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CONSTRUCTIVE AND INVESTIGATE THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

NGUYEN XUAN THAO AND NGUYEN THANH HONG

ABSTRACT. A generalized convolution with the weight function for the Fourier cosine and sine transforms is introduced. It’s properties and applications to solve systems of integral equations are considered.

1. Introduction

Let $F_s$ be the Fourier sine transform [2]

$$(F_s f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y) dy,$$

and $F_c$ be the Fourier cosine transform [2]

$$(F_c f)(c) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y) dy.$$ 

Convolution theory has been studied in 20th. Firstly, the convolutions for the Fourier; Laplace and Mellin transforms have investigated. Later on, the convolutions for the integral transforms Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have already investigated. The convolution of two functions $f$ and $g$ for the Fourier cosine transform is introduced in [7]

$$(f *_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(|x - y|) + g(x + y)]dy, \quad x > 0,$$

which satisfied the following factorization equality

$$F_c(f *_{F_c} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0.$$ 

In 1958, Vilenkin I.Ya introduced the first convolution with the weight function for the transform Mehler - Fock. In 1967, Kakichev V.A proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [4]). He constructed the convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier
sine transform which is of the form [4] and [10]

\[ (f \star_{F_s} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\text{sign}(x + y - 1)g(|x + y - 1|) + \text{sign}(x - y + 1)g(|x - y + 1|) \]
\[ - g(x + y + 1) - \text{sign}(x - y - 1)g(|x - y - 1|)]dy, \ x > 0, \]

and proved the following factorization identity [4], [10]

\[ F_s(f \star_{F_s} g)(y) = \sin y(Fsf)(y)(Fsg)(y), \ \forall y > 0. \]

The convolution with the weight function \( \gamma(y) = \cos y \) for the Fourier cosine transform of two functions \( f \) and \( g \) is introduced in [11]

\[ (f \star_{F_c} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(|y+x-1|) + g(|y-x-1|) + g(y+x+1) + g(|y-x+1|)]dy, \ x > 0 \]

and satisfy the factorization equality [11]

\[ F_c(f \star_{F_c} g)(y) = \cos y(Fcf)(y)(Fcg)(y), \ \forall y > 0. \]

In 1941, Churchill R.V introduced the first generalized convolution of two functions \( f \) and \( g \) for the Fourier sine and Fourier cosine transforms [7]

\[ (f \star_{F_s} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \ x > 0, \]

and proved the following factorization identity [7]

\[ F_s(f \star_{F_s} g)(y) = (Fsf)(y)(Fsg)(y), \ \forall y > 0. \]

In the nineties of the last century, Yakubovich S. B has introduced several generalized convolutions with index of the Mellin transform, Kontorovich-Lebedev transform, G-transform and H-transform. In 1998, Kakiichev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [5]). Up to now, based on this method, several new generalized convolutions for the integral transforms were established and investigated.

The generalized convolution of two functions \( f \) and \( g \) for the Fourier cosine and sine transforms is defined by [6]

\[ (f \star_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[\text{sign}(u - x)g(|u - x|) + g(u + x)]du, \ x > 0. \]

For this generalized convolution the following factorization equality holds [6]

\[ F_c(f \star_{F_c} g)(y) = (Fsf)(y)(Fsg)(y), \ \forall y > 0. \]
Another generalized convolution with the weight function $\gamma_1(y) = \sin y$ for the Fourier cosine and sine has been studied in [15]

\begin{equation}
(f \ast_2 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)]du, \ x > 0.
\end{equation}

It satisfies the factorization property [15]

\begin{equation}
F_c(f \ast_2 g)(y) = \sin y(F_s f)(y)(F_s g)(y), \ \forall y > 0.
\end{equation}

The generalized convolution of two functions $f$ and $g$ with the weight function $\gamma_1(y) = \sin y$ for the Fourier sine and cosine transforms has the form

\begin{equation}
(f \ast_1 g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)]du, \ x > 0,
\end{equation}

and satisfy the following factorization identity [14]

\begin{equation}
F_s(f \ast_1 g)(y) = \sin y(F_c f)(y)(F_c g)(y), \ \forall y > 0.
\end{equation}

In this paper we construct a new generalized convolution with a weight function for the Fourier cosine and sine transforms. Its properties and the relation with several well-known convolutions and generalized convolutions are considered. We also apply this notion to solve a system of integral equations.

\section{The Generalized Convolution}

**Definition 1.** A generalized convolution with the weight function $\gamma(y) = \cos y$ for the Fourier cosine and sine transforms of functions $f$ and $g$ is defined by

\begin{equation}
(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(y+x+1) + \text{sign}(y-x-1)g(|y-x-1|)
+ \text{sign}(y+x-1)g(|y+x-1|) + \text{sign}(y-x+1)g(|y-x+1|)]dy, \ x > 0.
\end{equation}

We denote by $L(\mathbb{R}_+)$ the space of all functions $f$ defined on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} |f(x)|dx < \infty$.

**Theorem 1.** Let $f$ and $g$ be functions in $L(\mathbb{R}_+)$ then the generalized convolution $(f \ast g)(x)$ defined by (2.1) also be a $L(\mathbb{R}_+)$ function. Moreover, the following factorization equality holds

\begin{equation}
F_c(f \ast g)(y) = \cos y(F_s f)(y)(F_s g)(y), \ \forall y > 0.
\end{equation}
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Proof. From the defining formula of the generalized convolution and the fact that $f, g \in L(\mathbb{R}_+)$ we have

\begin{align}
\int_0^{+\infty} |(f \ast g)(x)| dx &= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} |f(y)| \left[ |g(y + x + 1) + \text{sign}(y - x - 1)g(|y - x - 1|) \right. \\
&\quad + \left. \text{sign}(y + x - 1)g(|y + x - 1|) + \text{sign}(y - x + 1)g(|y - x + 1|) \right] dy dx \\
&\leq \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} |f(y)| \left[ \int_0^{+\infty} |g(y + x + 1)| dx + \int_0^{+\infty} |g(|y - x - 1|)| dx \\
&\quad + \int_0^{+\infty} |g(|y + x - 1|)| dx + \int_0^{+\infty} |g(|y - x + 1|)| dx \right] dy.
\end{align}

On the other hand,

\begin{align}
\int_0^{+\infty} |g(y + x + 1)| dx + \int_0^{+\infty} |g(|y - x - 1|)| dx &= \int_0^{+\infty} |g(t)| dt + \int_{y+1}^{+\infty} |g(|t|)| dt \\
&\quad = 2 \int_0^{+\infty} |g(t)| dt.
\end{align}

Similarly,

\begin{align}
\int_0^{+\infty} |g(|y + x - 1|)| dx + \int_0^{+\infty} |g(|y - x + 1|)| dx &= 2 \int_0^{+\infty} |g(t)| dt.
\end{align}

From (2.3), (2.4) and (2.5) one holds

\begin{align}
\int_0^{+\infty} |(f \ast g)(x)| dx &\leq \sqrt{\frac{2}{\pi}} \int_0^{+\infty} |g(t)| dt \int_0^{+\infty} |g(t)| dt \leq +\infty.
\end{align}

So $(f \ast g)(x)$ belong to $L(\mathbb{R}_+)$. Now we prove the factorization equality (2.2). We have

\begin{align}
\cos y(F_s f)(y)(F_s g)(y) &= \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} \cos y \sin uy \sin vy f(u)g(v) dudv \\
&= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u)g(v) [\cos y(u - v + 1) + \cos y(u - v - 1) - \cos y(u + v + 1) - \cos y(u + v - 1)] dudv.
\end{align}
Changing the variables gives

\[
\cos y(F_s f)(y)(F_s g)(y) = \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_{-u-1}^{+\infty} g(t + u + 1) \cos ytdt + \int_{-u+1}^{+\infty} g(t + u - 1) \cos ytdt \\
- \int_{u+1}^{+\infty} g(t - u - 1) \cos ytdt - \int_{u-1}^{+\infty} g(t - u + 1) \cos ytdt \right] du
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} f(u) \left[ \int_0^{+\infty} g(u + t + 1) \cos ytdt + \int_0^{+\infty} \text{sign}(u - t + 1) g(|u - t + 1|) \cos ytdt \\
+ \int_0^{+\infty} \text{sign}(u + t - 1) g(|u + t - 1|) \cos ytdt + \int_0^{+\infty} \text{sign}(u - t - 1) g(|u - t - 1|) \cos ytdt \right] du
\]

\[
= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} f(u)[g(u + t + 1) + \text{sign}(u - t + 1) g(|u - t + 1|) + \text{sign}(u + t - 1) g(|u + t - 1|) \\
+ \text{sign}(u - t - 1) g(|u - t - 1|)] du \cos ytdt
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (f \ast \bar{g})(t) \cos ytdt
\]

\[
= F_c(f \ast \bar{g})(y).
\]

It shows that

\[
F_c(f \ast \bar{g})(y) = \cos y(F_s f)(y)(F_s g)(y).
\]

The proof of Theorem is completes. \qed

**Theorem 2.** In the space \(L(\mathbb{R}_+)\) the generalized convolution \((2.1)\) is not associative and the relation with knowns convolutions and generalized convolutions as follows

a) \(f \ast (g \ast h) = g \ast (f \ast h) = h \ast (g \ast f)\), where \(f \ast g\) is defined by \((1.7)\).

b) \(f \ast (g \ast h) = (f \ast g) \ast h = (f \ast h) \ast g\), where \(f \ast g\) is the Fourier cosine convolution \((1.1)\).

c) \(f \ast \frac{e}{2} (g \ast h) = (f \ast \frac{g}{2}) \ast h = h \ast (f \ast \frac{g}{2})\), where \(f \ast g\) and \(f \ast \frac{g}{2}\) are defined by \((1.3)\) and \((1.11)\).

d) \(f \ast (g \ast h) = (f \ast \frac{g}{2}) \ast F_c \ast h = (f \ast \frac{g}{2} \ast F_c) \ast h\), where \(f \ast g\) is the Fourier cosine convolution with a weight function \((1.5)\).
Proof. a) From the factorization equality, we have
\[ F_s(f \ast (g \ast h))(y) = (F_s f)(y)(F_c(g \ast h))(y) = (F_s f)(y) \cos y(F_s g)(y)(F_s h)(y) = (F_s g)(y)(F_c(f \ast h))(y) = F_s(g \ast (f \ast h))(y). \]

Which implies that \( f \ast (g \ast h) = g \ast (f \ast h) \)

On the other hand,
\[ F_s(f \ast (g \ast h))(y) = (F_s f)(y) \cos y(F_s g)(y)(F_s h)(y) = (F_s h)(y) \cos y(F_s g)(y)(F_s f)(y) = F_s(g \ast (f \ast h))(y). \]

Then we obtain the part a). The parts b), c), d) can be obtained in a similar way. The Theorem is proved.

Theorem 3. In the space \( L(R_+) \) the generalized convolution (2.1) does not have a unit element.

Proof. Suppose that there exists a unit element \( e \) of the generalized convolution (2.1) in \( L(R_+) \).

It means that \( f \ast e = e \ast f = f \) for any function \( f \in L(R_+) \). It follows that \( F_c(f \ast e)(y) = (F_c f)(y), \forall y > 0 \).

Hence, \( \cos y(F_s e)(y)(F_s f)(y) = (F_s f)(y), \forall y > 0 \).

Choosing \( f(x) = e^{-x} \in L(R_+) \). From the fact that
\[ (F_c f)(y) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + y^2}, \quad (F_s f)(y) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + y^2}, \]

we obtain
\[ (F_s e)(y) = \frac{1}{y \cos y}. \]

It is a contradiction from the fact that \( \frac{1}{y \cos y} \notin L_\infty(R_+) \) while \( (F_s e)(y) \in L_\infty(R_+) \) since \( e \in L(R_+) \).

The Theorem is proved.

Let \( L(R_+, e^x) = \{ h, \text{ for all } e^x h(x) \in L(R_+) \} \).

Theorem 4 (Titchmarsh-type Theorem). Let \( f, g \in L(R_+, e^x) \). If \( (f \ast g)(x) \equiv 0 \) then either \( f \equiv 0 \) or \( g \equiv 0 \).

Proof. Suppose that \( (f \ast g)(x) = 0, \forall x > 0 \), in view of Theorem 1

\[ F_c(f \ast g)(y) = \cos y(F_s f)(y)(F_s g)(y) = 0, \forall y > 0. \]
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We have
\[ \left| \frac{d^n}{dy^n} [\sin(xy)f(x)] \right| = \left| f(x)x^n \sin(xy + n \frac{\pi}{2}) \right| \leq \left| f(x)x^n \right| = e^{-x}.x^n.\left| f_1(x) \right| \leq n!\left| f_1(x) \right|, \]
where $f_1(x) = e^x f(x) \in L(\mathbb{R}_+)$. Due to Weierstrass criterion, the integral $\int_0^{+\infty} \frac{d^n}{dy^n} [\sin(xy)f(x)]dx$ uniformly converges on $\mathbb{R}_+$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_s f)(y)$ is analytic. Similarly, $(F_s g)(y)$ is analytic. So from (2.6) we have $(F_s f)(y) \equiv 0$ or $(F_s g)(y) \equiv 0$. It completes the proof. □

3. APPLICATION TO SOLVE SYSTEMS OF INTEGRAL EQUATIONS

Consider a system of integral equations
\[
\begin{align*}
  f(x) + \lambda_1 \int_0^{+\infty} g(t)\varphi_1(x,t)dt + \lambda_2 \int_0^{+\infty} g(t)\eta_1(x,t)dt + \lambda_3 \int_0^{+\infty} g(t)\mu_1(x,t)dt &= h(x) \\
  \lambda_4 \int_0^{+\infty} \xi(t)[f(x+t) - f(|x-t|)]dt + \lambda_5 \int_0^{+\infty} g(t)\psi_1(x,t)dt + g(x) &= k(x),
\end{align*}
\]
(3.1)

here $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are complex numbers, $\varphi, \eta, \mu, \xi \in L(\mathbb{R}_+)$, and
\[
\begin{align*}
  \varphi_1(x,t) &= \frac{1}{2\sqrt{2\pi}} \left[ \varphi(t+x+1) + \text{sign}(t-x-1)\varphi(|t-x-1|) + \text{sign}(t+x-1)\varphi(|t+x-1|) \\
  &\quad + \text{sign}(t-x+1)\varphi(|t-x+1|) \right], \\
  \eta_1(x,t) &= \frac{1}{2\sqrt{2\pi}} \left[ \eta(|x+t-1|) + \eta(|x-t+1|) - \eta(x+t+1) - \eta(|x-t-1|) \right], \\
  \mu_1(x,t) &= \frac{1}{2\sqrt{2\pi}} \left[ \text{sign}(t-x)\mu(|t-x|) + \mu(t+x) \right], \\
  \psi_1(x,t) &= \frac{1}{2\sqrt{2\pi}} \left[ \psi(|x+t-1|) + \psi(|x-t+1|) - \psi(x+t+1) - \psi(|x-t+1|) \right],
\end{align*}
\]
and $\psi(x) = (v_1 * v_2)(x)$.

It shows that $\varphi_1, \eta_1, \mu_1, \psi$ and $\psi_1$ are also $L(\mathbb{R}_+)$ - functions

Theorem 5. With the condition
\[
F_e(\lambda_1\lambda_4 \varphi \ast \xi + \lambda_2\lambda_4 \xi \ast \eta + \lambda_3\lambda_4 \mu \ast \xi + \lambda_1\lambda_5 \varphi \ast (\nu_1 \ast \nu_2) + \lambda_2\lambda_5 \nu_1 \ast (\nu_2 \ast \eta) + \lambda_3\lambda_5 \mu \ast \psi)(y) \neq 1,
\]

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there exists a unique solution in \(L(\mathbb{R}^+)\) of system (3.1) which is of the form

\[
f = h - \lambda_1 \varphi \ast k - \lambda_2 \mu \ast \eta - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (\mu \ast \eta) \ast l - \lambda_3 (\mu \ast k) \ast l
\]
\[
g = k - \lambda_4 \xi \ast h - \lambda_5 \psi \ast \eta - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \ast \eta) \ast l
\]

Here \(l \in L(\mathbb{R}^+)\) and is defined by

\[
F_c l = \frac{F_c (\lambda_1 \lambda_4 \varphi \ast \mu + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \eta (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \eta (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)}{1 - F_c (\lambda_1 \lambda_4 \varphi \ast \mu + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \eta (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \eta (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)}
\]

**Proof.** System (3.1) can be rewritten in the form

\[
f(x) + \lambda_1 (g \ast \varphi)(x) + \lambda_2 (g \ast \eta)(x) + \lambda_3 (g \ast \mu)(x) = h(x)
\]

(3.2)

\[
\lambda_4 (\xi \ast f)(x) + \lambda_5 (f \ast \psi)(x) + g(x) = k(x).
\]

Using the respectively factorization equalities of above generalized convolutions we have

\[
(F_c f)(y) + \lambda_1 \cos (F_c g)(y)(F_c \varphi)(y) + \lambda_2 \sin (F_c g)(y)(F_c \eta)(y) + \lambda_3 (F_c g)(y)(F_c \mu)(y) = (F_c h)(y)
\]

\[
\lambda_4 (F_c \xi)(y)(F_c f)(y) + \lambda_5 \sin (F_c f)(y)(F_c \psi)(y) + (F_c g)(y) = (F_c k)(y).
\]

We have

\[
\Delta = \left| \begin{array}{ccc}
1 & \lambda_1 \cos (F_c \varphi)(y) + \lambda_2 \sin (F_c \eta)(y) + \lambda_3 (F_c \mu)(y) \\
\lambda_4 (F_c \xi)(y) + \lambda_5 \sin (F_c \psi)(y) & 1 \\
\end{array} \right|
\]

\[
= 1 - \lambda_1 \lambda_4 \cos (F_c \varphi)(y)(F_c \xi)(y) - \lambda_2 \lambda_4 \sin (F_c \eta)(y)(F_c \xi)(y) - \lambda_3 \lambda_4 \sin (F_c \mu)(y)(F_c \xi)(y) - \lambda_1 \lambda_5 \sin (F_c \varphi)(y)(F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 (F_c \eta)(y)(F_c \psi)(y) - \lambda_3 \lambda_5 \sin (F_c \mu)(y)(F_c \psi)(y) - \lambda_1 \lambda_4 \cos (F_c \varphi)(y)(F_c \xi)(y) - \lambda_2 \lambda_4 \sin (F_c \eta)(y)(F_c \xi)(y) - \lambda_3 \lambda_4 \sin (F_c \mu)(y)(F_c \xi)(y) - \lambda_1 \lambda_5 \sin (F_c \varphi)(y)(F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 (F_c \eta)(y)(F_c \psi)(y) - \lambda_3 \lambda_5 \sin (F_c \mu)(y)(F_c \psi)(y) \\
1 - \lambda_1 \lambda_4 \cos (F_c \varphi)(y)(F_c \xi)(y) - \lambda_2 \lambda_4 \sin (F_c \eta)(y)(F_c \xi)(y) - \lambda_3 \lambda_4 \sin (F_c \mu)(y)(F_c \xi)(y) - \lambda_1 \lambda_5 \sin (F_c \varphi)(y)(F_c \psi)(y) - \lambda_2 \lambda_5 \sin^2 (F_c \eta)(y)(F_c \psi)(y) - \lambda_3 \lambda_5 \sin (F_c \mu)(y)(F_c \psi)(y)
\end{array} \right|
\]

Then due to Wiener-Levi’s Theorem [10] there exists a function \(l \in L(\mathbb{R}^+)\) such that

\[
\frac{1}{\Delta} - 1 = \frac{F_c (\lambda_1 \lambda_4 \varphi \ast \mu + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \eta (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \eta (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)}{1 - F_c (\lambda_1 \lambda_4 \varphi \ast \mu + \lambda_2 \lambda_4 \xi \ast \eta + \lambda_3 \lambda_4 \mu \ast \xi + \lambda_1 \lambda_5 \varphi \ast \eta (\nu_1 \ast \nu_2) + \lambda_2 \lambda_5 \nu_1 \ast \eta (\nu_2 \ast \eta) + \lambda_3 \lambda_5 \mu \ast \psi)}
\]

\[
= (F_c l)(y),
\]
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furthermore

$$\begin{align*}
\Delta_1 &= \begin{vmatrix}
(F_c k)(y) & \lambda_1 \cos y(F_c \varphi)(y) + \lambda_2 \sin y(F_c \eta)(y) + \lambda_3 (F_c \mu)(y) \\
(F_s k)(y) & 1
\end{vmatrix} \\
&= (F_c h)(y) - \lambda_1 \cos y(F_c \varphi)(y)(F_s k)(y) - \lambda_2 \sin y(F_c \eta)(y)(F_s k)(y) - \lambda_3 (F_s \mu)(y)(F_s k)(y) \\
&= F_c \left( h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\eta}{2} - \lambda_3 \mu \ast k \right).
\end{align*}$$

Therefore

$$\begin{align*}
(F_s f)(y) &= \frac{\Delta_1}{\Delta} = F_c \left( h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\eta}{2} - \lambda_3 \mu \ast k \right)[1 + (F_c l)(y)] \\
&= F_c \left( h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\eta}{2} - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \frac{\eta}{2} \ast l) - \lambda_3 (\mu \ast k \ast l) \right)(y)
\end{align*}$$

It shows that

$$f = h - \lambda_1 \varphi \ast k - \lambda_2 k \frac{\eta}{2} - \lambda_3 \mu \ast k + h \ast l - \lambda_1 (\varphi \ast k) \ast l - \lambda_2 (k \frac{\eta}{2} \ast l) - \lambda_3 (\mu \ast k \ast l)$$

Similarly

$$\begin{align*}
\Delta &= \begin{vmatrix}
1 & (F_c h)(y) \\
\lambda_4 (F_s \xi)(y) + \lambda_5 \sin y(F_c \psi)(y) & (F_s k)(y)
\end{vmatrix} \\
&= (F_s k)(y) - \lambda_4 (F_s \xi)(y)(F_c h)(y) - \lambda_5 \sin y(F_c \psi)(y)(F_c h)(y) \\
&= F_s \left( k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\eta}{1} \ast h \right)(y).
\end{align*}$$

Then

$$\begin{align*}
(F_s g)(y) &= \frac{\Delta_2}{\Delta} = F_s \left( k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\eta}{1} \ast h \right)(y)[1 + (F_c l)(y)] \\
&= F_s \left( k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\eta}{1} \ast h + k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \frac{\eta}{1} \ast h) \ast l \right)(y).
\end{align*}$$

Hence

$$g = k - \lambda_4 \xi \ast h - \lambda_5 \psi \frac{\eta}{1} \ast h + k \ast l - \lambda_4 (\xi \ast h) \ast l - \lambda_5 (\psi \frac{\eta}{1} \ast h) \ast l.$$

The proof is complete.

REFERENCES


THE GENERALIZED CONVOLUTION WITH THE WEIGHT FUNCTION FOR THE \( F_c, F_s \) TRANSFORMS


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