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# Mathematical Definitions of Forest Energy and Forest Health for Forest Kinematic Model

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## Abstract

We are concerned with a forest kinematic model presented by Kuznetsov et al. [4]. In this report, we will survey some results obtained from investigation of this model equations (see [1,2,3]). By using Lyapunov function, we can define forest energy and represent the direction of the growth of forest. Moreover, on the basis of theoretical results combining with some numerical results, we will propose a mathematical quantity to measure the health of forest ecosystem.

## 1. Introduction

In the study of forest growth dynamics, the numerical simulations on the basis of suitable mathematical models are becoming one of indispensable methods. When we concerned with dynamics of forest ecosystem, age dependent tree relationship is more interesting than the individual of trees. By forest age structure dynamics we mean the space and time variation of tree numbers in different age classes, caused by various internal and external factors.

In this talk, we are concerned with the Age-Structured Continuous Space Model. Among others we consider a prototype model describing the growth of a forest by age-dependent trees relationships and by regeneration processes, which was proposed by Kuznetsov et al. [4]. They considered a mono-species ecosystem with only two age classes of trees, the young age class and the old age class, and model the regeneration process by seed production, seed dispersion and establishment of seeds. Their system of

equations reads

$$\begin{cases} \frac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial t} = fu - hv & \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x) & \text{in } \Omega. \end{cases} \quad (1.1)$$

Here,  $\Omega$  is a  $\mathcal{C}^2$  or convex, bounded domain in  $\mathbb{R}^2$ . The unknown functions  $u = u(x, t)$  and  $v = v(x, t)$  denote the tree densities of young and old age classes, respectively, at a position  $x \in \Omega$  and at time  $t \in [0, \infty)$ . The third unknown function  $w = w(x, t)$  denotes the density of seeds in the air at  $x \in \Omega$  and  $t \in [0, \infty)$ . The third equation describes the kinetics of seeds;  $d > 0$  is a diffusion constant of seeds, and  $\alpha > 0$  and  $\beta > 0$  are seed production and seed deposition rates, respectively. While the first and second equations describe the growth of young and old trees, respectively;  $0 < \delta \leq 1$  is an establishment rate of seeds,  $f > 0$  is an aging rate,  $h > 0$  is a mortality of old trees. And  $\gamma(v) > 0$  is a mortality of young trees which is allowed to depend on the old-tree density  $v$  and is expected to hit a minimum at a certain optimal value of  $v$ . We assume as in the paper [4] that the function  $\gamma(v)$  is given by a quadratic function

$$\gamma(v) = a(v - b)^2 + c, \quad (1.2)$$

where  $a, b, c > 0$  are all positive constants.

In the papers [1], we constructed a dynamical system  $(S(t), K, X)$  determined from the initial-boundary value problem (1.1). As the underlying space  $X$ , we set a space of the form

$$X = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u \in L^\infty(\Omega), v \in L^\infty(\Omega), w \in L^2(\Omega) \right\}. \quad (1.3)$$

It is necessary to handle the first and second ordinary differential equations in the Banach space  $L^\infty(\Omega)$ . Indeed, since  $\gamma(v)u$  contains a nonlinear term like  $v^2u$  (see (1.2)), the Banach space to be chosen must enjoy a norm property  $\|v^2u\| \leq C\|v\|^2\|u\|$ , namely, the space must be a Banach algebra. Moreover, even if the initial functions  $u_0, v_0$  and  $w_0$  are smooth, its solution  $(u, v, w)$  can tend to a discontinuous stationary solution as  $t \rightarrow \infty$  (see [2, Section 6]). That is, the continuous function space  $\mathcal{C}(\bar{\Omega})$  is not suitable. The phase space  $K$  consists of triplets of nonnegative functions of  $X$ , i.e.,

$$K = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; 0 \leq u \in L^\infty(\Omega), 0 \leq v \in L^\infty(\Omega), 0 \leq w \in L^2(\Omega) \right\}. \quad (1.4)$$

The nonlinear semigroup  $S(t)$  acts on  $K$  for  $0 \leq t < \infty$ . In [2], we found a Lyapunov function and investigated asymptotic behavior of trajectories  $S(t)U_0, U_0 \in K$ . Since

some  $S(t)U_0$  can converge to a discontinuous stationary solution even if the initial value  $U_0 \in K$  consists of smooth functions and since if so the trajectory  $S(t)U_0$  has an empty  $\omega$ -limit set in  $X$ , the dynamical system  $(S(t), K, X)$  never enjoys any compact attractor in general. By this reason we introduced three kinds of  $\omega$ -limit sets for  $U_0 \in K$ , i.e.,  $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset w^*\text{-}\omega(U_0) \neq \emptyset$ , here  $\omega(U_0)$  denotes the usual one,  $L^2\text{-}\omega(U_0)$  is an  $\omega$ -limit set with respect to the  $L^2$  topology and  $w^*\text{-}\omega(U_0)$  is that with respect to the weak\* topology of  $L^\infty(\Omega)$ . And we proved by utilizing the Lyapunov function that  $L^2\text{-}\omega(U_0)$  consists of stationary solutions only. So, roughly speaking, every trajectory  $S(t)U_0, U_0 \in K$ , converges asymptotically to some stationary solution of (1.1).

In the paper [3], we study the structure of stationary solutions of (1.1). The structure depends on the parameter  $h$  drastically. In fact, when  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ , where  $a, b$  and  $c$  are positive constants contained in  $\gamma(v)$  (see (1.2)), it is shown that there exist two homogeneous stationary solutions  $P_+$  (which is non zero solution) and the zero solution  $O = (0, 0, 0)$  and that  $P_+$  is stable and  $O$  is unstable. This means that in this case any forest starting from a non zero initial state holds alive. In the meantime, when  $\frac{f\alpha\delta}{c+f} < h < \infty$ , the zero solution  $O$  is a unique stationary solution and is globally stable, that is, every forest is going to vanish asymptotically. When  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$ , there exist three homogeneous stationary solutions  $P_\pm$  (which are non zero) and the zero solution  $O$ ; here,  $P_+$  and  $O$  are stable meanwhile  $P_-$  is unstable. This means that some forests can hold alive and others are going to vanish. What is more interesting is that, in this case, there exist many inhomogeneous stationary solutions. Especially when  $a$  and  $b$  are sufficiently large, one can construct an infinite number of discontinuous stationary solutions  $(\bar{u}, \bar{v}, \bar{w})$ 's,  $\bar{u}, \bar{v} \in L^\infty(\Omega)$  being discontinuous and  $\bar{w} \in H^2(\Omega)$  being continuous.

By using this results and combining with the Lyapunov function (kinetic energy of dynamical system), we can represent the direction of the growth of forest as in Figure 1.

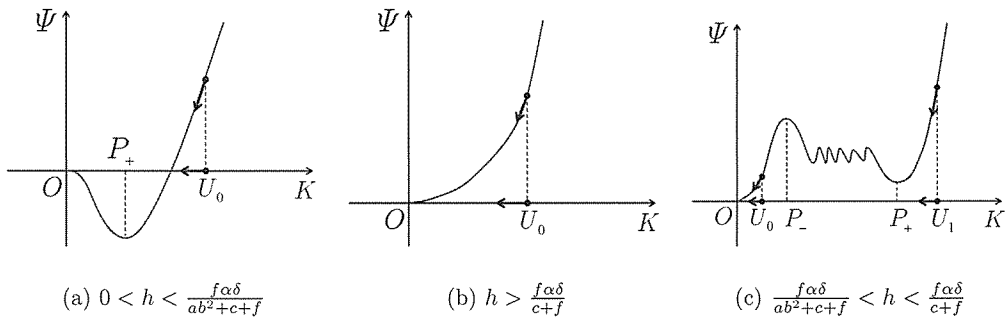


Figure 1: Graph of Lyapunov function

*Case 1.*  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ . There exists two homogeneous stationary solutions  $P_+$  and  $O$ , where  $P_+$  is stable and minimal energy; and  $O$  is unstable. As shown in Figure 1 (a), every trajectory starting from an initial state  $U_0 \neq 0$  always proceeds in the way that its energy decreases and tends to  $P_+$ .

*Case 2.*  $\frac{f\alpha\delta}{c+f} < h < \infty$ . The zero solution  $O$  is a unique stationary solution and minimal energy (see Figure 1 (b)). Therefore, every trajectory of dynamical system tends to 0.

*Case 3.*  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$ . There exists three homogeneous stationary solutions  $P_+$ ,  $P_-$  and 0, where  $P_+$  and 0 are stable and local minimal energy; and  $P_-$  is unstable. More precisely, as shown in Figure 1 (c), there exists so many stationary solutions (perhaps discontinuous) which are local minimal energy. It implies that some trajectory can tend to  $P_+$ , or 0 or some stationary solution depending on the initial condition.

## 2. Mathematical measurement of forest health

There are many definitions of “forest health” depending on the viewpoint of the user of the forest. Forest health reflects many concerns about the sustainability of forest ecosystems. The important meaning of forest health is that the ability of a forest to recover from natural and human-caused stresses or disturbances. On the basis of theoretical results, we will propose a mathematical quantity to measure the health of forest ecosystem which is described by (1.1). This definition is from the viewpoint of asymptotic behavior of solutions.

When  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ , we known there exist two homogeneous stationary solutions  $P_+$  which is stable and the zero solution  $O$  which is unstable. In addition, there is no nonnegative stationary solution other than homogeneous ones. This means that in this case any forest starting from a nonzero initial state holds alive. We can interpret this fact as follows. Let us consider a regeneration of trees of old age class. They produce seeds with rate  $\alpha$  and the seeds are established with rate  $\delta$  and become young trees, and then some young trees die with rate  $a(v-b)^2+c$  but others grow toward old trees with rate  $f$ ; so the net of aging rate is given by  $\frac{f}{a(v-b)^2+c+f}$ . In this way, on one hand, we see that the regeneration rate of trees of old age class is  $\frac{\alpha\delta f}{a(v-b)^2+c+f}$ . In the worst case, i.e.,  $v=0$ , we have a rate  $\frac{\alpha\delta f}{ab^2+c+f}$ . On the other hand, the death rate of old trees is give by  $h$ . Therefore, if  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ , then the regeneration rate always dominates the death rate, namely the forest is never extinct.

In the meantime, when  $\frac{f\alpha\delta}{c+f} < h < \infty$ , the zero solution is a unique stationary solution and is globally stable, that is, every forest is going to vanish asymptotically. As shown above, when  $v=b$ , we have an optimal regeneration rate  $\frac{f\alpha\delta}{c+f}$ , so  $\frac{f\alpha\delta}{c+f} < h < \infty$  means that the death rate  $h$  of old age trees is large than the optimal regeneration rate. That is, the forest cannot be alive in any form.

In the case when  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$  is valid, there exist three homogeneous stationary solutions  $P_{\pm}$  and the zero solution  $O$ ; here  $P_+$  and  $O$  are stable meanwhile  $P_-$  is unstable. We know also that in this case there are many stationary solutions (sometime infinite number of stationary solutions). This means that some forest can hold alive and others are going to vanish. In view of these facts we are naturally led to define a number  $\Phi$  in such a way that

$$h = \frac{f\alpha\delta}{ab^2\Phi + c + f}$$

or

$$\Phi(a, b, c, f, h, \alpha, \delta) = \frac{f\{\frac{\alpha\delta}{h} - 1\} - c}{ab^2}. \quad (2.1)$$

By the discussion above, if  $\Phi < 0$  then  $\frac{f\alpha\delta}{c+f} < h$  and the forest is going to vanish; if  $\Phi > 1$  then  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$  and any forest starting from a nonzero initial state holds alive; and if  $0 < \Phi < 1$ , then  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$  and some forests can hold alive and others are going to vanish depending on the initial states.

Then, in some sense, we can say that the quantity  $\Phi(a, b, c, f, h, \alpha, \delta)$  is a measurement for forest health. In this way we have defined  $\Phi$  from the arguments of mathematical analysis. But, as will be observed in the next section, such a definition can be justified by numerical results.

### 3. Numerical Results

This section is devoted to presenting some numerical results for the system (1.1). Throughout all numerical computations, we set  $\Omega = [0, 1] \times [0, 1]$ . The coefficients of system (1.1) are fixed as  $\beta = 1.0$ ,  $\delta = 1$ ,  $a = 1.0$ ,  $c = 0.2$ ,  $d = 0.05$  and  $b = 1$ . The rest coefficients  $\alpha$ ,  $f$ ,  $h$  will be fixed depending on the purpose of calculations. We consider a forest ecosystem in which there exists the stationary state  $P_+$ . It is already known that  $P_+$  is an exponentially stable equilibrium of the dynamical system. Hence, if there is nothing to influence, forest can hold alive in this state forever.

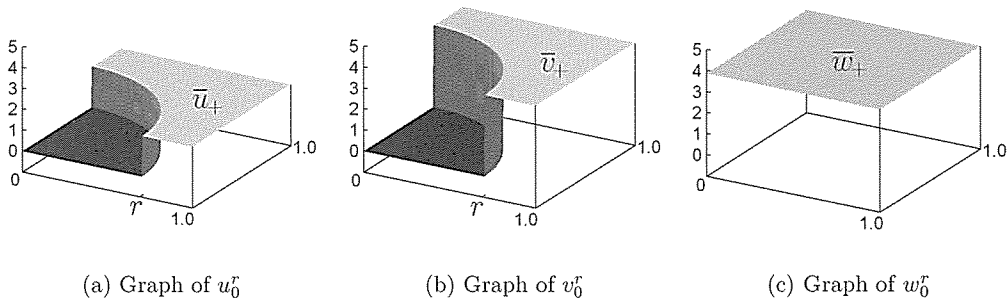


Figure 2: Initial function  $P_+^r = (u_0^r, v_0^r, w_0^r)$

Now we cut a part of forest (include young trees and old trees) in a quarter circle with radius  $r$  as shown in Figure 2, and observe what happen to the forest ecosystem. It is easy to see that if we cut a little part, say  $r$  is small, then forest can evolve towards recover all domain to stationary state  $P_+$ ; if we cut too much, say  $r$  is large, forest is going to vanish; and in some cases, forest can tend to a discontinuous state. The following numerical examples show this fact.

In the numerical computation, the coefficients are fixed as  $\alpha = 1.0$  and  $f = 1$ , but the mortality of old trees  $h$  and the radius  $r$  are variable. We consider two cases  $b = 1$  and

$b = 3$ . From the theoretical results it follows that, if  $b = 1$  then  $ab^2 < 3(c + f)$  and every stationary solution is continuous, therefore we can not expect that the solution will tend to a discontinuous stationary solution. Contradictorily, if  $b = 3$  then  $ab^2 > 3(c + f)$  and it is possible that some solution tends to a discontinuous stationary solution.

Now for each value of  $h$ , we calculate the values of  $r$  such that forest starting from initial state  $P_+^r$  is going to vanish, or recover to homogeneous stationary solution  $P_+$ , or tend to a inhomogeneous stationary solution. We performed numerical computations for sufficiently large time until the graph of solutions and the values of Lyapunov function are stabilized numerically. The relation between  $h$  and  $r$  is as shown in Figure 3.

*Case 1.*  $b = 1$ . All solutions tend to homogeneous stationary solutions  $P_+$  or  $O$ . There are two regions **R** and **E** as shown in 3 (a). If  $(h, r) \in \mathbf{E}$  then the forest is going to vanish; meanwhile if  $(h, r) \in \mathbf{R}$  then the forest is going to recover to  $P_+$ .

*Case 2.*  $b = 3$ . There are three regions **R**, **D** and **E**. If  $(h, r) \in \mathbf{D}$  then the forest tends to a discontinuous stationary solution.

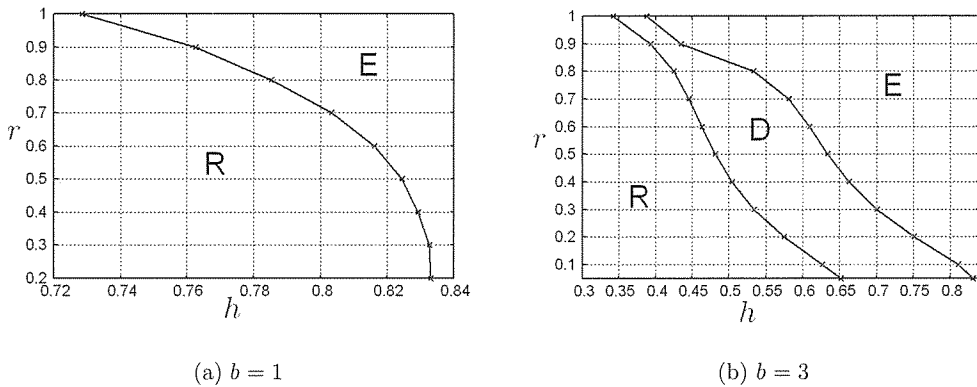


Figure 3: Relation between  $h$  and  $r$

Therefore we are led to define, for each forest ecosystem, the restitution radius by

$$R = \sup\{r > 0; \text{forest starting from } P_+^r \text{ is going to recover to } P_+\}. \quad (3.1)$$

This radius  $R = R(a, b, c, d, f, h, \alpha, \beta, \delta)$  depends on all parameters of model.

Now we present some numerical results to show the relation between the measurement of forest health  $\Phi$  which is defined in (2.1) and the restitution radius  $R$ . The main idea is that, first we fix an initial function  $P_+^r$  (see Figure 2); second we change values of parameters of the system (1.1) and calculate to find if the solution starting from  $P_+^r$  is going to tends to  $P_+$  or not; third, from these calculations, we can divide values of all parameters into regions **R**, **D** or **E** as same meaning as above. Finally, if values of  $\Phi$  in each region are separated independent of parameters then we can say  $\Phi$  is characteristic for restitution of forest with respect to initial function  $P_+^r$ .

The numerical calculation is performed in the following way. The initial function  $P_+^r$  is fixed with  $r = 0.5$ . For simplicity of calculations, we fix all parameters except two of them and find out the relation between the two parameters.

*Case 1.* Fix  $f = 1$ ,  $b = 1$ . The parameters  $h$  and  $\alpha$  are variable. In this case,  $ab^2 < 3(c + f)$  therefore all stationary solutions are continuous. For each value  $\alpha$  in  $[0.1, 1]$ , we find the values of  $h$  such that forest starting from  $P_+^{0.5}$  is going to vanish, or recovers to homogeneous stationary solution  $P^+$ .

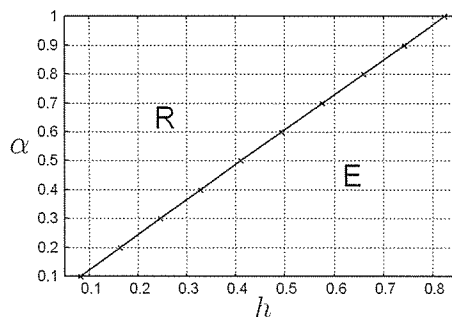


Figure 4: Relation between  $h$  and  $\alpha$

Figure 4 shows numerical plot of  $h$  and  $\alpha$ . It is easy to see that there is a linear relation between  $h$  and  $\alpha$ , and the line  $(h, \alpha)$  separates two regions **R** and **E**. Moreover, the values of  $\Phi$  in this line can be approximate by a constant  $C \in (0, 1)$ . Hence, if  $(h, \alpha)$  belong to the region for that  $\Phi > C$  then the forest starting from  $P_+^{0.5}$  tends to homogeneous stationary solution  $P^+$ . On the contrary, if  $(h, \alpha)$  belong to the region for that  $\Phi < C$  then the forest starting from  $P_+^{0.5}$  is going to vanish.

*Case 2.* Fix  $\alpha = 1$ ,  $b = 1$ . The coefficients  $h$  and  $f$  are variable. Let  $f \in [0.2, 1]$ , then  $ab^2 < 3(c + f)$  and therefore all stationary solutions are continuous. The graph of  $h$  and  $f$  is shown in Figure 5 (a). There are two regions **R** and **E** separated by a curve. Figure 5 (b) shows the graph of  $1/h$  and  $1/f$ . It is easy to see there is a linear relation between  $1/h$  and  $1/f$ . Moreover, values of  $\Phi$  in the curve separates two regions **R** and **E** can be approximate by the same constant  $C$  as in *Case 1*. Hence, if  $(h, f)$  belong to the region for that  $\Phi > C$  then the forest starting from  $P_+^{0.5}$  tends to homogeneous stationary solution  $P^+$ . On the contrary, if  $(h, f)$  belong to the region for that  $\Phi < C$  then the forest starting from  $P_+^{0.5}$  is going to vanish.

These numerical results show that, if the parameters of system are taken so that the restitution radius is constant with  $R = 0.5$ , then  $\alpha$  and  $h$  are proportional and  $1/f$  and  $1/h$  are in a linear relation. These then mean that the measurement of forest health is also constant with  $\Phi = C$ ,  $C$  being a suitable constant, for the change of parameters under the restriction  $R = 0.5$ . Some other numerical computations show inverse results, namely, if the parameters of system are taken so that the measurement of forest health is constant, then the restitution radius is constant. These observations suggest us that  $\Phi$  and  $R$  are connected intimately, probably there would exist a one-to-one correspondence among them. One could calculate the restitution radius from the measurement  $\Phi$  alone



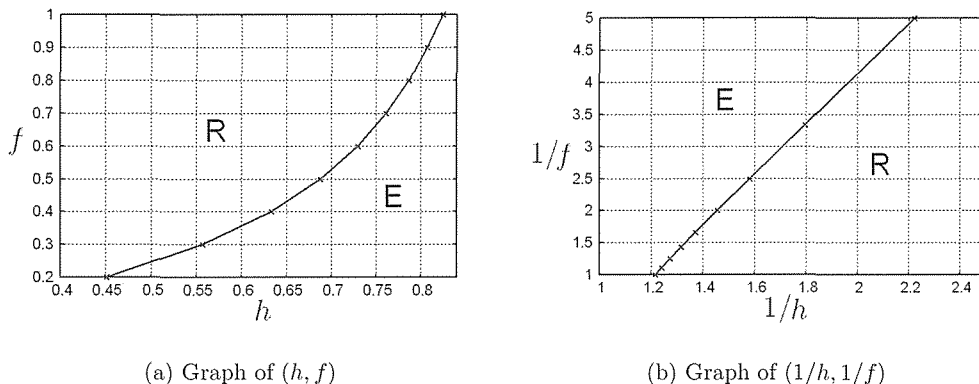


Figure 5: Relation between  $h$  and  $f$

which is determined by the ecological parameters appearing in the system. And one could therefore characterize the restitution radius from the ecological parameters alone.

It is now very important problem to know how the restitution radius is determined from the measurement  $\Phi$ . To know this, however, it is needed to perform more numerical computations and to analyse these results. For the moment it is only possible to say that  $R$  is an increasing function of  $\Phi$  and the diffusion coefficient  $d$  of seeds in the air also contributes to the correspondence  $\Phi \rightarrow R$ , although  $d$  does not appear in the definition of  $\Phi$ .

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