Characteristic Cauchy problem for first-order quasilinear equations

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1. Cauchy problem

Consider following first-order quasilinear equation

$$\sum_{j=1}^{n} a_j(x,u)\frac{\partial u}{\partial x_j} = a(x,u), \tag{1}$$

where $x = (x_1, x_2, ..., x_n) \in \Omega \subset R^n$.
Suppose there given a smooth (n-1)-dimensional surface $\gamma \subset R^n$

$$\gamma \equiv \{x = x^0(y') \in R^n; y' \equiv (y_1, y_2, ..., y_{n-1}) \in \omega \subset R^{n-1}\}, \tag{2}$$

where ω is a neighbourhood of some fixed point $y'^0 \in R^{n-1}$,

$$x^0(y') = (x_1^0(y'), x_2^0(y'), ..., x_n^0(y'))$$

and

$$\operatorname{rank}\begin{bmatrix} \frac{\partial x_1^0(y')}{\partial y_1} & \frac{\partial x_1^0(y')}{\partial y_2} & \cdots & \frac{\partial x_1^0(y')}{\partial y_{n-1}} \\ \frac{\partial x_2^0(y')}{\partial y_1} & \frac{\partial x_2^0(y')}{\partial y_2} & \cdots & \frac{\partial x_2^0(y')}{\partial y_{n-1}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_n^0(y')}{\partial y_1} & \frac{\partial x_n^0(y')}{\partial y_2} & \cdots & \frac{\partial x_n^0(y')}{\partial y_{n-1}} \end{bmatrix} = n-1, \quad \forall y' \in \omega. \tag{3}$$

Cauchy problem Look for a solution $u(x) \in C^1(\Omega)$ such that

$$u(x)\Big|_{x=x^0(y')} = u_0(y'), \tag{4}$$

where $u_0(y')$ is a given smooth function.

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2. Noncharacteristic condition

Definition 1. *We say that the Cauchy problem (1), (4) is not characteristic if the following condition holds*

$$A(y') \equiv \begin{vmatrix} \frac{\partial x_1^0(y')}{\partial y_1} & \frac{\partial x_1^0(y')}{\partial y_2} & \cdots & \frac{\partial x_1^0(y')}{\partial y_{n-1}} & a_1(x^0(y'), u_0(y')) \\ \frac{\partial x_2^0(y')}{\partial y_1} & \frac{\partial x_2^0(y')}{\partial y_2} & \cdots & \frac{\partial x_2^0(y')}{\partial y_{n-1}} & a_2(x^0(y'), u_0(y')) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n^0(y')}{\partial y_1} & \frac{\partial x_n^0(y')}{\partial y_2} & \cdots & \frac{\partial x_n^0(y')}{\partial y_{n-1}} & a_n(x^0(y'), u_0(y')) \end{vmatrix} \neq 0, \quad \forall y' \in \omega \tag{5}$$

3. Problem statement

Suppose

$$A(y'^0) = 0, \tag{6}$$

$$A(y') \neq 0, \forall y' \in \omega, y' \neq y'^0. \tag{7}$$

The question: What we can say about *the existence* of solution to the charateristic problem (1), (4) in a neighbourhood, or even in a semineighbourhood, of the point $x^0(y'^0)$?

4. A relation of the equation (1) with a linear homogenouse first-order equation

Theorem 1. *Suppose $\varphi(x,u) \in C^1$ is a solution of the following linear homogenouse first-order equation:*

$$\sum_{j=1}^{n} a_j(x,u)\frac{\partial \varphi}{\partial x_j} + a(x,u)\frac{\partial \varphi}{\partial u} = 0, (x,u) \in P \subset R^{n+1}_{x,u} \tag{8}$$

such that for some point $(x^0, u^0) \in P$ the following conditions hold

$$\varphi(x^0, u^0) = 0,$$

$$\frac{\partial \varphi(x^0, u^0)}{\partial u} \neq 0. \tag{9}$$

Then the relation

$$\varphi(x,u) = 0 \tag{10}$$

defines a C^1-solution $u(x)$ of the equation (1) in a neighbourhood of x^0, such that $u(x^0) = u^0$.

5. An extended surface

To consider the Cauchy problem for the equation (8), we extend the (n-1)-dimensional surface $\gamma \subset R^n_x$ to an n-dimensional surface $\Gamma \subset P \subset R^{n+1}_{x,u}$.
Denote $y = (y', y_n)$. We define the surface Γ by following equations:

$$\begin{cases} x = X^0(y) \equiv x^0(y') + y_n \dfrac{\partial x^0(y')}{\partial y_1} \times \dfrac{\partial x^0(y')}{\partial y_2} \times \ldots \times \dfrac{\partial x^0(y')}{\partial y_{n-1}}, \\ u = U^0(y) \equiv u_0(y') + A(y')y_n, \end{cases} \tag{11}$$

where the vector product $\frac{\partial x^0(y')}{\partial y_1} \times \frac{\partial x^0(y')}{\partial y_2} \times \ldots \times \frac{\partial x^0(y')}{\partial y_{n-1}}$ is a vector in R^n_x and is defined by the following formula

$$\frac{\partial x^0(y')}{\partial y_1} \times \frac{\partial x^0(y')}{\partial y_2} \times \ldots \times \frac{\partial x^0(y')}{\partial y_{n-1}} \equiv \begin{vmatrix} e_1 & e_2 & \ldots & e_n \\ \frac{\partial x^0_1(y')}{\partial y_1} & \frac{\partial x^0_2(y')}{\partial y_1} & \ldots & \frac{\partial x^0_n(y')}{\partial y_1} \\ \frac{\partial x^0_1(y')}{\partial y_2} & \frac{\partial x^0_2(y')}{\partial y_2} & \ldots & \frac{\partial x^0_n(y')}{\partial y_2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial x^0_1(y')}{\partial y_{n-1}} & \frac{\partial x^0_2(y')}{\partial y_{n-1}} & \ldots & \frac{\partial x^0_n(y')}{\partial y_{n-1}} \end{vmatrix}, \tag{12}$$

$e_1, e_2, ..., e_n$ are canonical unit vectors in R^n_x.

Theorem 2. *Suppose (3) holds. Then the surface $\Gamma \subset R^{n+1}$ is an n-dimensional surface, i.e.*

$$rank \begin{bmatrix} \frac{\partial X^0_1(y)}{\partial y_1} & \frac{\partial X^0_1(y)}{\partial y_2} & \ldots & \frac{\partial X^0_1(y)}{\partial y_n} \\ \frac{\partial X^0_2(y)}{\partial y_1} & \frac{\partial X^0_2(y)}{\partial y_2} & \ldots & \frac{\partial X^0_2(y)}{\partial y_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial X^0_n(y)}{\partial y_1} & \frac{\partial X^0_n(y)}{\partial y_2} & \ldots & \frac{\partial X^0_n(y)}{\partial y_n} \\ \frac{\partial U_0(y)}{\partial y_1} & \frac{\partial U_0(y)}{\partial y_2} & \ldots & \frac{\partial U_0(y)}{\partial y_n} \end{bmatrix} = n, \quad \forall y \in Q,$$

where Q is some neighbourhood of the point $y^0 = ({y'}^0, 0) \in R^n_y$.

6. An Cauchy problem for the equation (8)

We are looking for solution $\varphi(x,u)$ of (8) such that

$$\varphi(x,u)\Big|_{x=X^0(y),u=U^0(y)} = y_n. \tag{13}$$

Theorem 3. *The Cauchy problem (8), (13) is noncharacteristic if the following condition holds*

$$\tilde{A}(y') \equiv \begin{vmatrix} \frac{\partial x_1^0(y')}{\partial y_1} & \frac{\partial x_1^0(y')}{\partial y_2} & \cdots & \frac{\partial x_1^0(y')}{\partial y_{n-1}} & v_1(y') & a_1(x^0(y'),u_0(y')) \\ \frac{\partial x_2^0(y')}{\partial y_1} & \frac{\partial x_2^0(y')}{\partial y_2} & \cdots & \frac{\partial x_2^0(y')}{\partial y_{n-1}} & v_2(y') & a_2(x^0(y'),u_0(y')) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \frac{\partial x_n^0(y')}{\partial y_1} & \frac{\partial x_n^0(y')}{\partial y_2} & \cdots & \frac{\partial x_n^0(y')}{\partial y_{n-1}} & v_n(y') & a_n(x^0(y'),u_0(y')) \\ \frac{\partial u_0(y')}{\partial y_1} & \frac{\partial u_0(y')}{\partial y_2} & \cdots & \frac{\partial u_0(y')}{\partial y_{n-1}} & A(y') & a(x^0(y'),u_0(y')) \end{vmatrix} \neq 0, \quad \forall y' \in \omega, \tag{14}$$

where $v(y') \equiv (v_1(y'), v_2(y'), ..., v_n(y')) = \frac{\partial x^0(y')}{\partial y_1} \times \frac{\partial x^0(y')}{\partial y_2} \times ... \times \frac{\partial x^0(y')}{\partial y_{n-1}}$.

7. The solution to the Cauchy problem (8), (13)

The characteristic system for Cauchy problem (8), (13) is

$$\begin{cases} X_j'(t) = a_j(X(t),U(t)), j = 1,2,...,n, \\ U'(t) = a(X(t),U(t)), \\ \Phi'(t) = 0, \end{cases} \tag{15}$$

with the following initial conditions

$$\begin{cases} X(0) = X^0(y), \\ U(0) = U_0(y), \\ \Phi(0) = y_n. \end{cases} \tag{16}$$

We denote the solutions of the problem (15), (16) by $X(y,t), U(y,t), \Phi(y,t)$. We consider the following system of (n+1) equations with respect to (n+1) unknowns $(y_1, y_2, ..., y_n, t)$

$$\begin{cases} X(y_1,y_2,...,y_n,t) = x \\ U(y_1,y_2,...,y_n,t) = u. \end{cases} \tag{17}$$

We denote the solutions of the system (17) by $Y_1(x,u), Y_2(x,u), ..., Y_n(x,u), T(x,u)$.

Theorem 4. *The function $Y_n(x,u)$ is a solution to the problem (8), (13).*

Remark 1. *Suppose (14) holds. If we set*

$$W(y,t)=\begin{bmatrix} \frac{\partial X_1(y,t)}{\partial y_1} & \frac{\partial X_1(y,t)}{\partial y_2} & \cdots & \frac{\partial X_1(y,t)}{\partial y_n} & \frac{\partial X_1(y,t)}{\partial t} \\ \frac{\partial X_2(y,t)}{\partial y_1} & \frac{\partial X_2(y,t)}{\partial y_2} & \cdots & \frac{\partial X_2(y,t)}{\partial y_n} & \frac{\partial X_2(y,t)}{\partial t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_n(y,t)}{\partial y_1} & \frac{\partial X_n(y,t)}{\partial y_2} & \cdots & \frac{\partial X_n(y,t)}{\partial y_n} & \frac{\partial X_n(y,t)}{\partial t} \\ \frac{\partial U(y,t)}{\partial y_1} & \frac{\partial U(y,t)}{\partial y_2} & \cdots & \frac{\partial U(y,t)}{\partial y_n} & \frac{\partial U(y,t)}{\partial t} \end{bmatrix}, \tag{18}$$

then

$$W(Y(x^0(y'),u_0(y')),T(x^0(y'),u_0(y'))=\tilde{A}(y'),\quad \forall y'\in\omega \tag{19}$$

and therefore $W(Y(x,u),T(x,u))\neq 0$ in some neighbourhood of the point $(x^0(y'^0),u_0(y'^0))$.

Theorem 5. *Suppose (14) holds. Then the following formula holds*

$$\frac{\partial Y_n(x,u)}{\partial u}=-\frac{1}{W(Y(x,u),T(x,u))}\times \tag{20}$$

$$\begin{vmatrix} \frac{\partial X_1}{\partial y_1}(Y(x,u),T(x,u)) & \frac{\partial X_1}{\partial y_2}(Y(x,u),T(x,u)) & \cdots & \frac{\partial X_1}{\partial y_{n-1}}(Y(x,u),T(x,u)) & a_1(x,u) \\ \frac{\partial X_2}{\partial y_1}(Y(x,u),T(x,u)) & \frac{\partial X_2}{\partial y_2}(Y(x,u),T(x,u)) & \cdots & \frac{\partial X_2}{\partial y_{n-1}}(Y(x,u),T(x,u)) & a_2(x,u) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_n}{\partial y_1}(Y(x,u),T(x,u)) & \frac{\partial X_n}{\partial y_2}(Y(x,u),T(x,u)) & \cdots & \frac{\partial X_n}{\partial y_{n-1}}(Y(x,u),T(x,u)) & a_n(x,u) \end{vmatrix}$$

Remark 2. *From (6), (20) we have*

$$\frac{\partial Y_n(x^0(y'^0),u_0(y'^0))}{\partial u}=0. \tag{21}$$

8. Solvability of the Cauchy problem (1), (4)

Suppose

$$\frac{\partial^2 Y_n(x^0(y'^0),u_0(y'^0))}{\partial u^2}\neq 0. \tag{22}$$

From the implicit function theorem it follows that the equation

$$\frac{\partial Y_n(x,u)}{\partial u}=0 \tag{23}$$

defines a function $u=\psi(x)$, that satisfies the condition

$$\psi(x^0(y'^0))=u_0(y'^0).$$

We denote by L the n-dimensional surface in $R^{n+1}_{x,u}$ that is defined by

$$L \equiv \{(x, u) \in R^{n+1}; u = \psi(x), x \text{ is in a neighbourhood of } x^0(y'^0) \in \Omega \subset R^n\}. \quad (24)$$

and by M the following (n-1)-dimensional surface in $R^{n+1}_{x,u}$

$$M \equiv \{(x, u) \in R^{n+1}; x = x^0(y'), u = u_0(y')), y' \in \omega\} \quad (25)$$

Thenit is obviousely that

$$(x^0(y'^0)), u_0(y'^0)) \in L \cap M.$$

The surface L separates $R^{n+1}_{x,u}$, locally at the point $x^0(y'^0)$, into two parts L^+ and L^-. Namely,

$$L^+ \equiv \{(x, u) \in R^{n+1}; u > \psi(x)); x \text{ is in a neighbourhood of } x^0(y'^0) \in R^n\}. \quad (26)$$

$$L^- \equiv \{(x, u) \in R^{n+1}; u < \psi(x)); x \text{ is in a neighbourhood of } x^0(y'^0) \in R^n\}. \quad (27)$$

We denote

$$M^+ \equiv M \cap L^+,$$

$$M^- \equiv M \cap L^-.$$

Proposition 1. *Suppose (6), (7) hold and* $n \geq 3$. *Then either* $M^+ = \emptyset$ *or* $M^- = \emptyset$.

Suppose, for definiteness, that $M^+ \neq \emptyset$ and $M_1^+, M_2^+, ..., M_k^+$ are its connected components. Each surface M_j^+, j = 1, 2, ..., k, determines in a semineighbourhood of the point $x^0(y'^0)$ a classical C^1-solution $u_j(x)$ to Cauchy problem (1), (4).

Theorem 6. *Suppose* $n \geq 3$ *and all conditions (6), (7), (14) (22) hold. Then for solvability of the characteristic Cauchy problem (1), (4) in a semineighbourhood of the point* $x^0(y'^0)$, *it is necessary and sufficient that all functions* $u_j(x)$, j = *1, 2, ..., k, coincide each to other in that semineighbourhood.*

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