Determination of Dipolar Sources for Three-dimensional Poisson Equation

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Abstract We consider an identification problem for dipolar sources for three-dimensional Poisson equation. Unknowns are locations, moments, and number of dipoles. We propose a reliable identification method under the condition that the potential can be observed on the boundary. Our method is based on the weighted integral using harmonic functions as weighting functions. The effectiveness of our method is shown by numerical examples.

Keywords: Inverse source problem; Poisson equation; dipole; numerical method; weighted integral

Introduction

Inverse problems for partial differential equations have attracted the attention of many researchers in the last two decades [1]. Investigation of reliable numerical method for inverse problems becomes very important. The identification targets of these problems are classified into several categories such as source term, boundary and/or initial conditions, shape of domain, and coefficients in governing equation. Especially, inverse source problem appears in various fields of science and engineering, such as heart conduction, air pollution, cracks in structure, and electrical activity of human brain, and so on [2, 4, 6, 7].

Inverse source problem is to identify unknown source term from given or observed data on the boundary or in the outside of domain. A usual numerical approach is the combination of forward analysis and least squares method. However this approach needs a priori information for unknown parameters and requires much computation time.

In this paper, we consider an identification problem for three-dimensional Poisson equation where the source term is expressed by sum of dipoles. Our aim is to identify locations, moments, and number of dipoles from the data on the boundary. We propose a reliable identification method for the above inverse source problem without using a priori information about unknowns. Our approach is based on the weighted integral on the boundary using harmonic functions as weighting functions [3]. We give the mathematical expression for the approximation of locations and moments, and also show these practical error bounds and the proper choice of weighting functions. The effectiveness of our method is illustrated by numerical examples.

Mathematical Formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary Γ. We consider the following inverse source problem for the Poisson equation:

$$\begin{cases} \Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), & \mathrm{supp} f \subset \Omega, \quad \boldsymbol{x} \in \mathbb{R}^3, \\ u(\boldsymbol{x}) = \overline{u}(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma, \\ u(\boldsymbol{x}) \to 0, & |\boldsymbol{x}| \to +\infty. \end{cases} \tag{1}$$

Here, $f(\boldsymbol{x})$ denotes unknown source term expressed as

$$f(\boldsymbol{x}) = \sum_{i=1}^{N} \frac{\partial \delta}{\partial \boldsymbol{m}_i}(\boldsymbol{x} - \boldsymbol{p}_i), \quad \boldsymbol{p}_i \in \Omega, \quad \boldsymbol{m}_i \in \mathbb{R}^3, \quad |\boldsymbol{m}_i| \neq 0, \tag{2}$$

where N is the number of dipoles, and $\boldsymbol{p}_i$ and $\boldsymbol{m}_i$ are the location and moment of the i-th dipole. We assume that $\overline{u}(\boldsymbol{x})$ is given or observed at discrete points on Γ. Our purpose is to identify $\boldsymbol{p}_i$ and $\boldsymbol{m}_i$ $(i = 1, 2, \ldots, N)$ from the data of $\overline{u}(\boldsymbol{x})$ on Γ.

Weighted Integral

Multiply both sides of $\Delta u(\boldsymbol{x}) = f(\boldsymbol{x})$ by a given weighting function $w(\boldsymbol{x})$, then we have

$$\int_\Omega w(\boldsymbol{x}) \Delta u(\boldsymbol{x}) d\Omega = \int_\Omega w(\boldsymbol{x}) f(\boldsymbol{x}) d\Omega. \tag{3}$$

Using a harmonic function as $w(\boldsymbol{x})$ and applying Green's formula, eq.(3) becomes

$$I(w) \equiv -\int_\Gamma \left(w(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) - u(\boldsymbol{x}) \frac{\partial w}{\partial \boldsymbol{n}}(\boldsymbol{x}) \right) d\Gamma = \sum_{i=1}^{N} \frac{\partial w}{\partial \boldsymbol{m}_i}(\boldsymbol{p}_i), \tag{4}$$

where $\boldsymbol{n}$ denotes the outward unit normal vector to Γ. The boundary integral $I(w)$ can be determined using only $\overline{u}(\boldsymbol{x})$, since $u(\boldsymbol{x})$ is unique in the outside of Ω [7]. We compute $I(w)$ applying the charge simulation method. The outline of the calculation of $I(w)$ is shown in [3, 5].

Let $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ be an orthonormal basis such that $\boldsymbol{e}_0 \cdot \boldsymbol{e}_1 = 0$ and $\boldsymbol{e}_2 = \boldsymbol{e}_0 \times \boldsymbol{e}_1$. In the following, each component of $\boldsymbol{p}_i$, and $\boldsymbol{m}_i$ is denoted by

$$p_{ij} = \boldsymbol{p}_i \cdot \boldsymbol{e}_j, \quad m_{ij} = \boldsymbol{m}_i \cdot \boldsymbol{e}_j, \qquad j = 0, 1, 2.$$

For the components p_{kj} and m_{kj}, we assume the conditions:

(i) $p_{k0} - \max_{i \neq k} p_{i0} > 0.$

(ii) $\sqrt{m_{k0}^2 + m_{k1}^2} \neq 0, \quad \sqrt{m_{k0}^2 + m_{k2}^2} \neq 0.$

Under the above conditions, we consider the identification of $\boldsymbol{p}_k$ and $\boldsymbol{m}_k$ using $I(w)$.

Identification of Location

We divide our problem into two-dimensional problems on $\boldsymbol{e}_0\boldsymbol{e}_1$-plane and $\boldsymbol{e}_0\boldsymbol{e}_2$-plane, and use the following weighting functions with parameter $\lambda > 0$:

$$\begin{aligned} w_{1\ell}(\boldsymbol{x}) &= e^{\lambda(\boldsymbol{x}\cdot\boldsymbol{e}_0)} \cos\{\lambda(\boldsymbol{x}\cdot\boldsymbol{e}_\ell)\}, \\ w_{2\ell}(\boldsymbol{x}) &= e^{\lambda(\boldsymbol{x}\cdot\boldsymbol{e}_0)} \sin\{\lambda(\boldsymbol{x}\cdot\boldsymbol{e}_\ell)\}, \\ w_{3\ell}(\boldsymbol{x}) &= (\boldsymbol{x}\cdot\boldsymbol{e}_0) w_{1\ell}(\boldsymbol{x}) - (\boldsymbol{x}\cdot\boldsymbol{e}_\ell) w_{2\ell}(\boldsymbol{x}), \\ w_{4\ell}(\boldsymbol{x}) &= (\boldsymbol{x}\cdot\boldsymbol{e}_0) w_{2\ell}(\boldsymbol{x}) + (\boldsymbol{x}\cdot\boldsymbol{e}_\ell) w_{1\ell}(\boldsymbol{x}), \end{aligned} \qquad \text{for } \ell = 1, 2. \tag{5}$$

Let

$$S_{1\ell} = \frac{I(w_{1\ell})}{\lambda}, \quad S_{2\ell} = \frac{I(w_{2\ell})}{\lambda}, \quad S_{3\ell} = \frac{I(w_{3\ell}) - S_{1\ell}}{\lambda}, \quad S_{4\ell} = \frac{I(w_{4\ell}) - S_{2\ell}}{\lambda}. \tag{6}$$

Using the above notation, the k-th dipole's location on $\boldsymbol{e}_0\boldsymbol{e}_\ell$-plane ($\ell = 1, 2$) is approximated by

$$p_{k0} \simeq \frac{S_{1\ell}S_{3\ell} + S_{2\ell}S_{4\ell}}{S_{1\ell}^2 + S_{2\ell}^2} \equiv \widetilde{p}_{k0}^{(\ell)}, \quad p_{k\ell} \simeq \frac{S_{1\ell}S_{4\ell} - S_{2\ell}S_{3\ell}}{S_{1\ell}^2 + S_{2\ell}^2} \equiv \widetilde{p}_{k\ell}. \tag{7}$$

For the errors $a_\ell \equiv p_{k0} - \widetilde{p}_{k0}^{(\ell)}$ and $b_\ell \equiv p_{k\ell} - \widetilde{p}_{k\ell}$, the following inequality is obtained:

$$\begin{aligned} |a_\ell \boldsymbol{e}_0 + b_\ell \boldsymbol{e}_\ell| &= \sqrt{a_\ell^2 + b_\ell^2} \\ &\leq \frac{\sum_{i \neq k} e^{-\lambda(p_{k0} - p_{i0})} \sqrt{(p_{k0} - p_{i0})^2 + (p_{k\ell} - p_{i\ell})^2} \sqrt{m_{i0}^2 + m_{i\ell}^2}}{e^{-\lambda p_{k0}} \sqrt{S_{1\ell}^2 + S_{2\ell}^2}} \equiv \varepsilon_{pk}^{(\ell)}. \end{aligned}$$

Under the conditions (i) and (ii), $\varepsilon_{pk}^{(\ell)}$ exponentially converges to 0 as $\lambda \to \infty$. Then, we have the approximation for $\boldsymbol{p}_k$ as follows:

$$\widetilde{\boldsymbol{p}}_k = \frac{\widetilde{p}_{k0}^{(1)} + \widetilde{p}_{k0}^{(2)}}{2} \boldsymbol{e}_0 + \widetilde{p}_{k1}\boldsymbol{e}_1 + \widetilde{p}_{k2}\boldsymbol{e}_2 \equiv \widetilde{p}_{k0}\boldsymbol{e}_0 + \widetilde{p}_{k1}\boldsymbol{e}_1 + \widetilde{p}_{k2}\boldsymbol{e}_2.$$

Finally, we can obtain the following estimate for $|\boldsymbol{p}_k - \widetilde{\boldsymbol{p}}_k|$:

$$|\boldsymbol{p}_k - \widetilde{\boldsymbol{p}}_k| \leq \sqrt{(\varepsilon_{pk}^{(1)})^2 + (\varepsilon_{pk}^{(2)})^2} + \frac{1}{2}|\widetilde{p}_{k0}^{(1)} - \widetilde{p}_{k0}^{(2)}| \equiv \varepsilon_{pk}. \tag{8}$$

Identification of Moment

Using eq.(6), the k-th dipole's moment on $\boldsymbol{e}_0\boldsymbol{e}_\ell$-plane ($\ell = 1, 2$) is approximated by

$$\begin{aligned} m_{k0} &\simeq e^{-\lambda \widetilde{p}_{k0}} (S_{1\ell} \cos \lambda \widetilde{p}_{k\ell} + S_{2\ell} \sin \lambda \widetilde{p}_{k\ell}) \equiv \widetilde{m}_{k0}^{(\ell)}, \\ m_{k\ell} &\simeq e^{-\lambda \widetilde{p}_{k0}} (S_{2\ell} \cos \lambda \widetilde{p}_{k\ell} - S_{1\ell} \sin \lambda \widetilde{p}_{k\ell}) \equiv \widetilde{m}_{k\ell}. \end{aligned} \tag{9}$$

Then, we have the approximation for $\boldsymbol{m}_k$ as follows

$$\widetilde{\boldsymbol{m}}_k = \frac{\widetilde{m}_{k0}^{(1)} + \widetilde{m}_{k0}^{(2)}}{2} \boldsymbol{e}_0 + \widetilde{m}_{k1}\boldsymbol{e}_1 + \widetilde{m}_{k2}\boldsymbol{e}_2 \equiv \widetilde{m}_{k0}\boldsymbol{e}_0 + \widetilde{m}_{k1}\boldsymbol{e}_1 + \widetilde{m}_{k2}\boldsymbol{e}_2.$$

Finally, we can obtain the following estimate for $|\boldsymbol{m}_k - \widetilde{\boldsymbol{m}}_k|$:

$$|\boldsymbol{m}_k - \widetilde{\boldsymbol{m}}_k| \lesssim \sqrt{(\varepsilon_{mk}^{(1)})^2 + (\varepsilon_{mk}^{(2)})^2} + \sqrt{(\sigma_{mk}^{(1)})^2 + (\sigma_{mk}^{(2)})^2} + \frac{1}{2}|\widetilde{m}_{k0}^{(1)} - \widetilde{m}_{k0}^{(2)}| \equiv \varepsilon_{mk}. \tag{10}$$

Here, $\varepsilon_{mk}^{(\ell)}$ and $\sigma_{mk}^{(\ell)}$ are given by

$$\begin{aligned} \varepsilon_{mk}^{(\ell)} &= \sum_{i \neq k} e^{-\lambda(p_{k0} - p_{i0})} \sqrt{m_{i0}^2 + m_{i\ell}^2}, \\ \sigma_{mk}^{(\ell)} &= \lambda \sqrt{(\widetilde{m}_{k0}^{(\ell)})^2 + \widetilde{m}_{k\ell}^2} \left(\varepsilon_{pk}^{(\ell)} + \frac{1}{2}|\widetilde{p}_{k0}^{(1)} - \widetilde{p}_{k0}^{(2)}| \right). \end{aligned}$$

We have the identification of the k-th dipole's location and moment for an orthonormal basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ satisfying the conditions (i) and (ii). The other dipoles can be identified by changing $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ properly. For the dipoles not to satisfy the conditions for any $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$, it is necessary to remove the information of already identified dipoles from $\overline{u}(\boldsymbol{x})$. We call this process *deflation*. The derivation of eqs.(7)-(10) and *deflation* are shown in Inui, Yamatani and Ohnaka [3].

Choice of e_0

For the suitable choice of e_0, we use additional weighting functions such that

$$\begin{aligned} w_{5\ell}(\boldsymbol{x}) &= \{(\boldsymbol{x}\cdot\boldsymbol{e}_0)^2 - (\boldsymbol{x}\cdot\boldsymbol{e}_\ell)^2\}w_{1\ell}(\boldsymbol{x}) - 2(\boldsymbol{x}\cdot\boldsymbol{e}_0)(\boldsymbol{x}\cdot\boldsymbol{e}_\ell)w_{2\ell}(\boldsymbol{x}), \\ w_{6\ell}(\boldsymbol{x}) &= \{(\boldsymbol{x}\cdot\boldsymbol{e}_0)^2 - (\boldsymbol{x}\cdot\boldsymbol{e}_\ell)^2\}w_{2\ell}(\boldsymbol{x}) + 2(\boldsymbol{x}\cdot\boldsymbol{e}_0)(\boldsymbol{x}\cdot\boldsymbol{e}_\ell)w_{1\ell}(\boldsymbol{x}), \end{aligned} \qquad \text{for } \ell = 1,2, \tag{11}$$

and let

$$S_{5\ell} = \frac{I(w_{5\ell}) - 2S_{3\ell}}{\lambda}, \quad S_{6\ell} = \frac{I(w_{6\ell}) - 2S_{4\ell}}{\lambda}. \tag{12}$$

Here, we introduce the following unit vectors

$$\boldsymbol{e}_0' = \boldsymbol{e}_0\cos\theta - \boldsymbol{e}_\ell\sin\theta, \quad \boldsymbol{e}_\ell' = \boldsymbol{e}_0\sin\theta + \boldsymbol{e}_\ell\cos\theta.$$

Using $\boldsymbol{e}_0'$ and $\boldsymbol{e}_\ell'$ instead of $\boldsymbol{e}_0$ and $\boldsymbol{e}_\ell$ in eq.(5), we obtain $\widetilde{p}_{k0}'$ and $\widetilde{p}_{k\ell}'$ as the identified values of $\boldsymbol{p}\cdot\boldsymbol{e}_0'$ and $\boldsymbol{p}\cdot\boldsymbol{e}_\ell'$. Let $a_\ell'(\theta) = \boldsymbol{p}\cdot\boldsymbol{e}_0' - \widetilde{p}_{k0}'$ and $b_\ell'(\theta) = \boldsymbol{p}\cdot\boldsymbol{e}_\ell' - \widetilde{p}_{k\ell}'$. For the approximation error of location, the following expression holds:

$$\begin{aligned} \boldsymbol{E}_{pk}^{(\ell)}(\theta) &\equiv a_\ell'(\theta)\boldsymbol{e}_0' + b_\ell'(\theta)\boldsymbol{e}_\ell'. \\ &= \left\{a_\ell'(\theta)\cos\theta + b_\ell'(\theta)\sin\theta\right\}\boldsymbol{e}_0 + \left\{-a_\ell'(\theta)\sin\theta + b_\ell'(\theta)\cos\theta\right\}\boldsymbol{e}_\ell. \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{\partial \boldsymbol{E}_{pk}^{(\ell)}}{\partial\theta}(0) &= \frac{\partial}{\partial\theta}\left\{a_\ell'(\theta)\cos\theta + b_\ell'(\theta)\sin\theta\right\}\Big|_{\theta=0}\boldsymbol{e}_0 + \frac{\partial}{\partial\theta}\left\{-a_\ell'(\theta)\sin\theta + b_\ell'(\theta)\cos\theta\right\}\Big|_{\theta=0}\boldsymbol{e}_\ell \\ &= \left\{\frac{\partial a_\ell'}{\partial\theta}(0) + b_\ell\right\}\boldsymbol{e}_0 + \left\{\frac{\partial b_\ell'}{\partial\theta}(0) - a_\ell\right\}\boldsymbol{e}_\ell \\ &\equiv \alpha_\ell\boldsymbol{e}_0 + \beta_\ell\boldsymbol{e}_\ell \end{aligned} \tag{13}$$

Note that $a_\ell = a_\ell'(0)$ and $b_\ell = b_\ell'(0)$. Using $S_{5\ell}$ and $S_{6\ell}$, α_ℓ and β_ℓ are written by

$$\alpha_\ell = \lambda\left[\frac{S_{1\ell}S_{6\ell} - S_{2\ell}S_{5\ell}}{S_{1\ell}^2 + S_{2\ell}^2} - 2\widetilde{p}_{k0}^{(\ell)}\widetilde{p}_{k\ell}\right], \quad \beta_\ell = -\lambda\left[\frac{S_{1\ell}S_{5\ell} + S_{2\ell}S_{6\ell}}{S_{1\ell}^2 + S_{2\ell}^2} - (\widetilde{p}_{k0}^{(\ell)})^2 + \widetilde{p}_{k\ell}^2\right].$$

We choose $\boldsymbol{e}_0$ such that

$$\left|\frac{\partial \boldsymbol{E}_{pk}^{(\ell)}}{\partial\theta}(0)\right| = \sqrt{\alpha_\ell^2 + \beta_\ell^2}.$$

is sufficiently small for both $\ell = 1,2$. Unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are chosen under the condition (ii) using $\widetilde{\boldsymbol{m}}_k$ instead of $\boldsymbol{m}_k$.

Numerical Examples

Let $\Omega = \{\boldsymbol{x}; |\boldsymbol{x}| < 1\}$. The boundary condition $\overline{u}(\boldsymbol{x})$ is obtained at L points on the spherical surface $\Gamma = \{\boldsymbol{x}; |\boldsymbol{x}| = 1\}$. The arrangement of L points is approximately uniform, and $\overline{u}(\boldsymbol{x})$ is analytically generated. For $L = 248$, 510, and 998, we use $\lambda = 12$, 14, and 15, respectively.

We consider 25 examples in which the locations, moments, and number of dipoles are generated by uniform random number under the following conditions:

- $|\boldsymbol{p}_i| \le 0.7, \quad |\boldsymbol{p}_i - \boldsymbol{p}_j| \ge 0.3 \ (i \ne j)$.
- $0.1 \le |\boldsymbol{m}_i| \le 0.5$.
- $N \in \{2, 3, 4\}$.

Table 1 shows the number of examples for each N in 25 examples. Table 2 shows the relation between N and $\widetilde{N}$, where $\widetilde{N}$ is the number of identified dipoles. For $L = 510$ and 998, $\widetilde{N}$ is equal to N in all examples although N is not directly identified. For $L = 248$, the difference between N and $\widetilde{N}$ is at most 1. Figure 1 shows the error estimations of $\boldsymbol{p}_i$ and $\boldsymbol{m}_i$ for $L = 510$ and 998, where ε_{pi} and ε_{mi} are calculated by using identification values instead of true values in eqs.(8) and (10). The error estimations ε_{pi} and ε_{mi} have more reasonable results for $L = 998$ than for $L = 510$. We consider that these results are caused by the error of numerical integrations of $I(w)$.

Conclusion

We show a reliable identification method for dipolar sources for three-dimensional Poisson equation. Our approach is based on the weighted integral on the boundary. Numerical experiment shows that the identified locations and moments are obtained reasonably, and that their error estimates give practical error bounds. Furthermore, in almost all examples, the identified number of dipoles is equal to the true one although our method does not directly identify the number of dipoles. The results show that our method may be effective in dipolar source identification.

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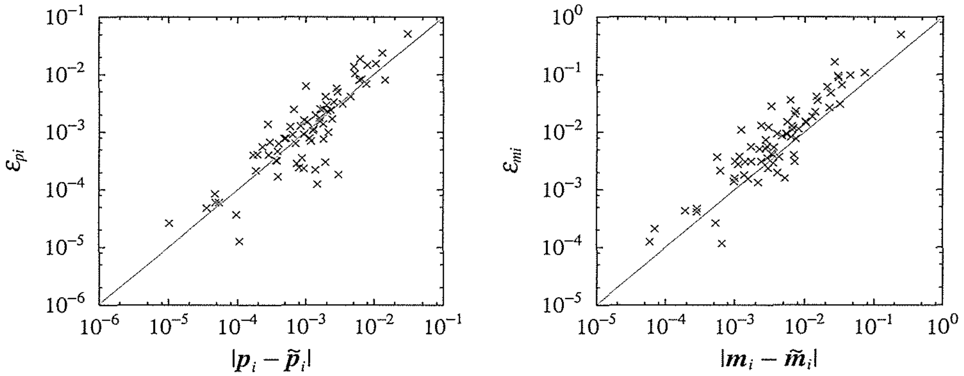
Table 1. The number of examples for each N.

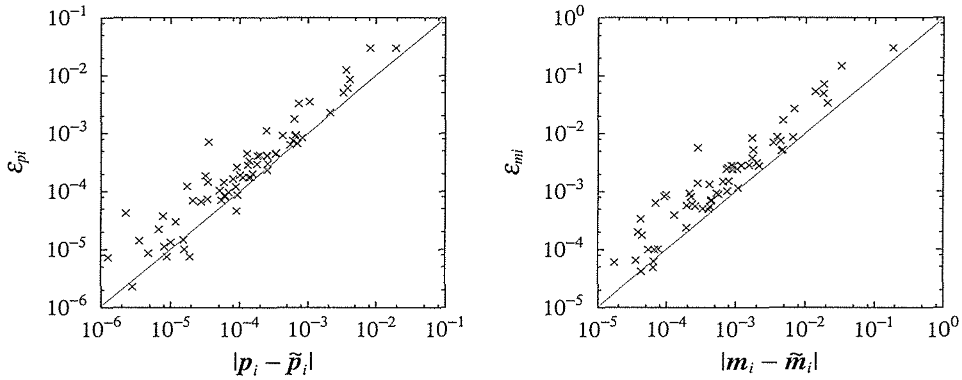
$N=2$	$N=3$	$N=4$
11	7	7

Table 2. The number of identified dipoles.

	$\widetilde{N}=N+1$	$\widetilde{N}=N$	$\widetilde{N}=N-1$
$L=248$	2	20	3
$L=510$	0	25	0
$L=998$	0	25	0



(a) $L = 510$



(b) $L = 998$

Figure 1. Error estimation of $\boldsymbol{p}_i$ and $\boldsymbol{m}_i$.