DYNAMICAL SYSTEM AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR FORESTRY KINEMATIC MODEL

L. H. Chuan* and A. Yagi**

* *Department of Environmental Technology, Osaka University, Suita, Osaka 565-0871, Japan*
** *Department of Applied Physics, Osaka University, Suita, Osaka 565-0871, Japan*

ABSTRACT

We are concerned with a forestry kinematic model presented by Kuznetsov et al. [4]. In this report, we will survey how to construct global solutions and a dynamical system for the model equations. We introduce three kinds of ω-limit sets, namely, $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \text{w}^*\text{-}\omega(U_0)$, for each point U_0. Using a Lyapunov function, we will then investigate basic properties of these ω-limit sets. Especially, it shall be explained that $L^2\text{-}\omega(U_0)$ consists of stationary solutions alone.

KEYWORDS

Forestry ecosystem, Dynamical system, Asymptotic behavior of solutions, ω-limit set, Lyapunov function.

INTRODUCTION

We study the initial-boundary values problem for a parabolic-ordinary system

$$(1) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in } \Omega. \end{cases}$$

This system has been introduced by Kuznetsov et al. [4] in order to describe the kinetics of forest from the viewpoint of the age structure. For simplicity they consider a prototype ecosystem of a mono-species and with only two age classes in a two-dimensional domain Ω.

The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and at time $t \in [0,\infty)$. The third unknown function $w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates respectively. While the first and second equations describe the growth of young and old trees respectively; $0 < \delta \leq 1$ is a seed establishment rate, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density v, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees.

On w, the Neumann boundary conditions are imposed on the boundary $\partial\Omega$. Nonnegative initial functions $u_0(x) \geq 0$, $v_0(x) \geq 0$ and $w_0(x) \geq 0$ are given in Ω.

Several authors have already been interested in such a model. Wu [8] studied the stability of travelling wave solutions. Wu and Lin [9] discussed the stability of stationary solutions. Lin and Liu [5] extended this result to a case when the model includes nonlocal effects.

In this report we intend to construct a global solution to (1) for each initial function $U_0 \in K$ and to construct a dynamical system determined from the problem. Furthermore, we are concerned with studying asymptotic behavior of solutions.

Throughout the report, Ω is a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$. According to [11], the Poisson problem $-d\Delta w + \beta w = v$ in Ω under the Neumann boundary conditions $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$ enjoys the optimal shift property that $v \in L^2(\Omega)$ always implies that $w \in H^2(\Omega)$. We assume as in [4] that the mortality of young trees is given by a square function of the form

$$\gamma(v) = a(v-b)^2 + c, \tag{2}$$

where a, b, $c > 0$ are positive constants. This means that the mortality takes its minimum when the old-age tree density is a specific value b. As mentioned, d, f, h, α, $\beta > 0$ are all positive constants and $0 < \delta \leq 1$.

MATERIALS AND MATHODS

We shall formulate the initial boundary value problem (1) as the Cauchy problem for an abstract semilinear equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0 \end{cases}$$

in the underlying product space $X = L^\infty(\Omega) \times L^\infty(\Omega) \times L^2(\Omega)$. Here, the linear operator A and the nonlinear operator F are defined by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \beta\delta w - \gamma(v)u \\ fu \\ \alpha v \end{pmatrix},$$

where Λ is a realization of the operator $-d\Delta + \beta$ in $L^2(\Omega)$ under the homogeneous Neumann boundary condition $\frac{\partial w}{\partial n} = 0$ on the boundary $\partial\Omega$. It is known that Λ is a positive definite self-adjoint operator of $L^2(\Omega)$ with $\mathcal{D}(\Lambda) = H^2_N(\Omega)$ (see [10, 11]), where $H^2_N(\Omega)$ is a closed subspace of $H^2(\Omega)$ consisting of functions w's satisfying the homogeneous Neumann boundary conditions on $\partial\Omega$. The initial value U_0 is taken from the space

$$K = \{(u_0, v_0, w_0); 0 \le u_0,\ v_0 \in L^\infty(\Omega) \text{ and } 0 \le w_0 \in L^2(\Omega)\}.$$

Then we can apply the general results in [7] to construct local solutions. Nonnegativity of local solutions and a priori estimates for local solutions will be established in ordinary manners. As an immediate consequence of a priori estimates, we can prove the existence and uniqueness of global solution. Moreover, from the Lipschitz continuity of solution in initial data, we can construct a dynamical system determined from (1).

In the next part, we investigate asymptotic behavior of each trajectory of the dynamical system. For this purpose, we will introduce three kinds of ω-limit set, namely, $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \mathrm{w}^*\text{-}\omega(U_0)$ for $U_0 \in K$. By finding a Lyapunov function for our dynamical system, we can obtain many results on these ω-limit sets.

RESULTS AND DISCUSSION

Theorem 1. *For any $U_0 \in K$, (1) possesses a unique global solution such that*

$$\begin{cases} 0 \le u, v \in \mathcal{C}([0,\infty); L^\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L^\infty(\Omega)), \\ 0 \le w \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H^2_N(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)). \end{cases}$$

For each $U_0 \in K$, there exists a unique global solution $U = U(t; U_0)$ to (1) and the solution is continuous with respect to the initial value. Therefore, we can define a semigroup $\{S(t)\}_{t\ge0}$ acting on K by $S(t)U_0 = U(t;U_0)$. Such that the mapping $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into K, where K is equipped with the distance induced from the universal space X. Hence, we have constructed a dynamical system $(S(t), K, X)$ determined from (1). Moreover, $(S(t), K, X)$ admits a bounded absorbing set $\mathcal{X} \subset \mathcal{D}(A) \cap K$.

In addition, we can prove that the functional

$$\Psi(U) = \int_\Omega \Big[\frac{\alpha}{2}(fu - hv)^2 + \frac{df\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v) + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw\Big]dx$$

is a Lyapunov function for the present dynamical system $(S(t), K, X)$.

As well known, the (usual) ω-limit set of $S(t)U_0$, $U_0 \in K$, is defined by

$$\omega(U_0) = \bigcap_{t\ge0} \overline{\{S(\tau)U_0;\ t \le \tau < \infty\}} \qquad \text{(closure in the topology of } X\text{)},$$

namely, $\overline{U} \in \omega(U_0)$ if and only if there exists a time sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 \to \overline{U}$ in the topology of X. There is some numerical simulation (see [6]) suggests that there exists a trajectory which starts from a continuous initial functions $U_0 = (u_0(x), v_0(x), w_0(x)) \in K$ but, as $t \to \infty$, converges to a discontinuous stationary solution $\overline{U} = (\overline{u}(x), \overline{v}(x), \overline{w}(x))$. If this phenomenon is true, then any sequence $S(t_n)U_0$ cannot converge to $\overline{U}$ in the topology of X, namely, it is possible that $\omega(U_0) = \emptyset$. This then suggests that our dynamical system never possesses a global attractor in the topology of X. By this reason we will content ourselves with constructing nonempty ω-limit sets in a suitable weak topology of X only.

We may equip X with the L^2 topology (resp. weak* topology) as follows. A sequence $\{(u_n, v_n, w_n)\}$ in X is said to be L^2 (resp. weak*) convergent to $(u_0, v_0, w_0) \in X$ as $n \to \infty$, if $u_n \to u_0$, $v_n \to v_0$ and $w_n \to w_0$ strongly in $L^2(\Omega)$ (resp. $u_n \to u_0$ and $v_n \to v_0$ weak* in $L^\infty(\Omega)$ and $w_n \to w_0$ strongly in $L^2(\Omega)$). Then, using these topologies we can define the L^2-ω-limit set and the w*-ω-limit set of $S(t)U_0$, $U_0 \in K$, by

$$L^2\text{-}\omega(U_0) = \bigcap_{t \geq 0} \overline{\{S(\tau)U_0;\ t \leq \tau < \infty\}} \qquad \text{(closure in the } L^2 \text{ topology of } X\text{)},$$

$$\mathrm{w}^*\text{-}\omega(U_0) = \bigcap_{t \geq 0} \overline{\{S(\tau)U_0;\ t \leq \tau < \infty\}} \qquad \text{(closure in the weak}^* \text{ topology of } X\text{)}.$$

We can prove the following results:

Theorem 2. *For each $U_0 \in K$, w^*-$\omega(U_0)$ is a nonempty set.*

Theorem 3. *For each $U_0 \in K$, $\omega(U_0) \subset L^2$-$\omega(U_0) \subset \mathrm{w}^*$-$\omega(U_0)$.*

Theorem 4. *Assume that $h > \frac{f\alpha\delta}{c+f}$. Then, $\omega(U_0) = L^2$-$\omega(U_0) = \mathrm{w}^*$-$\omega(U_0) = \{(0,0,0)\}$ for every $U_0 \in K$.*

Theorem 5. *Assume that $ab^2 < 3(c+f)$. Then, L^2-$\omega(U_0) = \mathrm{w}^*$-$\omega(U_0)$ for every $U_0 \in K$.*

Theorem 6. *For any $U_0 \in K$, L^2-$\omega(U_0)$ consists of equilibria of the dynamical system.*

For the proofs of all the theorems in this report, refer to [1] and [2].

CONCLUSIONS

We constructed a global solution to (1) for each triplet of initial functions and constructed a dynamical system determined from the problem. Furthermore, by finding a Lyapunov function for our dynamical system and using three kinds of ω-limit set, we can obtain some results about asymptotic behavior of solutions.

References

[1] L. H. Chuan and A. Yagi, *Dynamical system for forest kinematic model*, Adv. Math. Sci. Appl., to appear.

[2] L. H. Chuan and A. Yagi, *Asymptotic behavior of solutions for forest kinetic equations*, Funkcial. Ekvac., to appear.

[3] L. H. Chuan and A. Yagi, *Stationary solutions to forest equations*, Glasgow Math. J., submitted.

[4] Yu A. Kuznetsov, M. Ya. Antonovsky, V. N. Biktashev and A. Aponina, *A cross-diffusion model of forest boundary dynamics*, J. Math. Biol. **32** (1994), 219-232.

[5] Y. Lin and Y. Liu, *Qualitative analysis for a model of forest with diffusion and nonlocal effects*, Nonlinear Analysis **39** (2000), 217-229.

[6] H. Nakata, *Numerical simulations for forest boundary dynamics model*, Master's thesis, Osaka University (2004).

[7] K. Osaki and A. Yagi, *Global existence for a chemotaxis-growth system in* $\mathbb{R}^2$, Adv. Math. Sci. Appl. **12** (2002), 587-606.

[8] Y. Wu, *Stability of travelling waves for a cross-diffusion model*, J. Math. Anal. Appli. **215** (1997), 388-414.

[9] Y. Wu and Y. Lin, *The stability of steady states for a model with diffusion and spatial average*, J. Math. Anal. Appli. **232** (1999), 259-271.

[10] R. Dautray and J. L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 2, Springer-Verlag, Berlin, 1988.

[11] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, London, 1985.