)]dy, \ x > 0,
\end{equation}

and proved the following factorization identity [4], [10]

\begin{equation}
F_s(f_{\gamma}^{F_s}g)(y) = \sin y(Fsf)(y)(Fsg)(y), \ \forall y > 0.
\end{equation}

The convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine transform of two functions $f$ and $g$ is introduced in [11]

\begin{equation}
(f_{\gamma}^{F_c}g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} f(y)[g(|y+x-1|)+g(|y-x-1|)+g(y+x+1)+g(|y-x+1|)]dy, \ x > 0
\end{equation}

and satisfy the factorization equality [11]

\begin{equation}
F_c(f_{\gamma}^{F_c}g)(y) = \cos y(Fcf)(y)(Fcg)(y), \ \forall y > 0.
\end{equation}

In 1941, Churchill R.V introduced the first generalized convolution of two functions $f$ and $g$ for the Fourier sine and Fourier cosine transforms [7]

\begin{equation}
(f_{1}^{*}g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \ x > 0,
\end{equation}

and proved the following factorization identity [7]

\begin{equation}
F_s(f_{1}^{*}g)(y) = (Fsf)(y)(Fsg)(y), \ \forall y > 0.
\end{equation}

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, $G$-transform and $H$-transform. In 1998, Kachičev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions $f$ and $g$ for the Fourier cosine and sine transforms is defined by [6]

\begin{equation}
(f_{\frac{1}{2}}^{*}g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(u)[\text{sign}(u-x)g(|u-x|) + g(u+x)]du, \ x > 0.
\end{equation}

For this generalized convolution the following factorization equality holds [6]

\begin{equation}
F_c(f_{\frac{1}{2}}^{*}g)(y) = (Fsf)(y)(Fsg)(y), \ \forall y > 0.
\end{equation}
Another generalized convolution with the weight function \( \gamma_1(y) = \sin y \) for the Fourier cosine and sine has been studied in \([15]\) (1.11)

\[
(f \ast_2 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)]du, \quad x > 0.
\]

It satisfies the factorization property \([15]\) (1.12)

\[
F_c(f \ast_2 g)(y) = \sin y (F_s f)(y)(F_c g)(y), \quad \forall y > 0.
\]

The generalized convolution of two functions \( f \) and \( g \) with the weight function \( \gamma_1(y) = \sin y \) for the Fourier sine and cosine transforms has the form (1.13)

\[
(f \ast_1 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u-1|)]du, \quad x > 0,
\]

and satisfy the following factorization identity \([14]\) (1.14)

\[
F_s(f \ast_1 g)(y) = \sin y (F_c f)(y)(F_s g)(y), \quad \forall y > 0.
\]

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Its properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

2. THE GENERALIZED CONVOLUTION

**Definition 1.** A generalized convolution with the weight function \( \gamma(y) = \cos y \) for the Fourier cosine and sine transforms of functions \( f \) and \( g \) is defined by

(2.1) \[
(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(y + x + 1) + \text{sign}(y - x - 1)g(|y - x - 1|) + \text{sign}(y + x - 1)g(|y + x - 1|) + \text{sign}(y - x + 1)g(|y - x + 1|)]dy, \quad x > 0.
\]

We denote by \( L(\mathbb{R}_+) \) the space of all functions \( f \) defined on \( \mathbb{R}_+ \) such that \( \int_+ |f(x)|dx < \infty \).

**Theorem 1.** Let \( f \) and \( g \) be functions in \( L(\mathbb{R}_+) \) then the generalized convolution \( (f \ast g)(x) \) defined by (2.1) also be a \( L(\mathbb{R}_+) \) function. Moreover, the following factorization equality holds (2.2)

\[
F_c(f \ast g)(y) = \cos y (F_s f)(y)(F_c g)(y), \quad \forall y > 0.
\]
THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE $F_{c}, F_{s}$ TRANSFORMS

Proof. From the defining formula of the generalized convolution and the fact that $f, g \in L(\mathbb{R}_{+})$ we have

\begin{align*}
(2.3) \quad \int_{0}^{\infty} |(f \ast g)|(x)dx &= \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} \int_{0}^{\infty} |f(y)| |g(y + x + 1) + \text{sign}(y - x - 1)g(|y - x - 1|) + \text{sign}(y + x - 1)g(|y + x - 1|) + \text{sign}(y - x + 1)g(|y - x + 1|)|dydx \\
&= \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} |f(y)| \left[ \int_{0}^{\infty} |g(y + x + 1)|dx + \int_{0}^{\infty} |g(|y - x - 1|)|dx \\
&\quad + \int_{0}^{\infty} |g(|y + x - 1|)|dx + \int_{0}^{\infty} |g(|y - x + 1|)|dx \right]dy.
\end{align*}

On the other hand,

\begin{align*}
(2.4) \quad \int_{0}^{\infty} |g(y + x + 1)|dx + \int_{0}^{\infty} |g(|y - x - 1|)|dx &= \int_{y+1}^{\infty} |g(t)|dt + \int_{-y-1}^{\infty} |g(|t|)|dt \\
&= 2 \int_{0}^{\infty} |g(t)|dt.
\end{align*}

Similary,

\begin{align*}
(2.5) \quad \int_{0}^{\infty} |g(|y + x - 1|)|dx + \int_{0}^{\infty} |g(|y - x + 1|)|dx = 2 \int_{0}^{\infty} |g(t)|dt.
\end{align*}

From (2.3), (2.4) and (2.5) one holds

\begin{align*}
\int_{0}^{\infty} |(f \ast g)|(x)dx \leq \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} |g(t)|dt \int_{0}^{\infty} |g(t)|dt \leq +\infty.
\end{align*}

So $(f \ast g)|(x)$ belong to $L(\mathbb{R}_{+})$.

Now we prove the factorization equality (2.2). We have

\begin{align*}
\cos y(F_{s}f)(y)(F_{s}g)(y) &= \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \cos y \sin uy \sin vy f(u)g(v)dudv \\
&= \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u)g(v)[\cos y(u - v + 1) + \cos y(u - v - 1) - \cos y(u + v + 1) - \cos y(u + v - 1)]dudv.
\end{align*}
THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE \( F_c, F_s \) TRANSFORMS

Changing the variables gives

\[
\cos y(F_s f)(y)(F_s g)(y) = \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_0^{+\infty} g(t + u + 1) \cos ytdt + \int_0^{-u-1} \cos ytdt \\
- \int_{u+1}^{+\infty} g(t - u - 1) \cos ytdt - \int_{u-1}^{+\infty} g(t - u + 1) \cos ytdt \right] du
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_0^{+\infty} g(u + t + 1) \cos ytdt + \int_0^{+\infty} \operatorname{sgn}(u - t + 1) g(|u - t + 1|) \cos ytdt \\
+ \int_0^{+\infty} \operatorname{sgn}(u + t - 1) g(|u + t - 1|) \cos ytdt + \int_0^{+\infty} \operatorname{sgn}(u - t - 1) g(|u - t - 1|) \cos ytdt \right] du
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u)[g(u + t + 1) + \operatorname{sgn}(u - t + 1) g(|u - t + 1|) + \operatorname{sgn}(u + t - 1) g(|u + t - 1|) \\
+ \operatorname{sgn}(u - t - 1) g(|u - t - 1|)] du \cos ytdt
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (f \star g)(t) \cos ytdt
\]

\[
= F_c(f \star g)(y).
\]

It shows that

\[
F_c(f \star g)(y) = \cos y(F_s f)(y)(F_s g)(y).
\]

The proof of Theorem is completes.

\[\square\]

**Theorem 2.** In the space \( L(\mathbb{R}_+) \) the generalized convolution (2.1) is not associative and the relation with knowns convolutions and generalized convolutions as follows

a) \( f \star (g \star h) = g \star (f \star h) = h \star (g \star f) \), where \( f \star g \) is defined by (1.7).

b) \( f \star_1 (g \star h) = (f \star_1 g) \star h = (f \star h) \star_1 g \), where \( f \star g \) is the Fourier cosine convolution (1.1).

c) \( f \star_2 (g \star h) = (f \star_2 g) \star h = h \star_2 (f \star g) \), where \( f \star_2 g \) and \( f \star_2 g \) are defined by (1.3) and (1.11).

d) \( f \star_1 (g \star h) = (f \star_1 g) \star h = (f \star h) \star_1 g \), where \( f \star g \) is the Fourier cosine convolution with a weight function (1.5).
Proof. a) From the factorization equality, we have
\[ F_s(f \ast (g \ast h))(y) = \left(F_s(f)g\right)(y) = \left(F_s(f)g\right)(y) \cos y(F_sf)(y) = \left(F_s(f)g\right)(y) \cos y(F_sf)(y) \]
\[ = F_s(f \ast (g \ast h))(y). \]

Which implies that \( f \ast (g \ast h) = g \ast (f \ast h) \)

On the other hand,
\[ F_s(f \ast (g \ast h))(y) = \left(F_s(f)g\right)(y) = \left(F_s(f)g\right)(y) \cos y(F_sf)(y) = \left(F_s(f)g\right)(y) \cos y(F_sf)(y) \]
\[ = F_s(f \ast (g \ast h))(y). \]

Then we obtain the part a). The parts b), c), d) can be obtain in similar way. The Theorem is proved.

Theorem 3. In the space \( L(R_+) \) the generalized convolution (2.1) does not have a unit element.

Proof. Suppose that there exists a unit element \( e \) of the generalized convolution (2.1) in \( L(R_+) \). It means that \( f \ast e = e \ast f = f \) for any function \( f \in L(R_+) \). It follows that \( F_c(f \ast e)(y) = (F_c(f))(y), \forall y > 0 \).

Hence, \( \cos y(F_se)(y)(F_sJ)(y) = (F_c(f))(y), \forall y > 0 \).

Choosing \( f(x) = e^{-x} \in L(R_+) \). From the fact that
\[ (F_c(f))(y) = \sqrt{2 \pi} \frac{1}{1 + y^2}, \quad (F_s(f))(y) = \sqrt{2 \pi} \frac{y}{1 + y^2}, \]
we obtain
\[ (F_se)(y) = \frac{1}{y \cos y}. \]

It is contradiction from the fact that \( \frac{1}{y \cos y} \notin L_{\infty}(R_+) \) while \( (F_se)(y) \in L_{\infty}(R_+) \) since \( e \in L(R_+) \).

The Theorem is proved.

Let \( L(R_+, e^x) = \{ h, \text{ for all } e^xh(x) \in L(R_+) \} \).

Theorem 4 (Titchmarsh-type Theorem). Let \( f, g \in L(R_+, e^x) \). If \( (f \ast g)(x) \equiv 0 \) then either \( f \equiv 0 \) or \( g \equiv 0 \).

Proof. Suppose that \( (f \ast g)(x) = 0, \forall x > 0 \), in view of Theorem 1
\[ F_c(f \ast g)(y) = \cos y(F_s(f))(y)(F_s(g))(y) = 0, \forall y > 0. \]
We have
\[
\left| \frac{d^n}{dy^n} [\sin(xy) f(x)] \right| = \left| f(x) x^n \sin(xy + n \frac{\pi}{2}) \right| \\
\leq \left| f(x) x^n \right| \\
= e^{-x^2} x^n \cdot |f_1(x)| \leq n! |f_1(x)|,
\]
where \( f_1(x) = e^x f(x) \in L(\mathbb{R}+) \).

Due to Weierstrass criterion, the integral \( \int_0^{+\infty} \frac{d^n}{dy^n} [\sin(xy) f(x)] dx \) uniformly converges on \( \mathbb{R}+ \).

Therefore, based on the differentiability of integrals depending on parameter, we conclude that \((F_s f)(y)\) is analytic. Similarly, \((F_s g)(y)\) is analytic. So from (2.6) we have \((F_s f)(y) \equiv 0\) or \((F_s g)(y) \equiv 0\). It completes the proof. \(\square\)

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations
\[
f(x) + \lambda_1 \int_0^{+\infty} g(t) \varphi_1(x,t) dt + \lambda_2 \int_0^{+\infty} g(t) \eta_1(x,t) dt + \lambda_3 \int_0^{+\infty} g(t) \mu_1(x,t) dt = h(x)
\]
\[
\lambda_4 \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \xi(t)[f(x + t) - f(|x - t|)] dt + \lambda_5 \int_0^{+\infty} g(t) \psi_1(x,t) dt + g(x) = k(x),
\]
(3.1)
here \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) are complex numbers, \( \varphi, \eta, \mu, \xi \in L(\mathbb{R}+) \), and
\[
\varphi_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \varphi(t + x + 1) + \text{sign}(t - x - 1) \varphi(|t - x - 1|) + \text{sign}(t + x - 1) \varphi(|t + x - 1|) \right. \\
+ \text{sign}(t - x + 1) \varphi(|t + x + 1|),
\]
\[
\eta_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \eta(|x + t - 1|) + \eta(|x - t + 1|) - \eta(x + t + 1) - \eta(|x - t - 1|) \right],
\]
\[
\mu_1(x,t) = \frac{1}{\sqrt{2\pi}} \left[ \text{sign}(t - x) \mu(|t - x|) + \mu(t + x) \right],
\]
\[
\psi_1(x,t) = \frac{1}{2\sqrt{2\pi}} \left[ \psi(|x + t - 1|) + \psi(|x - t + 1|) - \psi(x + t + 1) - \psi(|x - t + 1|) \right],
\]
and \( \psi(x) = (\nu_1 * \nu_2)(x) \).

It shows that \( \varphi_1, \eta_1, \mu_1, \psi \) and \( \psi_1 \) are also \( L(\mathbb{R}+) \) - functions

**Theorem 5.** With the condition
\[
F_c(\lambda_1 \lambda_4 \varphi_1 \ * \ \xi + \lambda_2 \lambda_4 \xi_2 \ * \ \eta + \lambda_3 \lambda_4 \mu * \xi + \lambda_1 \lambda_5 \varphi_1 \ * (\nu_1 \ * \ \nu_2) + \lambda_2 \lambda_5 \nu_1 \ * (\nu_2 \ * \ \eta) + \lambda_3 \lambda_5 \mu \ * \ \psi)(y) \neq 1,
\]
there exists a unique solution in \( L(\mathbb{R}^+) \) of system (3.1) which is of the form

\[
f = h - \lambda_1 \varphi \ast k - \lambda_2 k \ast \eta - \lambda_3 \mu \ast l - \lambda_4 (\varphi \ast k) \ast F_c - \lambda_5 (\psi \ast h) \ast F_c
\]

\[
g = k - \lambda_4 \xi \ast h - \lambda_5 \psi \ast \eta - \lambda_4 (\xi \ast h) \ast F_c - \lambda_5 (\psi \ast h) \ast F_c
\]

Here \( l \in L(\mathbb{R}^+) \) and is defined by

\[
F_c l = \frac{F_c (\lambda_1 \lambda_4 \varphi \ast \xi + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)}{1 - F_c (\lambda_1 \lambda_4 \varphi \ast \xi + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)}
\]

**Proof.** System (3.1) can be rewritten in the form

\[
f(x) + \lambda_1 (g \ast \varphi)(x) + \lambda_2 (g \ast \eta)(x) + \lambda_3 (g \ast \mu)(x) = h(x)
\]

(3.2)

\[
\lambda_4 (\xi \ast f)(x) + \lambda_5 (f \ast \psi)(x) + g(x) = k(x).
\]

Using the respectively factorization equalities of above generalized convolutions we have

\[
(F_c f)(y) + \lambda_1 \cos y (F_s g)(y)(F_s \varphi)(y) + \lambda_2 \sin y (F_s g)(y)(F_s \eta)(y) + \lambda_3 (F_s g)(y) (F_s \mu)(y) = (F_c h)(y)
\]

(3.3)

\[
\lambda_4 (F_s \xi)(y)(F_c f)(y) + \lambda_5 \sin y (F_c f)(y)(F_c \psi)(y) + (F_s g)(y) = (F_s k)(y).
\]

We have

\[
\Delta = \begin{vmatrix}
1 & \lambda_1 \cos y (F_s \varphi)(y) + \lambda_2 \sin y (F_s \eta)(y) + \lambda_3 (F_s \mu)(y) \\
\lambda_4 (F_s \xi)(y) + \lambda_5 \sin y (F_c \psi)(y) & 1
\end{vmatrix}
\]

\[
= 1 - \lambda_1 \lambda_4 \cos y (F_s \varphi)(y)(F_s \xi)(y) - \lambda_2 \lambda_4 \sin y (F_s \eta)(y)(F_s \xi)(y) - \lambda_3 \lambda_4 \lambda_5 (F_s \mu)(y) (F_s \xi)(y) - \\
\lambda_1 \lambda_5 \sin y (F_s \varphi)(y)(F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 y (F_s \eta)(y)(F_c \psi)(y) - \lambda_3 \lambda_5 \lambda_5 \sin y (F_s \varphi)(y)(F_c \psi)(y) - \\
= 1 - F_c (\lambda_1 \lambda_4 \varphi \ast \xi + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)(y)
\]

\[
+ \lambda_3 \lambda_5 \mu \ast \psi)(y) \neq 0.
\]

Then due to Wiener-Levi's Theorem [10] there exists a function \( l \in L(\mathbb{R}^+) \) such that

\[
\frac{1}{\Delta} - 1 = \frac{F_c (\lambda_1 \lambda_4 \varphi \ast \xi + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)(y)}{1 - F_c (\lambda_1 \lambda_4 \varphi \ast \xi + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)(y)} = (F_c l)(y),
\]
furthermore

\[ \Delta_1 = \begin{vmatrix} \lambda_1 \cos y(F_\varphi)(y) + \lambda_2 \sin y(F_\mu)(y) \\ \lambda_3 \sin y(F_\mu)(y) + \lambda_4 \sin y(F_\mu)(y) \end{vmatrix} \frac{1}{(F_c k)(y)} = (F_c h)(y) - \lambda_1 \cos y(F_\varphi)(y) \]

\[ = F_c \left( h - \lambda_1 \varphi * k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu * k \right)(y). \]

Therefore

\[ (F_s f)(y) = \frac{\Delta_1}{\Delta} = F_c \left( h - \lambda_1 \varphi \frac{\gamma}{2} k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu * k \right)(y) [1 + (F_c l)(y)] \]

\[ = F_c \left( h - \lambda_1 \varphi \frac{\gamma}{2} k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu * k + h * l - \lambda_1 (\varphi \frac{\gamma}{2} k) * l - \lambda_2 (k \frac{\gamma}{2} \eta) * l - \lambda_3 (\mu * k) * l \right)(y). \]

It shows that

\[ f = h - \lambda_1 \varphi \frac{\gamma}{2} k - \lambda_2 k \frac{\gamma}{2} \eta - \lambda_3 \mu * k + h * l - \lambda_1 (\varphi \frac{\gamma}{2} k) * l - \lambda_2 (k \frac{\gamma}{2} \eta) * l - \lambda_3 (\mu * k) * l \]

Similarly

\[ \Delta = \begin{vmatrix} \lambda_4 (F_s \xi)(y) + \lambda_5 \sin y(F_c \psi)(y) \\ \lambda_6 \cos y(F_c \psi)(y) \end{vmatrix} \frac{1}{(F_c k)(y)} = (F_c k)(y) - \lambda_4 (F_s \xi)(y)(F_c h)(y) - \lambda_5 \sin y(F_c \psi)(y)(F_c h)(y) \]

\[ = F_s \left( k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma}{1} \right)(y). \]

Then

\[ (F_s g)(y) = \frac{\Delta_2}{\Delta} = F_s \left( k - \lambda_4 \xi * h - \lambda_5 \psi \frac{\gamma}{1} \right)(y) [1 + (F_c l)(y)] \]

\[ = F_s \left( k \frac{1}{1} - \lambda_4 \xi \frac{1}{1} + h - \lambda_5 \psi \frac{\gamma}{1} \right)(y) + F_s \left( k \frac{1}{1} - \lambda_4 (\xi \frac{1}{1}) * l - \lambda_5 (\psi \frac{\gamma}{1}) * l \right)(y). \]

Hence

\[ g = k - \lambda_4 \xi \frac{1}{1} + h - \lambda_5 \psi \frac{\gamma}{1} + k \frac{1}{1} - \lambda_4 (\xi \frac{1}{1}) * l - \lambda_5 (\psi \frac{\gamma}{1}) * l. \]

The proof is complete. \( \square \)

REFERENCES


THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE $F_c, F_s$ TRANSFORMS


E-mail address: thaonxbmai@yahoo.com, hongdhsp1@yahoo.com

Hanoi Water Resources University, 175 Tay Son, Dong Da, Hanoi, Vietnam,