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DYNAMICAL SYSTEM AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR FORESTRY KINEMATIC MODEL

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ABSTRACT

We are concerned with a forestry kinematic model presented by Kuznetsov et al. [4]. In this report, we will survey how to construct global solutions and a dynamical system for the model equations. We introduce three kinds of \( \omega \)-limit sets, namely, \( \omega(U_0) \subset L^2 - \omega(U_0) \subset \omega^* - \omega(U_0) \), for each point \( U_0 \). Using a Lyapunov function, we will then investigate basic properties of these \( \omega \)-limit sets. Especially, it shall be explained that \( L^2 - \omega(U_0) \) consists of stationary solutions alone.

KEYWORDS

Forestry ecosystem, Dynamical system, Asymptotic behavior of solutions, \( \omega \)-limit set, Lyapunov function.

INTRODUCTION

We study the initial-boundary values problem for a parabolic-ordinary system

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \beta \delta w - \gamma(v) u - f u \\
\frac{\partial v}{\partial t} &= f u - h v \\
\frac{\partial w}{\partial t} &= d \Delta w - \beta w + \alpha v \\
\frac{\partial w}{\partial n} &= 0 \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x)
\end{aligned}
\]

in \( \Omega \times (0, \infty) \),

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on \( \partial \Omega \times (0, \infty) \),
in \( \Omega \).

This system has been introduced by Kuznetsov et al. [4] in order to describe the kinetics of forest from the viewpoint of the age structure. For simplicity they consider a prototype
ecosystem of a mono-species and with only two age classes in a two-dimensional domain $\Omega$.

The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and at time $t \in [0, \infty)$. The third unknown function $w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0, \infty)$. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates respectively. While the first and second equations describe the growth of young and old trees respectively; $0 < \delta \leq 1$ is a seed establishment rate, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density $v$, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees.

On $w$, the Neumann boundary conditions are imposed on the boundary $\partial \Omega$. Nonnegative initial functions $u_0(x) \geq 0$, $v_0(x) \geq 0$ and $w_0(x) \geq 0$ are given in $\Omega$.

Several authors have already been interested in such a model. Wu [8] studied the stability of travelling wave solutions. Wu and Lin [9] discussed the stability of stationary solutions. Lin and Liu [5] extended this result to a case when the model includes nonlocal effects.

In this report we intend to construct a global solution to (1) for each initial function $U_0 \in K$ and to construct a dynamical system determined from the problem. Furthermore, we are concerned with studying asymptotic behavior of solutions.

Throughout the report, $\Omega$ is a bounded, convex or $C^2$ domain in $\mathbb{R}^2$. According to [11], the Poisson problem $-d\Delta w + \beta w = v$ in $\Omega$ under the Neumann boundary conditions $\frac{\partial w}{\partial n} = 0$ on $\partial \Omega$ enjoys the optimal shift property that $v \in L^2(\Omega)$ always implies that $w \in H^2(\Omega)$. We assume as in [4] that the mortality of young trees is given by a square function of the form

$$\gamma(v) = a(v - b)^2 + c,$$

where $a, b, c > 0$ are positive constants. This means that the mortality takes its minimum when the old-age tree density is a specific value $b$. As mentioned, $d, f, h, \alpha, \beta > 0$ are all positive constants and $0 < \delta \leq 1$.

### MATERIALS AND METHODS

We shall formulate the initial boundary value problem (1) as the Cauchy problem for an abstract semilinear equation

$$\begin{cases}
\frac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\
U(0) = U_0
\end{cases}$$

in the underlying product space $X = L^\infty(\Omega) \times L^\infty(\Omega) \times L^2(\Omega)$. Here, the linear operator $A$ and the nonlinear operator $F$ are defined by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad F(U) = \begin{pmatrix} \beta \delta w - \gamma(v)u \\ fu \\ \alpha v \end{pmatrix},$$
where \( \Lambda \) is a realization of the operator \(-d\Delta + \beta\) in \( L^2(\Omega) \) under the homogeneous Neumann boundary condition \( \partial_\omega \omega = 0 \) on the boundary \( \partial\Omega \). It is known that \( \Lambda \) is a positive definite self-adjoint operator of \( L^2(\Omega) \) with \( \mathcal{D}(\Lambda) = H^2_N(\Omega) \) (see [10, 11]), where \( H^2_N(\Omega) \) is a closed subspace of \( H^2(\Omega) \) consisting of functions \( w \)'s satisfying the homogeneous Neumann boundary conditions on \( \partial\Omega \). The initial value \( U_0 \) is taken from the space

\[
K = \{(u_0, v_0, w_0); 0 \leq u_0, v_0 \in L^\infty(\Omega) \text{ and } 0 \leq w_0 \in L^2(\Omega)\}.
\]

Then we can apply the general results in [7] to construct local solutions. Nonnegativity of local solutions and a priori estimates for local solutions will be established in ordinary manners. As an immediate consequence of a priori estimates, we can prove the existence and uniqueness of global solution. Moreover, from the Lipschitz continuity of solution in initial data, we can construct a dynamical system determined from (1).

In the next part, we investigate asymptotic behavior of each trajectory of the dynamical system. For this purpose, we will introduce three kinds of \( \omega \)-limit set, namely, \( \omega(U_0) \subset L^2-\omega(U_0) \subset w^*\omega(U_0) \) for \( U_0 \in K \). By finding a Lyapunov function for our dynamical system, we can obtain many results on these \( \omega \)-limit sets.

**RESULTS AND DISCUSSION**

**Theorem 1.** For any \( U_0 \in K \), (1) possesses a unique global solution such that

\[
\begin{align*}
0 \leq u, v \in C([0, \infty); L^\infty(\Omega)) \cap C^1((0, \infty); L^\infty(\Omega)), \\
0 \leq w \in C([0, \infty); L^2(\Omega)) \cap C((0, \infty); H^2_N(\Omega)) \cap C^1((0, \infty); L^2(\Omega)).
\end{align*}
\]

For each \( U_0 \in K \), there exists a unique global solution \( U = U(t; U_0) \) to (1) and the solution is continuous with respect to the initial value. Therefore, we can define a semigroup \( \{S(t)\}_{t \geq 0} \) acting on \( K \) by \( S(t)U_0 = U(t; U_0) \). Such that the mapping \( (t, U_0) \mapsto S(t)U_0 \) is continuous from \([0, \infty) \times K\) into \( K \), where \( K \) is equipped with the distance induced from the universal space \( X \). Hence, we have constructed a dynamical system \((S(t), K, X)\) determined from (1). Moreover, \((S(t), K, X)\) admits a bounded absorbing set \( X \subset \mathcal{D}(\Lambda) \cap K \).

In addition, we can prove that the functional

\[
\Psi(U) = \int_\Omega \left[ \frac{\alpha}{2} (fu - hv)^2 + \frac{df}{2} \frac{\beta}{\delta} |\nabla w|^2 + \frac{f}{\alpha} \Gamma(v) + \frac{f^2}{2} \beta^3 w^2 - (f \alpha \beta \delta) vw \right] dx
\]

is a Lyapunov function for the present dynamical system \((S(t), K, X)\).

As well known, the (usual) \( \omega \)-limit set of \( S(t)U_0, U_0 \in K \), is defined by

\[
\omega(U_0) = \bigcap_{t \geq 0} \{S(\tau)U_0; \tau \leq \tau < \infty\} \quad \text{(closure in the topology of } X),
\]

namely, \( \bar{U} \in \omega(U_0) \) if and only if there exists a time sequence \( \{t_n\} \) tending to \( \infty \) such that \( S(t_n)U_0 \rightarrow \bar{U} \) in the topology of \( X \). There is some numerical simulation (see [6]) suggests that there exists a trajectory which starts from a continuous initial functions \( U_0 = (u_0(x), v_0(x), w_0(x)) \in K \) but, as \( t \rightarrow \infty \), converges to a discontinuous stationary...
solution $\overline{U} = (\overline{u}(x), \overline{v}(x), \overline{w}(x))$. If this phenomenon is true, then any sequence $S(t_n)U_0$ cannot converge to $\overline{U}$ in the topology of $X$, namely, it is possible that $\omega(U_0) = \emptyset$. This then suggests that our dynamical system never possesses a global attractor in the topology of $X$. By this reason we will content ourselves with constructing nonempty $\omega$-limit sets in a suitable weak topology of $X$ only.

We may equip $X$ with the $L^2$ topology (resp. weak* topology) as follows. A sequence $\{(u_n, v_n, w_n)\}$ in $X$ is said to be $L^2$ (resp. weak*) convergent to $(u_0, v_0, w_0) \in X$ as $n \to \infty$, if $u_n \to u_0$, $v_n \to v_0$ and $w_n \to w_0$ strongly in $L^2(\Omega)$ (resp. $u_n \to u_0$ and $v_n \to v_0$ weak* in $L^\infty(\Omega)$ and $w_n \to w_0$ strongly in $L^2(\Omega)$). Then, using these topologies we can define the $L^2$-$\omega$-limit set and the $w^*$-$\omega$-limit set of $S(t)U_0$, $U_0 \in K$, by

$$L^2-\omega(U_0) = \bigcap_{t \geq 0} \{S(t)U_0; t \leq \tau < \infty\} \quad \text{(closure in the } L^2 \text{ topology of } X),$$

$$w^*-\omega(U_0) = \bigcap_{t \geq 0} \{S(t)U_0; t \leq \tau < \infty\} \quad \text{(closure in the weak* topology of } X).$$

We can prove the following results:

**Theorem 2.** For each $U_0 \in K$, $w^*\omega(U_0)$ is a nonempty set.

**Theorem 3.** For each $U_0 \in K$, $\omega(U_0) \subseteq L^2-\omega(U_0) \subseteq w^*\omega(U_0)$.

**Theorem 4.** Assume that $h > \frac{fa\delta}{c+f}$. Then, $\omega(U_0) = L^2-\omega(U_0) = w^*\omega(U_0) = \{(0,0,0)\}$ for every $U_0 \in K$.

**Theorem 5.** Assume that $ab^2 < 3(c+f)$. Then, $L^2-\omega(U_0) = w^*\omega(U_0)$ for every $U_0 \in K$.

**Theorem 6.** For any $U_0 \in K$, $L^2-\omega(U_0)$ consists of equilibria of the dynamical system.

For the proofs of all the theorems in this report, refer to [1] and [2].

**CONCLUSIONS**

We constructed a global solution to (1) for each triplet of initial functions and constructed a dynamical system determined from the problem. Furthermore, by finding a Lyapunov function for our dynamical system and using three kinds of $\omega$-limit set, we can obtain some results about asymptotic behavior of solutions.

**References**


