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DYNAMICS OF POPULATIONS DESCRIBED BY A LOTKAVOLTERA EQUATION UNDER RANDOM ENVIRONMENT

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ABSTRACT

In this paper we survey some results about the trajectory behavior of Lotka-Volterra competition system under random environment. This survey is based on our researches during the last years.

1. INTRODUCTION

We consider the Lotka - Volterra system

$$\begin{cases} \dot{x} = x(a(t) - b(t)x - c(t)y) \\ \dot{y} = y(d(t) - e(t)x - f(t)y), \end{cases} \tag{1.1}$$

where a, b, c, d, e, f are continuous functions. We suppose that a, b, c, d, e, f are bounded above and below by positive constants. This is a model of two competing species whose quantities at time t are $x(t)$ and $y(t)$. The functions a and d are the respective intrinsic growth rates; b and f measure the respective intraspecific competition within species x and y and the functions c, e measure the interspecific competitions between two species. The details of the ecological significance of such system are discussed by many authors.

It is known that for Equation (1.1) the quadrant plane $R^2_+ = \{(u, v) : 0 < u < \infty; 0 < v < \infty\}$ is invariant, i.e., if $(x(t), y(t))$ is a solution of (1.1) with $x(t_0) > 0$, $y(t_0) > 0$ for some $t_0 \in R$ then $x(t) > 0$, $y(t) > 0$ for any $t \in (-\infty, \infty)$.

The trajectory of (1.1) as $t \to \infty$ is rather complicated meanwhile it is easy to deal with numerical solutions. Therefore, we will add some further conditions on coefficients of the equation (1.1) to investigate he trajectory behavior of (1.1) at infinity. First, we look at the equation (1.1) by weaken Ahmad conditions in [1]. We suppose that

$$\limsup_{|t|\to\infty} \frac{a(t)}{b(t)} < \liminf_{|t|\to\infty} \frac{d(t)}{e(t)}, \tag{1.2}$$

$$\limsup_{|t|\to\infty} \frac{d(t)}{f(t)} < \liminf_{|t|\to\infty} \frac{a(t)}{c(t)}, \tag{1.3}$$

then we obtain

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Theorem 1.1. *Under the conditions* (1.2) *and* (1.3),

1. *The equation* (1.1) *has a unique solution defined on* $(-\infty, \infty)$ *which is bounded above and below by positive constants*
2. *This solution (denoted* $(x^*(t), y^*(t))$*) by is a forward attractive one in the sense that*

$$\lim_{t\to+\infty}(x(t) - x^*(t)) = 0; \quad \lim_{t\to+\infty}(y(t) - y^*(t)) = 0.$$

3. *This solution repels every other backward solution in the sense that if there is a* t_0 *such that either* $x(t_0) > x^*(t_0)$ *or* $y(t_0) > y^*(t_0)$ *then* $(x(t), y(t))$ *is explosive. For other cases*

$$\lim_{t\to-\infty} x(t) = 0; \quad \lim_{t\to-\infty} y(t) = 0.$$

The conditions (1.2) and (1.3) ensure that the vector field of (1.1) always gets into the interior from the boundary of R^2_+. Therefore, it is easy to understand this unique bounded solution is attractive. We can interpret theorem 1.1 in a ecosystem meaning as follows: if the interspecific and intrinsic coefficients satisfy the conditions (1.2) and (1.3), the quantities of two species in such eco-systems tend to equilibrium states.

Further, if these coefficients are random variables we have

Theorem 1.2. *The solution col*$(x(t), y(t))$ *mentioned in theorem 1.1 is a stationary process*

2. Non-autonomous Lotka-Volterra competition system under the bistable hypothesis

We are now interested in the case where the inequalities (1.2), (1.3) are conversed. In this case, the vector field of (1.1) is bistable. This means that We now consider the Lotka - Volterra equation (1.1) with the following hypotheses:

Hypotheses 2.1. 1. There exist two constants m, M such that

$$m < g < M \quad \text{for any} \quad g := a, b, c, d, e, f.$$

2.

$$\liminf_{|t|\to\infty} \frac{a(t)}{b(t)} > \limsup_{|t|\to\infty} \frac{d(t)}{e(t)}, \tag{2.1}$$

$$\liminf_{|t|\to\infty} \frac{d(t)}{f(t)} > \limsup_{|t|\to\infty} \frac{a(t)}{c(t)} \cdot \tag{2.2}$$

3.

$$\liminf_{|t|\to\infty} \frac{c(t)}{f(t)} > \limsup_{|t|\to\infty} \frac{b(t)}{e(t)} \cdot \tag{2.3}$$

Although this case is very sensible, we still are able to prove that there exists solution having a similar property as in theorem 1.1.

Let $t_0 \in R$ arbitrary, we consider the forward equation of (1.1), i.e., for $t > t_0$.

Proposition 2.2. *Every solution (x, y) of Equation (1.1) is bounded above on $[t_0, +\infty)$.*

We will show that under Conditions (2.1) and (2.2), either every solution of forward equation (1.1) is strictly positive or it has a coordinate tending to 0 as $t \to \infty$. Firstly, we consider two "marginal" equations

$$\dot{u}(t) = u(t)[a(t) - b(t)u(t)] \tag{2.4}$$

$$\dot{v}(t) = v(t)[d(t) - f(t)v(t)] \tag{2.5}$$

Suppose that $u(t, s, x)$ is the solution of the equation (2.4) satisfying the initial conditions $u(s, s, x) = x$ and $v(t, s, y)$ is the solution of (2.5) with $v(s, s, y) = y$.

Proposition 2.3. *For any $x \in R_+$, there exists a $T = T(x) > 0$ such that if $t > s > t_0$ and $t - s > T$ then*

$$u(t, s, x) > k_1 + \gamma/2; \qquad v(t, s, x) > k_2 + \gamma/2$$

We now turn to estimate solutions of (1.1). Denote by $(x(t, s, x_0, y_0), y(t, s, x_0, y_0))$ the solution of (1.1) satisfying $(x(s, s, x_0, y_0), y(s, s, x_0, y_0)) = (x_0, y_0)$.

Proposition 2.4. *There exist a neighborhood $\mathcal{U}$ of $R_+ \times \{0\}$ and a neighborhood $\mathcal{V}$ of $\{0\} \times R_+$ on $[0, \infty) \times [0, \infty)$ such that for any $s \geqslant t_0$*

a) If $(x_0, y_0) \in \mathcal{U}$ then

$$\lim_{t \to \infty} y(t, s, x_0, y_0) = 0; \quad \lim_{t \to \infty} [x(t, s, x_0, y_0) - u(t, s, x_0)] = 0$$

b) If $(x_0, y_0) \in \mathcal{V}$ then

$$\lim_{t \to \infty} x(t, s, x_0, y_0) = 0; \quad \lim_{t \to \infty} [y(t, s, x_0, y_0) - v(t, s, y_0)] = 0$$

We denote $\mathcal{A}$ the set of $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\lim_{t \to \infty} x(t, t_0, x, y) = 0$ and $\mathcal{B}$ the set of (x, y) such that $\lim_{t \to \infty} y(t, t_0, x, y) = 0$.

Proposition 2.5. *$\mathcal{A}$ and $\mathcal{B}$ are open sets. Moreover for any $x_0 > 0$ and $y_0 > 0$, the sets $\mathcal{A} \cap \{x_0\} \times \mathbb{R}_+$ and $\mathcal{B} \cap \mathbb{R}_+ \times \{y_0\}$ are two open intervals.*

Proposition 2.6. *On every line $x = x_0$, there exists at most one point (x_0, y_0) such that the solution starting at (x_0, y_0) at t_0 is bounded above and below by positive constants. A similar result can be formulated on every line $y = y_0$.*

Corollary 2.7. *If $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions of (1.1), bounded above and below by positive constants then inequality $x_1(t_0) < x_2(t_0)$ implies the inequality $y_1(t_0) < y_2(t_0)$.*

Summing up, we have

Proposition 2.8. *There exists a number* $a > 0$ *and a strictly increasing, continuous function* $\varphi : [0, a) \to \mathbb{R}^+$ *such that every solution starting at* $(x, \varphi(x))$; $0 < x < a$ *at* t_0 *is bounded above and below by positive constants. Furthermore, for any* $(x_0, y_0) \notin$ *graph* ϕ, *either* $\lim_{t\to\infty} x(t, t_0, x_0, y_0) = 0$ *or* $\lim_{t\to\infty} y(t, t_0, x_0, y_0) = 0$.

We now pass to study the behavior of solutions when $t \to -\infty$. Consider the backward system of (1.1)

$$\begin{cases} \dot{x} = x\big(a(t) - b(t)x - c(t)y\big) \\ \dot{y} = y\big(d(t) - e(t)x - f(t)y\big) \end{cases}, \quad t \leqslant -t_0$$

Proposition 2.9. *If* $\limsup_{t\to-\infty}[x(t) + y(t)] = +\infty$ *then* $\lim_{t\to\infty} x(t) = +\infty$ *and* $\lim_{t\to\infty} y(t) = +\infty$

Proposition 2.10. *There exist two backward invariant open sets, namely* $\mathcal{U}_1$ *and* $\mathcal{V}_1$ *such that* $(0, k_1) \times \{0\} \subset \mathcal{U}_1$; $\{0\} \times (0, k_2) \subset \mathcal{V}_1$ *and if* $(x_0, y_0) \in \mathcal{U}_1$ *or* $(x_0, y_0) \in \mathcal{V}_1$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$.

Corollary 2.11. *If* $\liminf_{t\to-\infty} x(t) = 0$ *or* $\liminf_{t\to-\infty} y(t) = 0$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$.

Summing up, we have:

Proposition 2.12. *There is a continuous strictly decreasing function* $\psi : [0, u_0] \to R_+$; $\psi(0) = v_0$ *and* $\psi(u_0) = 0$ *such that*

1) If $\psi(x(t_0)) < y(t_0)$ *or* $x(t_0) > u_0$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = \infty$.

2) If $\psi(x(t_0)) > y(t_0)$ *and* $x(t_0) < u_0$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$.

3) If $\psi(x(t_0)) = y(t_0)$ *then* $(x(t), y(t))$ *is bounded above and below by positive constant on* $(-\infty, -t_0]$.

Proposition 2.13. *Under the conditions (2.1), (2.2) and (2.3), System (1.1) has a unique solution bounded above and below by positive constant on* $(-\infty, \infty)$.

Thus, in this case there exists a positive solution which is bounded above and below by positive constants but this solution is not stable. The graph of this solution divides the sate space $\mathbb{R}_2^+$ into two parts such that if the initial condition belongs to one part, the first species will disappear meanwhile if the initial condition belongs to another part, the second species will be vanished.

3. Lotka-Volterra competition systems under the telegraph noises

In this section and the following section we study the trajectory behavior of Lotka-Volterra competition system under the telegraph noises. Up to the present, many models reveal the effect of environmental variability on the population dynamics in mathematical ecology. Especially a great effort has been expended to find the possibility of coexistence of competing species under the unpredictable or rather predictable (such as seasonal) environmental fluctuations. It is well recognized that the seasonality has similar effects to stochastic variation, but as is pointed out, the theory of coexistence in a seasonal environment needs further development to reveal a variety of possibilities that seasonal fluctuations may exhibit. Among these, one reviews the result of Lotka-Volterra competition system with periodic coefficients, and show the new modes of the possibilities of stable periodic solutions even when the stable coexistence cannot be realized in the corresponding classical Lotka-Volterra system with constant coefficients.

Here, we give the restrictive condition for the competition parameters, so there is no possibility of multiple periodic solutions. In our separate papers [3] and [4], we analyze the Lotka-Volterra competition system with constant coefficients and periodic coefficients under the telegraph noises, i.e., environmental variability causes the parameter switching between two systems. Then we consider the situation that the interacting populations experience pseudo-stochastic environmental fluctuations; completely predictable environmental fluctuations with unpredictable discontinuous change, such as seasonality in a year with "a cycle of three cold days and four warm days".

The noise that interferes into the model is a non-memorableness, i.e., its evolution depends only the present state. Further, it has the exponentially distributed spontaneous switching between only two environmental sub-states under the periodic environmental fluctuations, and shows the complex behavior in the transient due to stochastic dynamics and the existence of the oscillatory attractor in the limit.

Let us consider a Markov process $(\xi_t)_{t\geqslant 0}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in the set of two elements, say $E = \{+, -\}$. Suppose that (ξ_t) has the transition intencities $+ \xrightarrow{\alpha} -$ and $- \xrightarrow{\beta} +$ with $\alpha > 0, \beta > 0$. The process (ξ_t) has a unique stationary distribution

$$p = \frac{\beta}{\alpha+\beta};\ q = \frac{\alpha}{\alpha+\beta}. \tag{3.1}$$

We now consider the competition equation

$$\begin{cases} \dot{x} = x\big(a(\xi_t) - b(\xi_t)x - c(\xi_t)y\big) \\ \dot{y} = y\big(d(\xi_t) - e(\xi_t)x - f(\xi_t)y\big) \end{cases}, \tag{3.2}$$

where $g : E \to \mathbb{R}_+$ for $g = a, b, c, d, e, f$.

The process (ξ_t) interferes in the Equation (??) as a noise which is well-known as real noise form. Without the noise (ξ_t), i.e., $g(\xi_t, t) = g(t)$ for any $g = a, b, c, d, e, f$, Equation (3.2) is well studied.

In case the noise (ξ_t) intervenes virtually into Equation (3.2), it makes a switching between the deterministic periodic system

$$\begin{cases} \dot{x}^+(t) = x^+(t)(a(+) - b(+)x^+(t) - c(+)y^+(t)), \\ \dot{y}^+(t) = y^+(t)(d(+) - e(+)x^+(t) - f(+)y^+(t)), \end{cases} \tag{3.3}$$

and the deterministic periodic one

$$\begin{cases}\dot{x}^-(t) = x^-(t)(a(-) - b(-)x^-(t) - c(-)y^-(t)),\\ \dot{y}^-(t) = y^-(t)(d(-) - e(-)x^-(t) - f(-)y^-(t)).\end{cases} \tag{3.4}$$

Thus, the relationship of these two systems will determine the trajectory behavior of Equation (3.2).

Since whenever $x(t)$ and $y(t)$ are small (respectively: large), $x(t)\uparrow$ and $y(t)\uparrow$ (respect. $x(t)\downarrow$ and $y(t)\downarrow$) then there are two constants k_3, M satisfying $0 < k_3 < \min\{x^+, y^+, x^-, y^-\} < M$ ((x^-, y^-) is a unique solution of the equation $a(-) - b(-)x - c(-)y = 0; d(-) - e(-)x - f(-)y = 0$) and $t_0 > 0$ such that $x(t) < M, y(t) < M$ and either $x(t) \geqslant k_3$ or $y(t) \geqslant k_3$ for any $t \geqslant t_0$. Therefore, without loss of generality, we suppose that either $M > x(t) \geqslant k_3$ or $M > y(t) \geqslant k_3$ for any $t \geqslant 0$.

Proposition 3.1. *a) If*

$$\lambda := \int_{[d(+)/f(+), d(-)/f(-)]} \big(p(a(+) - c(+)v)\nu^+(v) + q(a(-) - c(-)v)\nu^-(v)\big)dv > 0,$$

then $\limsup_{t\to\infty} x(t, x, y) > 0$ *for* $P-$*a.s. and for any* $x > 0, y > 0$.

b) If

$$\theta := \int_{[a(+)/b(+), a(-)/b(-)]} \big(p(d(+) - e(+)u)\mu^+(u) + q(d(-) - e(-)u)\mu^-(u)\big)du > 0,$$

then $\limsup_{t\to\infty} y(t, x, y) > 0$ *for* $P-$*a.s. and for any* $x > 0, y > 0$.

To get the further information on the trajectory behavior of the solutions of (3.2), we need the following hypothesis:

Hypotheses 3.2. The coefficients of the equation (3.2) satisfy:

$$a) \qquad a(+)/b(+) < d(+)/e(+); \quad a(+)/c(+) > d(+)/f(+) \tag{3.5}$$

$$b) \qquad a(-)/b(-) > d(-)/e(-); \quad a(-)/c(-) < d(-)/f(-) \tag{3.6}$$

The inequalities (3.5) are only a special case of (1.2) (1.3) and the inequalities (3.6) are of (2.1) and (2.2).

Lemma 3.3. *For any $M > 0$ and small $\delta > 0, \delta_1 > 0$, there exists a $T_3^* = T_3^*(\delta, \delta_1) > 0$ such that $(x^+(t), y^+(t)) \in U_{\delta_1}$ for any $t \geqslant T_3^*$, provided that $\delta < x^+(0) < M, \delta < y^+(0) < M$.*

Lemma 3.4. *There exists ε_1 ($k_3 > \varepsilon_1 > 0$) such that if $(x^+(t_1), y^+(t_1)) \in [k_3, \infty) \times [0, \varepsilon_1]$ then $x^+(t) \geqslant k_3$ $\forall t > t_1$. Moreover, if $y^+(t_2) < \varepsilon_1$ then $\sup_{t_1 < t < t_2} y^+(t) < \varepsilon_1$.*

There is a similar result for the case $(x^+(t_1), y^+(t_1)) \in [0, \varepsilon_1] \times [k_3, \infty)$

Lemma 3.5. *Let $(x^-(t, x, y), y^-(t, x, y))$ be the solution of the equation*

$$\begin{cases}\dot{x}^-(t) = x^-(t)(a(-) - b(-)x^-(t) - c(-)y^-(t)),\\ \dot{y}^-(t) = y^-(t)(d(-) - e(-)x^-(t) - f(-)y^-(t)),\end{cases}$$

with $(x^-(0,x,y),y^-(0,x,y)) = (x,y)$. *For any* $\varepsilon_1 > 0$ *with* $[k_3,\infty)\times[0,\varepsilon_1] \in \mathcal{U}$ *(see (2.10)), we can find* $\varepsilon_2 > 0$ *such that if* $(x^-(t_1),y^-(t_1)) \in [k_3,\infty)\times[0,\varepsilon_2]$ *then* $x^-(t) \geqslant k_3$ *for any* $t \geqslant t_1$ *and* $\sup_{t>t_1} y^-(t) \leqslant \varepsilon_1$. *There is a similar result for the case* $(x^-(t_1),y^-(t_1)) \in [0,\varepsilon_2]\times[k_3,\infty)$.

Lemma 3.6. *There is* $S_1 > 0$ *such that if* $0 < x(t) < M$, $\varepsilon \leqslant y(t) < M$ *then* $\inf_{0<s<S_1} y(t+s) \geqslant \varepsilon/2$.

Proposition 3.7. *Suppose that Condition (3.10), (3.11) hold. Let* $\omega(x,y)$ *be the* $\omega-$*limit set of the solution* $(x(t,x,y),y(t,x,y))$ *of (3.2) with* $x>0, y>0$. *Then,* $(x^+,y^+) \in \omega(x,y)$.

Proposition 3.8. *Suppose that* $x > 0, y > 0$ *and* $\omega(x,y)$ *is the* $\omega-$ *limit set of the solution* $(x(t,x,y),y(t,x,y)$.

a) The positive orbit γ^- *of the solution* $(x^-(t,x^+,y^+),y^-(t,x^+,y^+))$ *of the equation (3.15) is a subset of* $\omega(x,y)$.

b) If $(x^+,y^+) \in \mathcal{A}$ *then the intervals* $\big[(a(-)/b(-),0);(a(+)/b(+),0)\big] \subset \omega(x,y)$.

c) If $(x^+,y^+) \in \mathcal{B}$ *then the intervals* $\big[(d(-)/f(-),0);(d(+)/f(+),0)\big] \subset \omega(x,y)$.

d) If $(x^+,y^+) \in \ell$ *then the part of* ℓ *linking* (x^+,y^+) *and* (x^-,y^-) *belongs to* $\omega(x,y)$. *Moreover, the positive orbit* γ^+ *of the solution* $(x^+(t,x^-,y^-),y^+(t,x^-,y^-))$ *of (3.14) is a subset of* $\omega(x,y)$. *In addition, if* $\gamma^+ \cap \mathcal{A} \neq \emptyset$ *then* $\big[(a(-)/b(-),0);(a(+)/b(+),0)\big] \subset \omega(x,y)$; *if* $\gamma^+ \cap \mathcal{B} \neq \emptyset$ *then* $\big[(d(-)/f(-),0);(d(+)/f(+),0)\big] \subset \omega(x,y)$; *if* $\gamma^+ \subset \ell$ *then* $\omega(x,y)$ *is the part of* ℓ *linking* (x^+,y^+) *and* (x^-,y^-).

We illustrate the above model by two following numerical examples

Case I: Systems do not satisfy the hypotheses (3.10) and (3.11). The solution $(x(t,x,y),y(t,x,y))$ has a component tending to 0.

Case II: The solution $(x(t,x,y),y(t,x,y)$ oscillates between the stable point (x^+,y^+) and the boundary point $(0,d(-)/f(-))$.

4. EVOLUTION OF PERIODIC POPULATION SYSTEMS

We consider the competition Lotka-Volterra equation

$$\begin{cases} \dot{x} = x\left(a(\xi_t,t) - b(\xi_t,t)x - c(\xi_t,t)y\right) \\ \dot{y} = y\left(d(\xi_t,t) - e(\xi_t,t)x - f(\xi_t,t)y\right) \end{cases}, \tag{4.1}$$

where $g : E \to \mathbb{R}_+$ for $g = a,b,c,d,e,f$ such that $g(i,\cdot)$ are continuous and periodic functions with period $T > 0$ for any $i \in E$. Suppose that m and M are two constants such that

$$m \leqslant g(i,t) \leqslant M,\ i \in E, t \in \mathbb{R} \quad \text{for } g = a,b,c,d,e,f.$$

We consider two marginal equations

$$\dot{x} = x\,(a(\xi_t, t) - b(\xi_t, t)x) \tag{4.2}$$
$$\dot{y} = y\,(d(\xi_t, t) - f(\xi_t, t)y). \tag{4.3}$$

Theorem 4.1. *a) If*

$$\lambda := \frac{1}{T}\mathbb{E}\left[\int_0^T (a(\xi_t, t) - c(\xi_t, t)v^*(t))\,dt\right] > 0 \tag{4.4}$$

then $\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s, x, y)\,ds > 0$ *for* $P-a.s.$ *and for any* $x > 0, y > 0$.

b) If

$$\gamma := \frac{1}{T}\mathbb{E}\left[\int_0^T (d(\xi_t, t) - e(\xi_t, t)u^*(t))\,dt\right] > 0 \tag{4.5}$$

then $\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s, x, y)\,ds > 0$ *for* $P-a.s.$ *and for any* $x > 0, y > 0$. *Here* $(x(t, x, y), y(t, x, y))$ *is a solution of Equation (4.1).*

Remark 4.2. a) We note that the average permanence implies that $\limsup_{t\to\infty} x(t) > 0$ and $\limsup_{t\to\infty} y(t) > 0$.

b) The conditions (4.4) and (4.5) are easily to be checked by simulation method based on the law of large numbers.

To get the further properties of the trajectories of the solutions of (4.1), we need some further hypothesis and we study it under the concrete cases.

Hypotheses 4.3. The coefficients of the equation (3.3) and (3.4) satisfy:

$$\sup_{0<t<T} \frac{a(i,t)}{b(i,t)} < \inf_{0<t<T} \frac{d(i,t)}{e(i,t)}; \tag{4.6}$$
$$\inf_{0<t<T} \frac{a(i,t)}{c(i,t)} > \sup_{0<t<T} \frac{d(i,t)}{f(i,t)}; \tag{4.7}$$
$$\inf_{0<t<T} \frac{b(i,t)}{e(i,t)} > \sup_{0<t<T} \frac{c(i,t)}{f(i,t)}, \tag{4.8}$$

for any $i \in E$.

By (4.6), (4.7), there are constants $k_1^{\pm}, k_2^{\pm}$ such that

$$\frac{a(+,t)}{b(+,t)} < k_1^+ < \frac{d(+,s)}{e(+,s)}; \qquad \frac{a(+,t)}{c(+,t)} > k_2^+ > \frac{d(+,s)}{f(+,s)}, \tag{4.9}$$
$$\frac{a(-,t)}{b(-,t)} < k_1^- < \frac{d(-,s)}{e(-,s)}; \qquad \frac{a(-,t)}{c(-,t)} > k_2^- > \frac{d(-,s)}{f(-,s)} \tag{4.10}$$

for any $t, s \in \mathbb{R}$ and $i \in E$. In case $k_1^+ = k_1^-$ and $k_2^+ = k_2^-$, by virtue of the results in [1, 2] with a slight modification, we can prove that, without noises, there exists a unique periodic solution of (4.1) taking values in $(0, \infty) \times (0, \infty)$. Under these hypotheses, Equation

$$\begin{cases} \dot{x}^+(t) = x^+(t)(a(+,t) - b(+,t)x^+(t) - c(+,t)y^+(t)), \\ \dot{y}^+(t) = y^+(t)(d(+,t) - e(+,t)x^+(t) - f(+,t)y^+(t)), \end{cases} \tag{4.11}$$

and

$$\begin{cases} \dot{x}^-(t) = x^-(t)(a(-,t) - b(-,t)x^-(t) - c(-,t)y^-(t)), \\ \dot{y}^-(t) = y^-(t)(d(-,t) - e(-,t)x^-(t) - f(-,t)y^-(t)). \end{cases} \tag{4.12}$$

(4.11) and (4.12) have a unique periodic solution with period T whose orbit, namely γ^+ (res. γ^-), attracts any solution starting at $(0,\infty)\times(0,\infty)$, that is:

$$\lim_{t\to\infty} \rho((x^+(t), y^+(t)), \gamma^+) = 0; \quad (\text{ res. } \lim_{t\to\infty} \rho((x^-(t), y^-(t)), \gamma^-) = 0). \tag{4.13}$$

Here we denote $\rho(x, A) = \inf\{\|x - z\| : z \in A\}$.

Lemma 4.4. *For any small $\delta_1 > 0, \delta_2 > 0$, there exists a $T_1^* = T_1^*(\delta_1, \delta_2) > 0$ such that $(x^i(t), y^i(t)) \in U_{\delta_1}(\gamma^i)$ for any $t \geqslant T_1^*$, provided that $(x^i(0), y^i(0)) \in K_{\delta_2}$. Here $U_{\delta_1}(\gamma^i)$ is the δ_1-neighborhood of γ^i and $i \in E$.*

Lemma 4.5. *There is a $T_2^* > 0$ such that $x^i(t^*) \leqslant k_1^i; y^i(t^*) \leqslant k_2^i$ for a $t^* \in [0, T_2^*]$, provided $(x^i(0), y^i(0)) \in K_0$. Here $i \in E$.*

Corollary 4.6. *For any $0 < \varepsilon_1 < \varepsilon$ and $0 \leqslant t_1 < t_2$, there exists an ε_2 $(\varepsilon_1 > \varepsilon_2 > 0)$ such that for any $i \in E$,*

a) If $r_2 \geqslant y^i(t_1) \geqslant \varepsilon_1$ (res. $r_1 \geqslant x^i(t_1) \geqslant \varepsilon_1$) then $r_2 \geqslant y^i(t) \geqslant \varepsilon_2$ (res. $r_1 \geqslant x^i(t) \geqslant \varepsilon_2$) for any $t > t_1$.

b) If $(x^i(t_1), y^i(t_1)) \in [\varepsilon, r_1]\times[0, \varepsilon_2]$ then $x^i(t) \in [\varepsilon, r_1]$ $\forall t > t_1$. In additional condition $y^i(t_2) < \varepsilon_1$ we have $\sup_{t_1<t<t_2} y^i(t) \leqslant \varepsilon_1$.

There is a similar result for the case $(x^i(t_1), y^i(t_1)) \in [0, \varepsilon_2] \times [\varepsilon, r_2]$.

Lemma 4.7. *For any $i \in E$, denote $\{(x^i(t), y^i(t)) : t_1 \leqslant t \leqslant t_2\} = \gamma^i_{[t_1,t_2]}$. Then, for any $\delta_1 > 0$, there is $\delta_2 > 0$ such that if $(x^i(t_1), y^i(t_1)) \in U_{\delta_2}(\gamma^i)$ we have $\gamma^i_{[t_1,t_1+2T]} \cap U_{\delta_1}(x^*, y^*) \neq \emptyset$ for any $(x^*, y^*) \in \gamma^i$.*

Theorem 4.8. *Suppose that the conditions (4.4), (4.5), (4.6), (4.7), (4.8) hold. Let $\omega(x, y)$ be the $\omega-$ limit set of the solution $(x(t,x,y), y(t,x,y))$ of Equation (4.1) with $x > 0, y > 0$. Then, γ^+ and γ^- are two subsets of $\omega(x, y)$.* (see Fig. 1.)

Lemma 4.9. *Let $(x^*, y^*) \in \gamma^-$ and $\delta_1 > 0$, we have for (x_n, y_n) defined by (3.9)*

$$\mathbf{P}\{(x_n, y_n) \in U_{\delta_1}(x^*, y^*) \quad \text{i.o.} \quad n\} = 1. \tag{4.14}$$

There is a similar result for the orbit γ^+.

Theorem 4.10. *Any forward orbit of Equation (3.3), starting at a point on γ^-, is a subset of $\omega(x, y)$ with $x > 0, y > 0$. Similarly, any forward orbit of Equation (3.4), starting at a point on γ^+, is a subset of $\omega(x, y)$.* (see Fig. 1.)

Lemma 4.11. *Under the conditions (4.6) and (4.7), there exists a function* $\varphi : [0, a^*) \to \mathbb{R}^+$ *such that if* $y < \varphi(x)$ *then* $\lim_{t\to\infty} y^-(t, x, y) = 0$ *and* $\lim_{t\to\infty}(x^-(t, x, y) - u^-(t, x)) = 0$*. Conversely, if* $y > \varphi(x)$ *then* $\lim_{t\to\infty} x^-(t, x, y) = 0$ *and* $\lim_{t\to\infty}(y^-(t, x, y) - v^-(t, y)) = 0$*. Thus, there is a "neutral" curve, namely* ℓ*, such that if* $(x_0, y_0) \in \ell$ *then the solution* $(x^-(t, x_0, y_0), y^-(t, x_0, y_0))$ *is bounded above and below by positive constants. Further, there is a unique periodic orbit contained in* ℓ*, say* γ^-*, which is visited by any solution starting at a point on* ℓ*, i.e., for any* $\delta > 0, (x^*, y^*) \in \gamma^-$*, there is a* $t > 0$ *such that* $(x^-(t, x_0, y_0), y^-(t, x_0, y_0)) \in U_\delta(x^*, y^*)$ *for any* $(x_0, y_0) \in \ell$*.*

Since whenever $x(t)$ and $y(t)$ of the solution of (4.1) are small (respectively: whenever at least one of $x(t)$ or $y(t)$ is large), $x(t) \uparrow$ and $y(t) \uparrow$ (respect. $x(t) \downarrow$ and $y(t) \downarrow$) then for sufficiently small ε chosen , we can find $t_0 \geqslant 0$ such that $x(t) < M/m, y(t) < M/m$ and either $x(t) \geqslant \varepsilon$ or $y(t) \geqslant \varepsilon$ for any $t \geqslant t_0$. Therefore, without loss of generality, we suppose that $x(t) < M/m, y(t) < M/m$ and either $x(t) \geqslant \varepsilon$ or $y(t) \geqslant \varepsilon$ for any $t \geqslant 0$.

Denote

$$A = \{(x, y) : y < \varphi(x), x > 0\}$$
$$B = \{(x, y) : y > \varphi(x), x > 0\}.$$

with the convention $\varphi(x) = \infty$ if $x \geqslant a^*$.

Lemma 4.12. *For any compact set* $K \subset A$ *(res.* $K \subset B$*) and any* δ_3- *neighborhood* $U_{\delta_3}(\gamma_U^- \times \{0\})$ *of* $\gamma_U^- \times \{0\}$ *(res.* $V_{\delta_3}(\{0\} \times \gamma_V^-)$ *neighborhood of* $\{0\} \times \gamma_V^-$*), there is a* $T_3^* = T_3^*(\delta_3, K) > 0$ *such that* $(x^-(t, x, y), y^-(t, x, y)) \in U_{\delta_3}(\gamma_U^- \times \{0\})$ *for any* $t > T_3^*$ *(respect.* $(x^-(t, x, y), y^-(t, x, y)) \in V_{\delta_3}(\{0\} \times \gamma_V^-)$ *for any* $t > T_3^*$*) and for any* $(x, y) \in K$*.*

Lemma 4.13. *Let* $(x^-(t, x, y), y^-(t, x, y))$ *be the solution of the equation (3.4). For any* $\varepsilon_1 > 0$ *with* $[\varepsilon, \infty) \times [0, \varepsilon_1] \subset A$*, we can find* $\varepsilon_3 > 0$ *such that if* $(x^-(t_1), y^-(t_1)) \in [\varepsilon, \infty) \times [0, \varepsilon_3]$ *then* $x^-(t) \geqslant \varepsilon$ *for any* $t \geqslant t_1$ *and* $\sup_{t \geqslant t_1} y^-(t) \leqslant \varepsilon_1$*. There is a similar result for the case* $(x^-(t_1), y^-(t_1)) \in [0, \varepsilon_1] \times [\varepsilon, \infty)$*.*

Lemma 4.14. *There is* $S_1 > 0$ *such that if* $\varepsilon \leqslant y(t)$ *(res.* $\varepsilon \leqslant x(t)$*) then* $\inf_{0<s<S_1} y(t + s) \geqslant \varepsilon/2$ *(res.* $\inf_{0<s<S_1} x(t + s) \geqslant \varepsilon/2$*) for any* $t \geqslant 0$*.*

Lemma 4.15. *The support of the periodic solution* $u^*(t)$ *of (4.2) is the smallest interval, namely* Γ_x*, containing* γ_U^+ *and* γ_U^-*. Similarly, the support of the periodic solution* $v^*(t)$ *of (4.3) is the smallest interval, namely* Γ_y*, containing* γ_V^+ *and* γ_V^-*. Here,* γ_U^+, γ_V^+ *are respectively two periodic orbits of* $\dot{u}^+ = u^+(a(+, t) - b(+, t)u^+); \dot{v}^+ = v^+(d(+, t) - f(+, t)v^+)$*.*

Theorem 4.16. *Suppose that the conditions (4.9) and (4.10) hold. Let* $\omega(x, y)$ *be the* $\omega-$ *limit set of the solution* $(x(t, x, y), y(t, x, y))$ *of (??) with* $x > 0, y > 0$*. Then,* $\gamma^+ \subset \omega(x, y)$*, where* γ^+ *is a unique periodic solution of (2.5).*

Theorem 4.17. *Suppose that conditions (4.4), (4.5) hold. Let* $\omega(x, y)$ *be the* $\omega-$ *limit set of the solution* $(x(t, x, y), y(t, x, y))$ *with* $x > 0, y > 0$*.*

a) Every orbit $\gamma^-(x^*, y^*)$ *of the solution of Equation (3.4), starting at any point of* γ^+ *is a subset of* $\omega(x, y)$*.*

b) If $\gamma^+ \cap \mathcal{A} \neq \emptyset$ then $\Gamma_x \times \{0\} \subset \omega(x, y)$.

c) If $\gamma^+ \cap \mathcal{B} \neq \emptyset$ then $\{0\} \times \Gamma_y \subset \omega(x, y)$.

d) If $C := \gamma^+ \cap \ell \neq \emptyset$ then the smallest connected part of ℓ, containing γ^- (a unique periodic orbit contained in ℓ, see Lemma 4.2) and C belongs to $\omega(x, y)$.

We illustrate the above model by two following numerical examples.

Example I: An example of the system satisfying $\gamma^+ \cap \mathcal{A} = \emptyset$ but $\gamma^+ \cap \mathcal{B} \neq \emptyset$ (see Theorems 4.7 and 4.9 (a), (b), and Fig.2.).

Example II: An example of the system satisfying $\gamma^+ \cap \mathcal{A} \neq \emptyset$ and $\gamma^+ \cap \mathcal{B} \neq \emptyset$ (see Theorems 4.7 and 4.9 (a)-(d), and Fig.3.).

5. Discussion

To conclusion this paper, we consider an ecology system of two competing species. Suppose that the evolution of every species depends on the quantity of rainfall for every period. If the rainfall is sufficient, their competition potential is equal and they develop periodically. Whenever the rainfall is small, the second species becomes very weak and its amount gets smaller with increasing of time although the influence of the other environment elements is still seasonally (periodically). However, in case the rainfall is in a stationary regime, the quantity of every species oscillates between the good situation and bad situation. There is no species to be disappeared.

There are some questions here. In the proof of Theorem 4.8 we suppose that the conditions (4.4) and (4.5) hold. However, if we use the Liapunov function $V^+ = my^\gamma; V^- = ny^\gamma$ with m, n, γ chosen conveniently, we can prove that if $0 < y(0) < \varepsilon; 0 < x(0) < r_1$ with a positive probability then $(x(t), y(t))$ has to get out the domain $0 < y \leqslant \varepsilon$, i.e., $\exists t^* > 0$ such that $y(t^*) > \varepsilon$, with a positive probability. Thus, we use this assumption perhaps only for the technical reason. By suggestion of Fig. 1, our conjecture is that Theorem 4.8 is still true without these assumptions. Further, the existence of a Markov periodic solution with period T, that attracts the other solutions of Equation (4.1), starting in $\mathbb{R}_+ \times \mathbb{R}_+$ under the conditions (4.6), (4.7), (4.8) is still open.

Also Fig. 1 suggests that the system composed of two stable subsystems is permanent under the conditions (4.4) and (4.5). Note that Theorem 2.3 implies only that the system is *average* permanent. To consider this is our future problem.

Practically, when the quantity of a species is small, we consider that it perishes. Thus we see that in an eco-system, if two species compete under the influence of random environment, one of them must be vanished.

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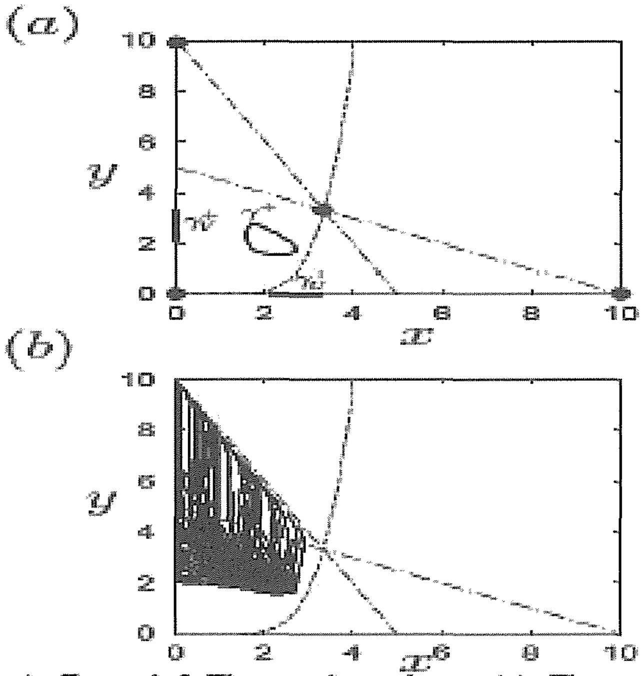

FIGURE 1. Example I. The x-y phase planes. (a): The periodic solutions (γ^+, γ_U^+ and γ_V^+) of System (2.5) with the following parameters are plotted: $a(+,t) = 10$, $b(+,t) = \sin(t) + 4$, $c(+,t) = 1$, $d(+,t) = 10$, $e(+,t) = 1$, $f(+,t) = \sin(t + \pi/2) + 4$. For System (2.6) with the following parameters, null clines (dot-dashed lines), equilibrium points (solid dots) and a neutral curve ℓ (a broken line) are shown: $a(-,t) = 2$, $b(-,t) = 0.2$, $c(-,t) = 0.4$, $d(-,t) = 2$, $e(-,t) = 0.4$, $f(-,t) = 0.2$. (b): The solution of System (2.4) switching between the above Systems (2.5) and (2.6) with the initial condition $(x(0), y(0)) = (4, 0.2)$ is plotted for $t \in [800, 1000]$.