PSEUDO-DIFFERENTIAL OPERATORS RELATED WITH ORTHONORMAL EXPANSIONS OF GENERALIZED FUNCTIONS AND APPLICATION TO DUAL SERIES EQUATIONS

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ABSTRACT. The aim of the present work is to introduce some functional spaces for investigating pseudo-differential operators involving orthogonal expansions of generalized functions and their application to dual series equations.

1. Introduction

The purpose of the present work is to introduce some functional spaces for investigating pseudo-differential operators of the form

$$A[u](x) = \sum_{n=0}^{\infty} a(n)\hat{u}(n)\psi_n(x), \tag{1.1}$$

where $\{\psi_n(x)\}_{n=0}^{\infty}$ is an orthonormal sequence of functions in $L_2, \hat{u}(n)$ denotes a value of the generalized function u on the function $\overline{\psi_n(x)}$, $a(n)$ is a known function and is called the symbol of the operator $A[u]$.

Quite a number of problems of mechanics and mathematical physics are reduced to the investigation of the operators in the form (1.1) and to resolution of correlative dual series equations (see [2,4]). Formal techniques for solving such equations have been developed vigorously, but their solvability so far as we know has been considered comparatively weakly(see[2,4]).

Our work is constructed as follows. In Sections 2 we recall some definitions and results from the theory of orthonormal series expansions for generalized functions [5], in Sections 3 and 4 we construct some functional spaces for the investigation of the pseudo- differential operator (1.1). These spaces are constructed by a way analogous to that used for the construction of Sobolev- Slobodeskii spaces based on the Fourier transform in [1]. We present these results for investigation of dual series equations in the Section 5.

2. Integral transform of generalized functions

We denote by J a certain interval of the real axis and by $\mathcal{N}$ the linear differential operator of the form

$$\mathcal{N} = \theta_0(x)D^{n_1}\theta_1(x)D^{n_2}...D^{n_m}\theta_m(x),$$

where $D = d/dx, n_k$ are positive integer numbers, $\theta_k(x)$ are infinitely differentiable functions on J and $\theta_k(x) \neq 0, \forall x \in J$. We also require that

$$\mathcal{N} = \overline{\theta_m(x)}(-D)_{n_m}...(-D)^{n_2}\overline{\theta_1(x)}(-D)^{n_1}\overline{\theta_0(x)},$$

where $\overline{\theta_k(x)}$ denotes the complex-conjugate of the function $\theta_k(x)$. Besides, one supposes that there exist a sequence $\{\lambda_n\}_0^\infty$ of real numbers are called eigenvalues of the operator $\mathcal{N}$ and a sequence $\{\psi_n(x)\}$ of infinitely differentiable functions from $L_2(J)$ are called eigenfunctions of the operator $\mathcal{N}$, for which $|\lambda_n| \to \infty$ when $n \to \infty(|\lambda_0| \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant ...)$ and

$$\mathcal{N}\psi_n(x) = \lambda_n\psi_n(x), \quad n = 0, 1, ...$$

Suppose that functions $\psi_n(x)$ generate an orthonormal sequence in $L_2(J)$ with the scalar product and norm

$$(u, v) = \int_J u(x)\overline{v(x)}dx, \quad ||u|| = \sqrt{(u, u)}.$$

Besides, we assume also that $\lambda_n = 0(n^q), n \to \infty$.

Definition 2.1 Denote by $\mathcal{A}$ the space of test functions $\varphi(x)$ such that:

1) $\varphi(x) \in C^\infty(J)$,
2) $\forall k = 0, 1, 2...; \alpha_k(\varphi) := ||\mathcal{N}^k\varphi|| < \infty$,
3) $(\mathcal{N}^k\varphi, \psi_n) = (\varphi, \mathcal{N}^k\psi_n)$.

The sequence$\{\varphi_n(x)\}_{n=0}^\infty$ of functions from $\mathcal{A}$ is called convergent in $\mathcal{A}$ to zero, if $\alpha_k(\varphi_n) \to 0$ when $n \to \infty, \forall k = 0, 1, 2, ...$

Obviously, $\mathcal{A}$ is a linear space and $\psi_n(x) \in \mathcal{A}$. In [5] it was shown that $\mathcal{A}$ is a complete space and besides, $\mathcal{D}(J) \subset \mathcal{A} \subset L_2(J)$, where $\mathcal{D}(J)$ is the space of basic functions [5].

Theorem 2.1. . *It $\varphi \in \mathcal{A}$ then*

$$\varphi(x) = \sum_{n=0}^\infty(\varphi, \psi)\psi(x),$$

where the series converges in $\mathcal{A}$.

Theorem 2.2. . *The series* $\sum_{n=0}^{\infty} a_n \psi_n(x)$ *converges in* $\mathcal{A}$ *if and only if the series* $\sum_{n=0}^{\infty} |a_n|^2 |\lambda_n|^{2k}$ *converges for any non- negative integer number* k.

Definition 2.2. A generalized function is called any continuous line functional on the space $\mathcal{A}$. We denote by $\mathcal{A}'$ the set of all generalized functions and by $< f, \varphi >$ a value of the generalized function $f \in \mathcal{A}'$ on the test function $\varphi \in \mathcal{A}$. The value of $f \in \mathcal{A}'$ on $\overline{\varphi} \in \mathcal{A}$ we denote by (f, φ). Like this, $(f, \varphi) =< f, \overline{\varphi} >$.

In [5] it was shown that the space $\mathcal{A}'$ is complete and $L_2(J) \subset \mathcal{A}' \subset \mathcal{D}'(\mathcal{J})$, where $\mathcal{D}'(\mathcal{J})$ is the conjugate space of $\mathcal{D}(\mathcal{J})$. Hence, every function $f(x) \in L_2(J)$ determines a regular functional f by the formula

$$(f, \varphi) = \int_J f(x)\overline{\varphi(x)}dx, \quad \varphi \in \mathcal{A} \subset L_2(J). \tag{2.1}$$

Theorem 2.3. *The series* $\sum_{n=0}^{\infty} b_n \psi_n(x)$ *converges in* $\mathcal{A}'$ *if and only if there exists a non-negative integer number* q, *such that the series* $\sum_{\lambda_n \neq 0}^{\infty} |b_n|^2 |\lambda_n|^{-2q}$ *converges.*

Theorem 2.4. *If* $f \in \mathcal{A}'$ *then* f *is expanded to the series*

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\psi_n(x), \tag{2.2}$$

where the series is convergent in $\mathcal{A}'$.

Theorem 2.5. *If* $f, g \in \mathcal{A}'$ *and* $(f, \psi_n) = (g, \psi_n), \forall n$, *then* $f = g$ *in the sense of* $\mathcal{A}'$.

Remark 2.6. *If* $f \in \mathcal{A}'$ *and* $F_n = (f, \psi_n)$, *then there exists a integer number* r, *such that* $F_n = 0(|\lambda_n|^r)$ *when* $n \to \infty$.

Definition 2.3. We consider the orthonormal expantion (2.2) as the inverse formula, defining a certain integral transform of genertalized functions, wich is given by the formula

$$\hat{f}(n) := S[f](n) := (f, \psi_n)\,, f \in \mathcal{A}'\,, n = 0, 1, 2, ... \tag{2.3}$$

Note that when $f \in L_2(J)$ in virtue of (2.1) formula (2.3) has the form

$$\hat{f}(n) = \int_J f(x)\overline{\psi_n(x)}dx.$$

The inverse mapping S^{-1} is given by the formula (2.2) and may be represented in the form

$$S^{-1}[\hat{f}(n)](x) := \sum_{n=0}^{\infty} \hat{f}(n)\psi_n(x) = f. \tag{2.4}$$

Definition 2.4. The generalized differential operator $\mathcal{N}'$ is defined by the following equality

$$(\mathcal{N}'f, \varphi) = (f, \mathcal{N}\varphi), \quad f \in \mathcal{A}', \quad \varphi \in \mathcal{A}. \tag{2.5}$$

In the sequel we shall identify $\mathcal{N}'$ with $\mathcal{N}$ and understand the generalized differential operator $\mathcal{N}$ in the sense (2.5). Thus, the operator $\mathcal{N}$ defines a continuous mapping from $\mathcal{A}'$ into $\mathcal{A}'$. Therefore, for any generalized function $f \in \mathcal{A}'$ there exist derivatives $\mathcal{N}^k f$, besides

$$S[\mathcal{N}^k f] = (\mathcal{N}^k f, \psi_n) = (f, \mathcal{N}^k \psi_n) = \lambda_n^k S[f](n). \tag{2.6}$$

The formula (2.6) may be used for solving differential equations in the form

$$P(\mathcal{N})u = f, \tag{2.7}$$

where $P(x)$ is a certain polynomial with constant coefficients. Indeed, applying the operator S to the equation (2.7) and using (2.6), we have

$$P(\lambda_n)\hat{u}(n) = \hat{f}(n). \tag{2.8}$$

Assume that $P(\lambda_n) \neq 0 (\forall n)$, from (2.8) it follows that

$$\hat{u}(n) = \frac{\hat{f}(n)}{P(\lambda_n)}. \tag{2.9}$$

Applying to (2.9) the operator S^{-1} defined by the formula (2.4), one gets

$$u(x) = S^{-1}\Big[\frac{\hat{f}(n)}{P(\lambda_n)}\Big](x) = \sum_{n=0}^{\infty} \frac{\hat{f}(n)}{P(\lambda_n)}\psi_n(x). \tag{2.10}$$

3. The space H_s

Definition 3.1. Let s be a real number. Denote by H_s the set of generalized functions $f \in \mathcal{A}'$, such that

$$||f||_s^2 := \sum_{n=0}^{\infty} (1 + |n|)^{2s} |\hat{f}(n)|^2 < \infty, \tag{3.1}$$

where $\hat{f}(n) = S[f](n)$. The scalar product in H_s is defined by the formula

$$(3.2) \qquad (f,g)_s := \sum_{n=0}^{\infty}(1+|n|)^{2s}\hat{f}(n)\overline{\hat{g}(n)}.$$

Consider some examples of the space H_s. If $s = 0$ then from (3.1) it follows that $\{\hat{f}(n)\} \in l_2 : \sum_{n=0}^{\infty}|f_n|^2 < \infty$, therefore, $f(x) = S^{-1}[\hat{f}(n)](x) \in L_2(J)$. Let $s = m$ be a positive integer number, $J = (-\pi,\pi)$ and S the finite Fourier transform , then H_s turns to the Sobolev space $W_2^m(-\pi,\pi)$.

Note that, in virtue of Theorems 2.1 and 2.2, we have $\mathcal{A} \subset H_s$ for any $s \in \mathbb{R}$. Hence, for any $u \in H_s$ and $\varphi \in \mathcal{A}$, in virtue of Cauchy-Schwarz inequality we have

$$(3.3) \qquad |(u,\varphi)| = |(u, \sum_{n=0}^{\infty}\hat{\varphi}\psi_n(x))| = |_{n=0}^{\infty}\hat{u}(n)\overline{\hat{\varphi}(n)}| \leqslant ||u||_s||\varphi||_{-s}.$$

Definition 3.2. Let α be a real number. Denote by σ_α the class of functions $a(n)$ satisfying the condition

$$(3.4) \qquad |a(n)| \leqslant C(1+|n|)^{\alpha}, \forall n = 0,1,2,...$$

where C is a certain positive constant. We shall say that the function $a(n)$ belongs to the class σ_α^0 if $a(n) \in \sigma_\alpha$ and $a(n) \geq 0$. Finally, the function $a(n)$ belongs to the class σ_α^+ if $a(n)^\pm \in \sigma_{\pm\alpha}$, respectively.

Theorem 3.1. *Assume that $a(n) \in \sigma_\alpha, u \in H_s, \hat{u}(n) = S[u](n)$. Then the pseudo-differential operator*

$$(3.5) \qquad A[u](x) := S^{-1}[a(n)\hat{u}(n)](x) := \sum_{n=0}^{\infty} a(n)\hat{u}(n)\psi_n(x)$$

is bounded from H_s into $H_{s-\alpha}$. If $a(n) \in \sigma_{-\beta}$, where $\beta > 1/2$, then the operator A is completely continuous in H_s.

Proof. In virtue of Remark 2.6 and (3.4) $a(n)\hat{u}(n)$ is the slow growth at infinity. Due to Theorem 2.3 the series (3.5) converges in $\mathcal{A}'$ to certain function $v := A[u] \in \mathcal{A}'$. We show that $v \in H_{s-\alpha}$. Indeed, applying the operator S to both parts (3.5), we have

$$(3.6) \qquad \hat{v}(n) = \widehat{A[u]}(n) = a(n)\hat{u}(n).$$

Multiplying by$(1+|n|)^{s-\alpha}$ both parts (3.6), taking into account that $(1+|n|)^{-\alpha}|a(n)| \leqslant C$ for all n, we have

$$||v||^2_{s-\alpha} = ||A[u]||^2_{s-\alpha} \leqslant C\sum_{n=0}^{\infty}(1+|n|)^{2s}|\hat{u}(n)|^2 = C||u||^2_s. \tag{3.7}$$

The inequality (3.7) shows that $A[u](x) \in H_{s-\alpha}$. Now we assume that $\alpha = -\beta, \quad \beta > 1/2$. Let δ_{ij} be the Kronecker symbol. We rewrite (3.6) in the form

$$\hat{v}(n) = \sum_{j=0}^{\infty} a(j)\hat{u}(j)\delta_{nj}. \tag{3.8}$$

Multiply by $(1+|n|)^s$ both parts (3.8) and denote $f_n = (1+|n|)^s\hat{v}(n), g_n = (1+|n|)^s\hat{u}(n), f = \{f_n\}, g = \{g_n\}$. Obviously, $f, g \in l_2$ and we have

$$f_n = \sum_{j=0}^{\infty} g_j a(j)\delta_{nj}\frac{(1+|n|)^s}{(1+|j|)^s}. \tag{3.9}$$

Then (3.9) defines certain linear continuous operator $L : f = Lg$ from l_2 into l_2. In virtue of Cauchy- Schwarz inequality, we have

$$||Lg||^2_{l_2} \leqslant \sum_{j=0}^{\infty}|g_j|^2\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\left|a(n)\delta_{nj}\frac{(1+|n|)^s}{(1+|j|)^s}\right|^2 =$$

$$= \sum_{j=0}^{\infty}|g_j|^2\sum_{n=0}^{\infty}|a(n)|^2 \leqslant \sum_{j=0}^{\infty}|g_j|^2\sum_{n=0}^{\infty}\frac{C^2}{(1+|n|)^{2\beta}}(\beta > 1/2).$$

Thus, we have

$$||L||^2 \leqslant \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\left|a(n)\delta_{nj}\frac{(1+|n|)^s}{(1+|j|)^s}\right|^2 = \sum_{n=0}^{\infty}|a(n)|^2 < \infty. \tag{3.10}$$

Now we prove that the operator L is completely continuous. Indeed, let $\{\alpha_m(j)\}, (m = 1, 2, ...)$ be a complete orthonormal basic in $l_2(0 \leqslant j < \infty)$:

$$(\alpha_m, \alpha_k) := \sum_{j=0}^{\infty}\alpha_{mj}\overline{\alpha_{kj}} = \delta_{mk}.$$

Then $\{\alpha_m(j)\alpha_k(n)\}^{\infty}_{m,k=1}$ is a complte orthonormal basic in $l_2([0 \leqslant j < \infty) \times [0 \leqslant n < \infty))$. Denote

$$A(n, j) := \delta_{nj}a(j)\frac{(1+|n|)^s}{(1+|j|)^s}$$

and rewrite (3.9) in the form

$$f_n = L[g](n) = \sum_{j=0}^{\infty} A(n,j) g_j.$$

In virtue of (3.9), we have $A(n,j) \in l_2([0 \leqslant j < \infty) \times [0 \leqslant n < \infty))$, hence there is the orthnormal expansion

$$A(n,j) = \sum_{m,k=1}^{\infty} \lambda_{mk} \alpha_m(n) \alpha_k(j).$$

For arbitrary element $g = \{g_j\} \in l_2$, we put

$$A_N(n,j) = \sum_{m,k=1}^{N} \lambda_{mk} \alpha_m(n) \alpha_k(j).$$

$$L_N[g](n) = \sum_{j=0}^{\infty} A_N(n,j) g_j = \sum_{m=1}^{N} \alpha_m(n) \Big(\sum_{k=1}^{N} \lambda_{mk} \beta_k \Big),$$

where

$$\beta_k = \sum_{j=0}^{\infty} \alpha_k(j) g_j.$$

It is clear that, the operator L_N is completely continuous in l_2. Since $A_N(n,j)$ is a partial sum of the Fourier series of funtions $A_N(n,j)$, we have

$$\sum_{n,j=0}^{\infty} |A(n,j) - A_N(n,j)|^2 \to 0, \quad (N \to \infty).$$

Therefore, applying the estimation (3.10) to the operator $L - L_n$, we have

$$||L - L_N|| \to 0, \quad (N \to \infty).$$

Thus, L is a completely continuous operator. Like this, there exists the subsequence $\{f_{n'}\}$ converging in l_2, therefore, there exists a subsequence $\{\hat{v}(n')\} = \{S[Au](n')\}$ converging in $\hat{H}_s := S[H_s]$, this means that one has found a sequence $\{v_{n'}\} = \{A[u](n')\}$ converging in H_s. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let H_s^* be the conjugate space of the space H_s. Then H_s^* is isomorphic to the space H_{-s}. Besides, a value of a functional $f \in H_{-s}$ on an element $u \in H_s$ is given by the formula*

$$(u,f)_0 = \sum_{n=0}^{\infty} \hat{u}(n) \overline{\hat{f}(n)}, \tag{3.11}$$

where $\hat{f}(n) = S[f](n) = (f, \psi_n)$, $\hat{u}(n) = S[u](n) = (u, \psi_n)$.

Proof. According to Riesz Theorem on the general form of linear continuous functional in Hilbert spaces any functional $\phi(u), u \in H_s$ is given by an element $v \in H_s$ and its norm $||\phi|| = \sup_{||u||_s=1} |\phi(u)|$ equals to $||v||_s$.

Denote

$$\hat{f}(n) = (1+|n|)^{2s}\hat{v}(n), \quad f = S^{-1}[\hat{f}]. \tag{3.12}$$

Then $f \in H_{-s}, ||f||_{-s} = ||v||_s$ and $(u,v)_s = (u,f)$, where

$$(u,f) = \sum_{n=0}^{\infty} \hat{u}(n)\overline{\hat{f}(n)}. \tag{3.13}$$

Like this, (3.12) establishes an isomorphism between H_s^* and H_{-s}, besides, the value of the functional $f \in H_s$ on the element $u \in H_s$ is given by the formula (3.13).The proof of Theorem 3.2 is complete. □

In virtue of Theorem 3.2, we put $H_s^* \simeq H_{-s}$.

4. The spaces $H_s^\circ(\Omega)$ and $H_s(\Omega)$

Let Ω be a certain subset of J. Let us introduce the following definitions.

Definition 4.1. Denote by $H_s^\circ(\Omega)$ the space defined as the closure of the set $C_0^\infty(\Omega)$ of infinitely differentiable functions with a compact support in $\overline{\Omega}$ with respect to the norm (3.1). The norm in $H_s^\circ(\Omega)$ is fefined by the same (3.1).

Thus, $H_s^\circ(\Omega)$ is a subspace of H_s.

Definition 4.2 The space $H_s(\Omega)$ is defined as the set of generalized functions f from $\mathcal{D}'(\Omega)$ having extensions $lf \in H_s$. The norm in $H_s(\Omega)$ is defined by the formula

$$||f||_{H_s(\Omega)} := \underset{l}{inf}||lf||_s, \tag{4.1}$$

where the infimum is taken over all possible extentions $lf \in H_s$.

Lemma 4.1. *Assume that $u \in H_s^\circ(\Omega), v \in H_{-s}^\circ(\Omega'), \Omega \cup \Omega' = J$. Then $(u,v) = 0$, where (u,v) denotes the value of the generalized function u on the elment $\overline{v}$. Contrarily, if $v \in H_{-s}$ and $(u,v) = 0$ for all $u \in H_s^\circ(\Omega)$, then $v \in H_{-s}^\circ(\Omega')$.*

Proof. Assume that $u \in H_s^\circ(\Omega), v \in H_{-s}^\circ(\Omega')$. According to the definition of the support of generalized functions we have $(u,\varphi) = 0, \quad \forall \varphi \in C_0^\infty(\Omega')$. Sinse the set $C_0^\infty(\Omega')$ is dense in $H_{-s}^\circ(\Omega')$, therefore from the inequality (3.3) it follows that $(u,v) = 0$ for all $u \in H_s^\circ(\Omega), v \in H_{-s}^\circ(\Omega')$.

Now assume that $v \in H_{-s}$ and $(u,v)=0, \quad \forall u \in H_s^\circ(\Omega)$. Then, in particular, $(v,\varphi)=\overline{(\varphi,v)}=0$ for any $\varphi \in C_0^\infty(\Omega)$, this means $suppv \subset \overline{\Omega'}$, that is $v \in H_{-s}^\circ(\Omega')$. The proof of Lemma 4.1 is complete. □

Theorem 4.2. *Let $u \in H_s^\circ(\Omega), f \in H_{-s}(\Omega)$ and lf be an extension of the function f from Ω to J belonging to $H_{-s}(\Omega)$, then the series*

$$[u,f] := (u,lf)_0 := \sum_{n=0}^{\infty} S[u](n)\overline{S[lf]}(n) \tag{4.2}$$

does not depend on the choice of the extension lf. Therefore, this series defines a linear continuous functional on $H_s^\circ(\Omega)$. Conversely, for every linear continuous functional $\phi(u)$ on $H_s^\circ(\Omega)$ there exists an element $f \in H_{-s}(\Omega)$ such that $\Phi(u)=[u,f]$ and $||\phi|| = ||f||_{H_{-s}(\Omega)}$.

Proof. Obviously the series (4.2) is convergent . Let $l'f$ be an another extension of the function f. Then we have $lf - l'f \equiv 0$ on Ω . Due to Lemma 4.1 we have $(u, lf - l'f) \equiv 0, \forall u \in H_s^\circ(\Omega)$ and $\forall f \in H_{-s}(\Omega)$. From (4.2) it follows $|(u,lf)_0| \leqslant ||u||_s||lf||_{-s}$. Sinse (u,lf) does not depend on the choice of lf then

$$|(u,lf)_0| \leqslant ||u||_s \inf_l ||lf||_{-s} = ||u||_s||f||_{H_{-s}(\Omega)}. \tag{4.3}$$

Thus, every element $f \in H_{-s}(\Omega)$ gives a continuous functional on $H_s^\circ(\Omega)$ by the formula (4.2). Let $\Phi(u)$ be a linear continuous functional on $H_s^\circ(\Omega)$. The space $H_s^\circ(\Omega) \subset H_s$ is a Hilbert space with scalar product (3.2). Therefore, due to Riesz Theorem there exists a function $v \in H_s^\circ(\Omega)$, such that $\phi(u) = (u,v)_s$. We put $\hat{f}_0(n) = (1+|n|)^{2s}\hat{v}(n), f_0 = S^{-1}[\hat{f}(n)]$. Then $f_0 \in H_{-s}, \quad pf_0 = f \in H_{-s}(\Omega)$, where p denotes the resriction operator to Ω. We have $\phi(u)=(u,v)_s = (u,f_0)$ and $||\phi|| = ||v||_s = ||f_0||_{-s} \geq ||f||_{H_{-s}(\Omega)}$. On the other hand, in virtue of (4.3) we have $||\phi|| = \sup_{||u||_s=1} |\phi(u)| \leqslant ||f||_{H_{-s}(\Omega)}$. Like this, $||\phi|| = ||f||_{H_{-s}(\Omega)}$. The proof of Theorem 4.2 is complete . □

Let $H_s^{\circ *}(\Omega)$ be the conjugate space of the space $H_s^\circ(\Omega)$. In virtue of Theorem 4.2 we put $H_s^{\circ *}(\Omega) \simeq H_{-s}(\Omega)$.

Theorem 4.3. *Assume that $b(n) \in \sigma_{2s-\beta}(\beta > 1/2), u \in H_s^\circ(\Omega)$ and p is the resriction operator to Ω. Consider the following pseudo-differential operator*

$$B[u] = pS^{-1}[b(n)\hat{u}(n)](x), \quad \hat{u}(n) = S[u](n).$$

Then the operator B from $H_s^\circ(\Omega)$ to $H_{-s}(\Omega)$ is completely continuous.

Proof. It is not difficult to show that the operator B is continuous operator from $H^\circ_s(\Omega)$ into $H_{-s+\beta}(\Omega)$. We put

$$A[u] = S^{-1}[b(n)\hat{u}(n)](x), \quad f = pJ_{-s}A[u], Lf = plf, \tag{4.4}$$

where J_{-s} denotes the embedding operator to H_{-s}, l and p are the extension and restriction operators respectively. We have $L[f] = B[u]$, besides, the operator L is bounded from $H_{-s}(\Omega)$ into $H_{-s+\beta}(\Omega)$. Let $l_0L[f]$ be a certain continuous extension of $L[f]$ (in view of Han- Banach Theorem). Denote by Λ_β the pseudo-differential operator of the form (1.1) with the symbol $(1+|n|)^\beta$. We have

$$L[f] = p\Lambda_{-\beta}\Lambda_\beta l_0 L[f].$$

According to Theorem 3.1 the operator $\Lambda_{-\beta}(\beta > 1/2)$ is completely continuous in $H_{-s}(J)$, $\Lambda_\beta l_0 L$ and p are continuous operators, then L is completely continuous in $H_{-s}(\Omega)$. The proof of Theorem 4.3 is complete. □

5. Dual series equations

5.1. **Preparation.** Let J_1 and J_2 be certain subsets of J, such that $J_1 \cup J_2 = J$. In this section we shall consider the following dual series equation:

$$p_1 S^{-1}[a(n)\hat{u}(n)] = f_1(x), \quad x \in J_1, \tag{5.1}$$

$$p_2 S^{-1}[\hat{u}(n)](x) = f_2(x), \quad x \in J_2, \tag{5.2}$$

where $\hat{u}(n)$ is a function to be found, the function $a(n)$ is given and is called the symbol of the dual equation (5.1)-(5.2), $f_1(x) \in \mathcal{D}'(J_1)$ and $f_2(x) \in \mathcal{D}'(J_2)$ are given distributions on J_1 and J_2 respectively, finally, p_1 and p_2 are restriction operators to J_1 and J_2 respectively.

We shall investigate the dual equation (5.1)-(5.2) under the following assumptions

$$a(n) \in \sigma^o_{2\alpha}, \quad f_1(x) \in H_{-\alpha}(J_1), \quad f_2(x) \in H_\alpha(J_2) \tag{5.3}$$

and we shall find the function $\hat{u}$ in the form $\hat{u} = S[u]$, where $u \in H_\alpha$.

Theorem 5.1. *(Uniqueness). Under the assumptions* (5.3) *the dual equation* (5.1)-(5.2) *has at most one solution* $u = S^{-1}[\hat{u}] \in H_\alpha$.

Proof. To prove the theorem it suffices to show that the homogeneous dual equation

$$p_1 S^{-1}[a(n)\hat{u}(n)] = 0 \quad x \in J_1,$$

$$p_2 S^{-1}[\hat{u}(n)](x) = u(x) = 0, \quad x \in J_2$$

has only the trivial solution.

Since $u \in H^{\circ}_{\alpha}(J_1)$ the last dual equation may be rewritten as

$$(Au)(x) = 0, \quad x \in J_1, \tag{5.4}$$

where

$$(Au)(x) := p_1 S^{-1}[a(n)\hat{u}(n)](x), \quad x \in J_1. \tag{5.5}$$

Since $Au \in H_{-\alpha}(J_1) \simeq H^{\circ *}_{\alpha}(J_1)$(see Theorem 4.2) we obtain from (4.2)

$$[u, Au] = \sum_{0}^{\infty} S[u](n)\overline{S[l_1 Au](n)}, \tag{5.6}$$

where $l_1 Au$ is an arbitrary extension of Au from J_1 onto $J : l_1 Au \in H_{-\alpha}$. Since the series on the right-hand side of (5.6) does not depend upon the choice of $l_1 Au$ (see Theorem 4.2) we can take

$$l_1 Au = l_1 p_1 S^{-1}[a(n)\hat{u}(n)](x) = S^{-1}[a(n)\hat{u}(n)](x).$$

Then we have

$$[u, Au] = \sum_{0}^{\infty} a(n)|\hat{u}(n)|^2 = 0$$

if the function $u(x) = S^{-1}[\hat{u}(n)](x)$ satisfies the equation (5.4). From this it follows that $u \equiv \hat{u} \equiv 0$ since $a(n) \geq 0 (a(n) \not\equiv 0)$. The proof of Theorem 5.1 is complete. □

Lemma 5.2. *The dual equation* (5.1)-(5.2) *is equivalent to the following equation*

$$p_1 S^{-1}[a(n)\hat{v}(n)](x) = f_1(x) - p_1 S^{-1}[a(n)\widehat{l_2 f_2}(n)](x), \tag{5.7}$$

where $v = S^{-1}[\hat{v}] \in H^{\circ}_{\alpha}(J_1)$ *satisfies the condition*

$$v + l_2 f_2 = u \in H_{\alpha} \tag{5.8}$$

($l_2 f_2 \in H_{\alpha}$ being an arbitrary extension of the function f_2 from J_2 onto J).

Proof. Assume that $u \in H_{\alpha}$ satisfies the dual equation (5.1)-(5.2) and $l_2 f_2 \in H_{\alpha}$ is an arbitrary extension of the function $f_2 \in H_{\alpha}(J_2)$. Taking $v = u - l_2 f_2$ we get $v \in H^{\circ}_{\alpha}(J_1)$. Putting (5.8) into (5.1) we have (5.7). The right-hand side of (5.7) belongs to $H_{-\alpha}(J_1)$ in view of Theorem 3.1 and Theorem 3.2.

Conversely, assume that $v \in H^{\circ}_{\alpha}(J_1)$ satisfies the equation (5.7). Then obviously, the function u defined by (5.8) belongs to H_{α}. We shall prove that this function satisfies the dual equation (5.1)-(5.2) in the sense of distributions. Indeed, in transfering the second member in the right-hand side of (5.7) to the left-hand side and using (5.8) we obtain the equality (5.1). Finally, from (5.8) it follows the equality (5.2). The proof of Lemma 5.2 is complete. □

Denote

$$h(x) = f_1(x) - p_1 S^{-1}[a(n)\widehat{l_2 f_2}(n)](x). \tag{5.9}$$

Using (5.5) we can rewrite (5.7) in the form

$$(Av)(x) = h(x), \quad x \in J_1. \tag{5.10}$$

Our purpose now is to establish the existence of solution of the equation (5.10) in th space $H^\circ_\alpha(J_1)$. We shall consider the following cases.

5.2. **The case** $a(n) = a^+(n) \in \sigma^+_{2\alpha}$. It is clear that in this case the norm and scalar product in H_α defined by (3.1) and (3.2) respectively are equivalent to the following

$$||v||^2_{a^+} = \sum_{n=0}^{\infty} a^+(n)|\hat{v}(n)|^2, \tag{5.11}$$

$$(v, w)_{a^+} = \sum_{n=0}^{\infty} a^+(n)\hat{v}(n)\overline{\hat{w}(n)}. \tag{5.12}$$

We shall also write A^+v instead of Av.

Theorem 5.3. *(Existence). If* $h \in H_{-\alpha}(J_1), a(n) = a^+(n) \in \sigma^+_{2\alpha}$ *then the equation* (5.10) *has an unique solution* $v \in H^\circ_\alpha(J_1)$.

Proof. By an argument similar to that used in the proof of the Theorem 4.2 we can show that

$$[w, A^+v] = \sum_{n=0}^{\infty} a^+(n)\hat{w}(n)\overline{\hat{v}(n)} = (w, v)_{a^+}$$

for arbitrary functions v and w belonging to $H^\circ_\alpha(J_1)$, where $[w, A^+v]$ is defined by the formula (4.2). Therefore, if $v \in H^\circ_\alpha(J_1)$ satisfies the equation (5.10) then the following equality holds

$$(w, v)_{a^+} = [w, h], \quad \forall w \in H^\circ_\alpha(J_1). \tag{5.13}$$

We shall demonstrate that if (5.13) holds for any $w \in H^\circ_\alpha(J_1)$ then the function v will satisfy the equation (5.10) in the sense of $\mathcal{D}'(J_1)$.

In fact, noting that (5.13) holds for $w = \varphi \in C_0^\infty(J_1)$ we get from (2.4) and (4.2):

$$[\varphi, h] = \sum_0^\infty S[\varphi]\overline{S[l_1 h](n)} = \overline{(l_1 h, \varphi)},$$

$$(\varphi, v)_{a^+} = \sum_0^\infty S[\varphi](n)\overline{S[S^{-1}[a^+(n)\hat{v}(n)]} = \overline{(S^{-1}[a^+(n)\hat{v}(n)], \varphi)}.$$

Hence we have

$$(S^{-1}[a^+(n)\hat{v}(n)], \varphi) = (l_1 h, \varphi), \quad \forall\varphi \in C_0^\infty(J_1),$$

i. e.

$$p_1 S^{-1}[a^+(n)\hat{v}(n)](x) = p_1 l_1 h(x) = h(x), \quad x \in J_1.$$

We now return to the relation (5.13). Since $[w, h]$ is a linear continuous functional on the Hilbert space $H_\alpha^\circ(J_1)$, then by virtue Riesz Theorem there exists an unique element $v_0 \in H_\alpha^\circ(J_1)$ such that

$$[w, h] = (w, v_0)_{a^+}, \quad \forall w \in H_\alpha^\circ(J_1)$$

and moreover

$$||v_0||_{a^+} \leqslant C||h||_{H_{-\alpha}(J_1)}, \tag{5.14}$$

where C is a positive constant. The proof of Theorem 5.3 is complete. □

Remark 1. It is easily seen that the inverse operator $(A^+)^{-1}$ is bounded from $H_{-\alpha}(J_1)$ onto $H_\alpha^\circ(J_1)$. This follows from The Theorem 5.3 and the inequality (5.14).

Remark 2. The solution u of the dual series equation (5.1)-(5.2) expressed in terms of the solution v of the equation (5.7) by the formula (5.8) does not depend on the choice of the extension $l_2 f_2$. This fact follows from the uniqueness of solution of the dual equation (5.1)-(5.2). Hence, we can choose the extension $l_2 f_2$ such that

$$||l_2 f_2||_\alpha \leqslant C_o||f_2||_{H_\alpha(J_2)},$$

where C_o is a certain positive constant.

In this case, from (5.8), (5.9) and (5.14) it is easy to obtain the following estimate

$$||u||_\alpha \leqslant C(||f_1||_{H_{-\alpha}(J_1)} + ||f_2||_{H_\alpha(J_2)}), \tag{5.15}$$

where $C = constant > 0$. Therefore, the solution of the dual equation (5.1)-(5.2) depends continuously upon the functions given on the right-hand side.

5.3. **The case** $a(n) \in \sigma^o_{2\alpha}$. Assume in addition that there is a function $a^+(n) \in \sigma^+_{2\alpha}$ such that

$$b(n) := a(n) - a^+(n) \in \sigma_{2\alpha-\beta}, \quad \beta > 1/2. \tag{5.16}$$

We now represent the operator A defined by (5.5) in the form $A = A^+ + B$, where

$$A^+v := p_1 S^{-1}[a^+(n)\hat{v}], \quad Bv := p_1 S^{-1}[b(n)\hat{v}]. \tag{5.17}$$

Theorem 5.4. *(Existence). Under the condition* (5.16) *for every* $f_1 \in H_{-\alpha}(J_1)$ *and* $f_2 \in H_\alpha(J_2)$ *the dual series equation* (5.1)-(5.2) *has an unique solution* $u \in H_\alpha$.

Proof. According to Lemma 5.2 the dual series equation (5.1)-(5.2) is equivalent to the equation (5.5). In virtue of Remark 1 the operator $(A^+)^{-1}$ is bounded from $H_{-\alpha}(J_1)$ into $\overset{\circ}{H}_\alpha(J_1)$ and in virtue of Theorem 4.3 the operator B is completely continuous from $\overset{\circ}{H}_\alpha(J_1)$ into $H_{-\alpha}(J_1)$. Therefore, the operator $A = A^+ + B$ is a Fredholm and from the uniqueness of solution it follows that the dual series equation (5.1)-(5.2) has a unique solution $u \in H_\alpha$. The proof is complete. □

Example. Consider the following problem [5]. Find a function $v(x, y)$ satisfying the Laplace equation

$$v_{xx} + v_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < \infty$$

with boundary conditions:

i) If $x \to +0$, or $x \to \pi - 0$, then $v(x, y)$ uniformly converges to zero on $Y \leqslant y < \infty, \forall Y > 0$.

ii) If $y \to \infty$, then $v(x, y)$ uniformly converges to zero on $0 < x < \pi$.

iii) If $y \to +0$, then $v(x, y) \to f(x) \in \mathcal{D}'(0, a)$ on $0 < x < a$ and $v_y(x, y) \to g(x) \in \mathcal{D}'(a, \pi)$ on $a < x < \pi$.

It is not difficult to show that the function $v(x, y)$ has the form

$$v(x, y) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{\hat{u}(n)}{n} e^{-ny} \sin nx,$$

where $\hat{u}(n)(n = 1, 2, ...)$ are determined by the following dual series equation

$$\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{\hat{u}(n)}{n} \sin nx = f(x), \quad 0 < x < a, \tag{5.18}$$

$$\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \hat{u}(n) \sin nx = -g(x), \quad a < x < \pi. \tag{5.19}$$

We put

$$u(x) := S^{-1}[\hat{u}(n)](x) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\hat{u}(n)\sin nx, \quad 0 < x < \pi,$$

$$\hat{u}(n) = S[u](n) = \big(u, \sqrt{\frac{2}{\pi}}\sin nx\big).$$

According to Theorem 5.3 we have that, the dual series equation (5.18)-(5.19) have an unique solution $u(x) \in H_{-1/2} \equiv H_{-1/2}(0,\pi)$. For simplicity, assume that $g(x) \equiv 0$ and the function $u(x)$ is represented in the form

$$u(x) = \frac{w(x)}{\sqrt{a^2 - x^2}}, \quad 0 < x < a,$$

where

$$\int_0^a \frac{|w(x)|^2}{\sqrt{a^2 - x^2}}dx < \infty.$$

Then one can show that the function $u(x)$ is a solution of the following integral equation

$$\frac{1}{\pi}\int_0^a \ln\left|\frac{\sin(x+t)}{\sin(x-t)}\right|u(t)dt = f(x), \quad 0 < x < a. \tag{5.20}$$

The integral equation (5.20) can be resolved by the method of orthogonal polynomials [3].

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