THE AR- PROPERTY AND THE FIXED POINT PROPERTY FOR COMPACT MAPS OF A SOME CONVEX SUBSET IN THE SPACE $l_p(0 < p < 1)$

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Abstract. We know that every convex subset in locally convex linear metric space is an absolute retract and it has the fixed point property for compact maps. However it is not know whether a convex subset of a non-locally convex space has the property? The aim of this paper is to prove the AR- property of bound convex subsets in the space $l_p(0 < p < 1)$ (non-locally convex linear metric space) and prove that normal simple in the space l_p has the fixed point property for compact maps.

1 Introduction

A topological space $\mathbf{Y}$ is called an absolute retract whether

a) Y is metrizable and

b) for an metriczable $\mathbf{X}$ and closed $A \subset \mathbf{X}$ each $f : A \mapsto \mathbf{Y}$ is extendable over $\mathbf{X}$.
The class of absolute retracts is denoted by AR.

In 1951 Dugundji proved the following theorem, see [1]

Theorem 1.1 (Dugundji). Every convex subset of a locally convex linear metric space is a absolute retract.

It is not know that whether a convex subset of a non-locally convex space has the property?

For every $p \in (0,1)$, we will denote

$$l_p = \{x = (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} \mid x_n \mid^p < +\infty\}$$

$\forall x = (x_n), y = (y_n)$; let $\|x\| = \sum_{n=1}^{\infty} \mid x_n \mid^p, d(x,y) = \sum_{n=1}^{\infty} \mid x_n - y_n \mid^p$. The space l_p is a non- locally convex linear metric space.

In 2, we will prove that every bound convex subset in the space l_p is a absolute retract.

Let $\mathbf{X}, \mathbf{Y}$ be topological spaces. A continous map $F : \mathbf{X} \to \mathbf{Y}$ is called compact if $F(\mathbf{X})$ is contained in a compact subset of $\mathbf{Y}$.

A topological $\mathbf{X}$ is called has the fixed point property for compact maps iff every compact map $F : \mathbf{X} \to \mathbf{X}$, there exits a point $x_0 \in \mathbf{X}$:

$$F(x_0) = x_0.$$

We know that every convex subset in locally convex linear topological space has the fixed point property for compact maps (see [2], page 71).

In [3], we will prove that normal simplex in the space l_p (non-locally convex linear metric space) has the fixed point property for compact maps.

We know that every AR has the fixed point property for compact maps(Borsuk Theorem, see [2],p.94). Therefore every convex subset in locally convex linear metric space has the fixed point property for compact maps; where normal simplex in the space l_p is unbound convex set.

2 The AR - property of bound convex subset in the space $l_p(0 < p < 1)$

The main result of 2 is

Theorem 2.1. Every bound convex subset in the space l_p is an AR.

To prove this theorem, we have to use following lemmas

Lemma 2.2. Every bound convex subset in the space l_p is totally bounded

Proof. See [3]. ☐

Let K is a convex subset in the linear space $\mathbf{X}$, a map f : from K into the linear space $\mathbf{Y}$ is called affine map if

$$\forall n \in \mathbb{N}; \forall x_1, x_2, \dots, x_n \in K; \forall \alpha_1, \alpha_2, \dots, \alpha_n \geqslant 0 : \alpha_1 + \alpha_2 + \dots + \alpha_n = 1,$$

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n).$$

We have

Lemma 2.3. Every compact convex subset in the space $l_p (0 < p < 1)$ is homemorphic to a compact convex subset in the locally convex linear metric space $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \ldots$.

Proof. The space $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \ldots$ with metric d is defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{2^{-n} \mid x_n - y_n \mid}{1 + \mid x_n - y_n \mid}, \ \forall x = (x_n), y = (y_n) \in \mathbb{R}^\infty$$

(see [1],p.36)

$\forall r > 0, \exists n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} 2^{-n} < \frac{r}{2}$. We have $V = \{x = (x_n) \in \mathbb{R}^\infty : \mid x_n \mid < \frac{r}{2.2^n}; \forall n \in \{1, 2, \ldots, n_0\}$ is open convex neighbourhood of zero in $\mathbb{R}^\infty$ and $V \subset B(0,r)$. Therefore $\mathbb{R}^\infty$ is the locally convex linear metric space.

$\forall n \in \mathbb{N}$, let $p_n : l_p \mapsto \mathbb{R}$ is defined by

$$p_n(x) = x_n$$

where $x = (x_1, x_2, \ldots, x_n \ldots) \in l_p$.

We have $\forall x = (x_1, x_2, \ldots, x_n \ldots), y = (y_1, y_2, \ldots, y_n \ldots) \in l_p :$

$$d(x,y) = \sum_{i=1}^{\infty} \mid x_i - y_i \mid^p \geqslant \mid x_n - y_n \mid^p .$$

Hence $p_n : l_p \mapsto \mathbb{R}$ is continous and

$$P : l_p \mapsto \mathbb{R}^\infty$$

is defined by

$$P(x) = (p_1(x), p_2(x), \ldots, p_n(x), \ldots), \forall x = (x_1, x_2, \ldots, x_n, \ldots) \in l_p$$

is continuos linear map.

Now, let K is a abitrary compact convex subset in the l_p, then restrict of P o K is a continuous affine map and injection. We see that K is compact. Therefore this map is a homemorphic from K on compact convex subset $f(K)$ in $\mathbb{R}^\infty$. □

Proof of Theorem 2.1. Let A is a bound convex subset in the space l_p, by Lemma 2.3, A is total bound subset $\Rightarrow K = clA$ is compact convex subset of l_p; let B is a closed subset of a metric space $\mathbf{X}$ and $g : B \to A$ is a continuos map $\Rightarrow P \circ g : B \to \mathbb{R}^\infty$ is continuous and $P \circ g(B) \subset P(A)$(where P is defined in Lemma 2.3)

Using Dugundji Theorem, there exists $H : \mathbf{X} \to \mathbb{R}^\infty$ is continous extension of $P \circ g$ and $H(\mathbf{X}) \subset convP \circ g(B) \subset convP(A) = P(A)$.

Let $P^* : K \mapsto P(K)$ is homeomorphic defined by $P^*(x) = P(x), \forall x \in K$. Therefore $(P^*)^{-1} \circ H : \mathbf{X} \mapsto A$ is a continous extension of $g \Rightarrow A$ is a AR $\square$

3 The fixed point property for compact maps of normal simplex in the space $l_p (0 < p < 1)$

In $l_p(0 < p < 1)$. Let

$$e_1 = (1, 0, 0, .., 0, ..)$$

$$e_2 = (0, 1, 0, .., 0, ..)$$

$$e_3 = (0, 0, 1, 0, .., 0, ..)$$

$$...........$$

$$e_n = (0, 0, .., 1, 0, ..)$$

$$...........$$

and $A = \overline{conv\{e_1, e_2, .., e_n, ..\}}$, A is the normal simplex in the space $l_p(0 < p < 1)$. We see that A is a unbound convex subset in the space $l_p(0 < p < 1)$. We will prove that A has fixed point property for compact maps.

Theorem 3.1. A has fixed point property for compact maps.

We need prove following lemmas.

Lemma 3.2. Let (X,d) is a metric space, $f : X \to X$ is a compact map without has fixed point. Then $\exists \epsilon_0 > 0 : d(x, f(x)) \geqslant \epsilon_0; \forall x \in X$

Proof. Let K is a compact in X such that $f(X) \subset K$. We need prove that $\exists \epsilon_0 > 0 : d(x, f(x)) \geqslant \epsilon_0; \forall x \in X$. Suppose contrary to our claim, that $\forall \epsilon > 0, \exists x \in X : d(x, f(x)) < \epsilon$.
$\forall n \in \mathbb{N}$, choose $\epsilon = \frac{1}{n} \Rightarrow \exists x_n \in X : d(x_n, f(x_n)) < \frac{1}{n}; \forall n \in \mathbb{N}$. We have $\{f(x_n)\} \subset K$ and K is compact $\Rightarrow$ there exists a subsequence $\{f(x_{m_n})\}$ of

$\{f(x_n)\}$ and $\exists y_0 \in K : \lim\limits_{n\to\infty} d(y_0, f(x_{m_n})) = 0(1)$.
We have $d(x_n, f(x_n)) < \frac{1}{n}; \forall n \in \mathbb{N} \Rightarrow d(x_{m_n}, f(x_{m_n})) < \frac{1}{m_n} \leqslant \frac{1}{n}; \forall n \in \mathbb{N}$
$\Rightarrow \lim\limits_{n\to\infty} d(x_{m_n}, f(x_{m_n})) = 0(2)$.
(1), (2) $\Rightarrow \lim\limits_{n\to\infty} d(y_0, x_{m_n})) = 0 \Rightarrow \lim\limits_{n\to\infty} d(f(y_0), f(x_{m_n})) = 0$.
$\Rightarrow f(y_0) = y_0 \Rightarrow$ This contracdicts our assumption.

□

Definition 3.3. Let A is a convex subset in a linear metric space (X,d), A is called admissible if $\forall \epsilon > 0$, every compact subset K in A, there exists continuous function $h : K \to A$ such that $d(x, f(x)) < \epsilon; \forall x \in K$ and h(K) is contained a finite dimensional linear subspace of X

Lemma 3.4. Let A is a convex subset in a linear metric space (X,d). If A is admissible, the A has fixed point property for compact maps.

Proof. Suppose contrary, that there exists compact map $f : A \to A$ and $f(x) \neq x; \forall x \in A$. Using lemma 3.2, then $\exists \epsilon_0 > 0 : d(x, f(x)) \geqslant \epsilon_0; \forall x \in A$ Let K is a compact map in A such that $f(A) \subset K$. We see that A is admissible, there exists continuous function $g : K \to A : d(g(x), x) < \frac{\epsilon_0}{4}; \forall x \in K$ and $g(K)$ is contained in a finite dimensional linear subspace of X. $\Rightarrow g(K) \subset L \cap A \Rightarrow L \cap A$ is a convex in a finite dimensional linear metric space L.
Consider $g \circ f|_{L\cap A} : L \cap A \to L \cap A$. We know that every finite dimensional linear metric space is a locally convex linear metric space $\Rightarrow L \cap A$ is an AR $\Rightarrow L \cap A$ has fixed point property for compact maps and $g \circ f|_{L\cap A}(L \cap A) \subset g(K)$, g(K) is a compact set $\Rightarrow \exists x_0 \in L \cap A : g \circ f(x_0) = x_0 \Rightarrow d(g(f(x_0)), x_0) < \frac{\epsilon_0}{4}$
$\Rightarrow d(f(x_0), x_0) < \frac{\epsilon_0}{4}$.
This contracdicts our assumption.

□

We only give main ideas of the proof of theorem 3.1. By Lemma 3.4, we only prove that A is admissible.
$\forall \epsilon > 0$. Let K is a arbitray compact in A. $\forall n \in \mathbb{N}, \forall x = (x_1, x_2, .., x_n, ..) \in K$.
Let $f_n(x) = (x_1, x_2, .., x_n, 1 - x_1 - x_2 - .. - x_n, 0, 0..)$. We see that $f_n(K)$ is contained in a finite dimensional linear subspace of l_p and when n is sufficient large, we have: $d(f_n(x), x) < \epsilon; \forall x \in K$.
Therefore A has fixed point property for compact maps.

References

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