ON THE GENERALIZED CONVOLUTION WITH A WEIGHT-FUNCTION FOR THE FOURIER COSINE, MELLIN AND FOURIER SINE INTEGRAL TRANSFORMS

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ABSTRACT. A generalized convolution with a weight-function for the Fourier cosine, Mellin and Fourier sine integral transforms is introduced. Its properties are studied, the application to solving a system of integral equations is outlined.

1 Introduction

The convolution for integral transforms were studied in the 20^{th} century, at first the convolution for Fourier transform (see, eg.[3, 21]), for the Laplace transform (see [26]), for the Mellin transform [21] and after that the convolution for the Hilbert transform [4, 24], the Hankel transform [7, 25], for the Kontorovich- Lebedev transform [7, 29], for the Stieltjes transform [22], the convolution with weight- function for the Fourier cosine transform [14].

The convolutions have applications to solving integral equations, evaluating integrals, summing a series [5, 6, 10, 21, 23, 24, 28].
The convolution of two functions f and g for Mellin integral transform M was introduced in [21]

$$(f * g)(x) = \int_0^{+\infty} f(u)g(\frac{x}{u})\frac{du}{u},$$

which satisfies the factorization identity

$$M(f * g)(y) = (Mf)(y)(Mg)(y), \forall y \in \mathbb{C}.$$

where the Mellin integral transform is [21]

$$\widetilde{f}(y) = (Mf)(y) = \int_0^{+\infty} f(x)x^{y-1}dx, \quad y \in \mathbb{C}.$$

Furthermore, the inverse Mellin transform is defined [2]

$$(M^{-1}\widetilde{f})(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(u)x^{-u}du, \quad x > 0.$$

Key words and phrases. Convolution, Fourier sine and Mellin transforms, integral equations
$MSC(2000) : 44A_{35}, 42_{38}, 45E_{10}$.

Let F_S and F_C be the Fourier cosine and sine transforms, respectively :

$$(F_S f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) sin(xy) dx, \quad y > 0$$

$$(F_C f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) cos(xy) dx, \quad y > 0.$$

In 1941, Churchill R.V gave out the convolution of functions f and g for Fourier cosine integral transform [3]

$$(f \underset{F_C}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y) \left[g\left(|x-u|\right) + g\left(x+u\right)\right] du, \ x > 0$$

which satisfies

$$F_C(f \underset{F_C}{*} g)(y) = (F_C f)(y)(F_C g)(y), \ \forall y > 0. \tag{1}$$

At the same time, he also gave out the convolution of functions f and g for Fourier sine and Fourier cosine integral transforms

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x-u|) - g(x+u)] du, \quad x > 0,$$

for which the factorization property holds

$$F_S(f * g)(y) = (F_S f)(y)(F_C g)(y), \quad \forall y > 0, \tag{2}$$

A convolution with the weight function $\beta(y) = \sin y$ of functions f and g for the Fourier sine integral transform F_S was studied in [7], [12]:

$$(f \overset{\beta}{\underset{F_S}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u) \big[\text{sign}(x+u-1) g(|x+u-1|) - g(x+u+1) +$$

$$+ \text{sign}(x-u-1) g(|x-u-1|) - \text{sign}(x-u+1) g(|x-u+1|)\big] du, \ x > 0$$

for which the factorization property holds :

$$F_S(f \overset{\beta}{\underset{F_S}{*}} g)(y) = \sin y (F_S f)(y)(F_S g)(y), \ \forall y > 0. \tag{3}$$

In the first of 90s of the last century, Yakubovich S.B. published some papers on special cases of generalized convolutions for integral transforms according to index, such as integral transforms of Mellin type [27], integral transforms of Kontorovich - Lebedev type [29], the $G-$transform [20]. For instance, for integral transforms of Mellin type

$$(K_i f)(x) = \int_0^{+\infty} f(t) k_i(\frac{x}{t}) \frac{dt}{t} (i = 0, 1, 2),$$

the convolution has the form

$$(f * g)(x) = \frac{1}{(2\pi i)^2} \int_{\sigma_s} \int_{\sigma_t} \frac{k_1^*(s) k_2^*(t)}{k_0^*(s+t)} f^*(s) g^*(t) x^{-s-t} ds dt, \quad x > 0,$$

where k_i^*, f^*, g^* respective are Mellin transforms of k_i, f, g $(i = 0, 1, 2)$, for which the factorization property holds

$$K_0(f * g)(y) = (K_1 f)(y)(K_2 g)(y).$$

In 1998, Kakichev and Nguyen Xuan Thao proposed a constructive method of defining the generalized convolution of two functions f and g, for three arbitrary integral transforms K_1, K_2, K_3 with the weight - function $\gamma(x)$ [8], for which the factorization property holds

$$K_1(f \overset{\gamma}{*} g)(y) = \gamma(y)(K_2 f)(y)(K_3 g)(y).$$

In recent years, several generalized convolutions were published, for instance: the generalized convolutions for Stieltjes, Hilbert and Fourier cosine- sine integral transforms [11], the generalized convolution for $H-$ transform [9], the generalized convolution for $I-$ transform [18], the generalized convolution with a weight-function for Fourier, Fourier cosine - sine transforms[15], the generalized convolutions for Kontorovich-Lebedev, Fourier sine and cosine transforms [19]... For example, the generalized convolution for Fourier cosine and sine transforms has the form[13]

$$(f \underset{T}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[sign(u-x) g(|u-x|) + g(u+x)] du, \quad x > 0,$$

which satisfies the factorization identity

$$F_C(f \underset{T}{*} g)(y) = (F_S f)(y)(F_S g)(y), \quad \forall y > 0. \tag{4}$$

The generalized convolution with the weight-function $\beta(y) = \sin y$ of the functions f and g for the Fourier cosine and sine integral transforms was studied in [17]

$$(f \overset{\beta}{\underset{1}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u) \big[g(|x+u-1|) + g(|x-u+1|) -$$

$$- g(x+u+1) - g(|x-u-1|) \big] du, \ x > 0$$

and the factorization property holds

$$F_C(f \overset{\beta}{\underset{1}{*}} g)(y) = \sin y (F_S f)(y)(F_C g)(y), \ \forall y > 0. \tag{5}$$

The generalized convolution with the weight-function $\beta(y) = \sin y$ for the Fourier sine and cosine transforms has been defined [16] by the indentity

$$(f \overset{\beta}{\underset{2}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u) \big[g(|x+u-1|) + g(|x-u-1|) -$$

$$- g(x+u+1) - g(|x-u+1|) \big] du, \ x > 0$$

with the factorization property

$$F_C(f \overset{\beta}{\underset{2}{*}} g)(y) = \sin y (F_C f)(y)(F_C g)(y), \ \forall y > 0. \tag{6}$$

In this paper we give out a notion of the generalized convolution with a weight-function of two functions for the Fourier cosine, Mellin and Fourier sine integral transforms. We will prove some of its properties as well as point out a relationship to the convolution (1). Finally, we will aplly this notion to solving a system of integral equations.

2 The generalized convolution for the Fourier cosine, Mellin and Fourier sine integral transforms

Let

$$\begin{aligned}\theta_1(x,u,v) = & \frac{1}{2\pi^2 i}\Gamma(-u)\Big\{[1+(x-v)^2]^{\frac{u}{2}}\sin[u.\arctan(x-v)] - \\ & - [1+(x+v)^2]^{\frac{u}{2}}\sin[u.\arctan(x+v)]\Big\}\end{aligned} \tag{7}$$

Definition. *The generalized convolution with the weight function $\gamma(y) = e^{-y}y^{-1}$ of functions f and g for the Fourier cosine, Mellin and Fourier sine integral transforms is defined*

$$(f \overset{\gamma}{*} g)(x) = \int\limits_{c-i\infty}^{c+i\infty}\int\limits_{0}^{+\infty} \theta_1(x,u,v)f(u)g(v)dudv, \quad x > 0, \tag{8}$$

where u is the complex variable and $c = Reu < -2, \quad t = Imu$.

Let $L(ch\frac{\pi Imu}{2}, \mathbb{R}) = \Big\{f : \int\limits_{-\infty}^{+\infty} ch\frac{\pi Imu}{2}|f(u)|dt < +\infty\Big\}$

and $L(\mathbb{R}_+) = \Big\{g : \int\limits_{0}^{+\infty} |g(v)|dv < +\infty\Big\}$.

Theorem 1. *Let $f(u) \in L(ch\frac{\pi Imu}{2}, \mathbb{R})$ $(c < -2)$ and $g \in L(\mathbb{R}_+)$. Then the convolution $(f \overset{\gamma}{*} g)$ belongs to $L(\mathbb{R}_+)$ and satisfies the factorization equality*

$$F_C(f \overset{\gamma}{*} g)(y) = \gamma(y)(M^{-1}f)(y)(F_S g)(y), \quad \forall y > 0. \tag{9}$$

Proof. First of all we show that

$$|\Gamma(z)| \leqslant \Gamma(Rez), \quad \forall z \in \mathbb{C} \text{ such that } Rez > 0 \tag{10}$$

and

$$|\sin z| \leqslant \cosh(\text{Im } z), \quad \forall z \in \mathbb{C} \tag{11}$$

Indeed,

$$|\Gamma(z)| = \Big|\int\limits_0^{+\infty} e^{-x}x^{z-1}dx\Big| \leqslant \int\limits_0^{+\infty} e^{-x}|x^{z-1}|dx$$

$$= \int\limits_0^{+\infty} e^{-x}|e^{(z-1)\ln x}|dx = \int\limits_0^{+\infty} e^{-x}e^{(\Re z-1)\ln x}dx$$

$$= \int\limits_0^{+\infty} e^{-x}x^{\Re z-1}dx = \Gamma(\Re z).$$

and

$$|\sin z| \leqslant \frac{|e^{iz}|+|e^{-iz}|}{2}$$

$$= \frac{|e^{-Imz}|+|e^{Imz}|}{2} = ch(Imz).$$

By the inequalities (10) and (11) , $\forall u \in \mathbb{C}$ such that $c < -2$, we have

$$|\theta_1(x,u,v)| \leqslant \frac{1}{2\pi^2}\Gamma(-c)\Big\{\Big|\big[1+(x-v)^2\big]^{\frac{u}{2}}\Big|ch[Imu.arctan(x-v)]+ \tag{12}$$

$$+\Big|\big[1+(x+v)^2\big]^{\frac{u}{2}}\Big|ch[Imu.arctan(x+v)]\Big\}$$

Since $c < -2$, we obtain

$$\Big|\big[1+(x\pm v)^2\big]^{\frac{u}{2}}\Big| \leqslant 1. \tag{13}$$

Because $|arctan(x\pm v)| \leqslant \frac{\pi}{2}$, it follows that

$$ch[Imu.arctan(x\pm v)] \leqslant ch(\frac{\pi}{2}Imu). \tag{14}$$

From $(12),(13),(14)$ we get

$$\theta_1(x,u,v) \leqslant \frac{1}{\pi^2}\Gamma(-c)ch\frac{\pi Imu}{2}. \tag{15}$$

Base on (15), with $c < -2$ we have

$$\Big|(f \overset{\gamma}{*} g)(x)\Big| = \Big|\int\limits_{c-i\infty}^{c+i\infty}\int\limits_0^{+\infty} \theta_1(x,u,v)f(u)g(v)dudv\Big|$$

$$\leqslant \int\limits_{-\infty}^{+\infty}\int\limits_0^{+\infty} |\theta_1(x,u,v)||f(u)||g(v)|dtdv$$

$$\leqslant \frac{1}{\pi^2}\Gamma(-c)\int\limits_{-\infty}^{+\infty}\int\limits_0^{+\infty} ch\frac{\pi Imu}{2}|f(u)||g(v)|dtdv < +\infty.$$

We have

$$\int_0^{+\infty}\Big|(f\overset{\gamma}{*}g)(x)\Big|dx \leqslant \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}|\theta_1(x,u,v)||f(u)||g(v)|dxdtdv$$

$$= \int_{-\infty}^{+\infty}\int_0^{+\infty}\Big(\int_0^{+\infty}|\theta_1(x,u,v)|dx\Big)|f(u)||g(v)|dtdv$$

$$\leqslant \frac{1}{2\pi^2}\Gamma(-c)\int_{-\infty}^{+\infty}\int_0^{+\infty}\Big(\int_0^{+\infty} ch\frac{\pi Imu}{2}\{[1+(x-v)^2]^{\frac{c}{2}}+[1+(x+v)^2]^{\frac{c}{2}}\}dx\Big)|f(u)||g(v)|dtdv$$

$$\leqslant \frac{1}{2\pi^2}\Gamma(-c)\int_{-\infty}^{+\infty}\int_0^{+\infty}\Big(\int_0^{+\infty} ch\frac{\pi Imu}{2}\{[1+(x-v)^2]^{-1}+[1+(x+v)^2]^{-1}\}dx\Big)|f(u)||g(v)|dtdv$$

$$= \frac{1}{2\pi^2}\Gamma(-c)\int_{-\infty}^{+\infty}\int_0^{+\infty} ch\frac{\pi Imu}{2}\{\text{artan }(x-v)\Big|_0^{+\infty}+\text{artan }(x+v)\Big|_0^{+\infty}\}|f(u)||g(v)|dtdv$$

$$= \frac{1}{2\pi}\Gamma(-c)\int_{-\infty}^{+\infty}\int_0^{+\infty} ch\frac{\pi Imu}{2}|f(u)||g(v)|dtdv < +\infty$$

So $\big(f\overset{\gamma}{*}g\big)\in L(\mathbb{R}_+)$.

To prove that the generalized convolution (8) satisfies the factorization equality (9), first of all we make the following observation: if $u\in\mathbb{C}$ and $v\in\mathbb{R}_+$ are fixed numbers, then $\theta_1(x,u,v)\in L(\mathbb{R}_+)$.

Indeed,

$$\int_0^{+\infty} |\theta_1(x,u,v)|dx \leqslant \frac{1}{2\pi^2}\Gamma(-c)\int_0^{+\infty}\Big\{\Big|\big[1+(x-v)^2\big]^{\frac{u}{2}}\Big||\sin[u.arctan(x-v)]|$$

$$+\Big|\big[1+(x+v)^2\big]^{\frac{u}{2}}\Big||\sin[u.arctan(x+v)]|\Big\}dx$$

$$\leqslant \frac{1}{2\pi^2}\Gamma(-c)\int_0^{+\infty}\Big\{\big[1+(x-v)^2\big]^{-1}ch[t.arctan(x-v)]$$

$$+\big[1+(x+v)^2\big]^{-1}ch[t.arctan(x+v)]\Big\}dx$$

$$=\frac{1}{2\pi^2}\Gamma(-c)\Big\{\int_0^{+\infty} ch[t.arctan(x-v)]d(arctan(x-v))+$$

$$+\int_0^{+\infty} ch[t.arctan(x+v)]d(arctan(x+v))\Big\}.$$

Therefore, if $t = Imu \neq 0$, then

$$\frac{1}{2\pi^2}\Gamma(-c)\Big\{\int_0^{+\infty} ch[t.arctan(x-v)]d(arctan(x-v))+$$

$$+\int_0^{+\infty} ch[t.arctan(x+v)]d(arctan(x+v))\Big\} =$$

$$\frac{1}{2\pi^2 t}\Gamma(-c)\Big\{sh\big[t.arctan(x-v)\big]\Big|_0^{+\infty} + sh\big[t.arctan(x+v)\big]\Big|_0^{+\infty}\Big\}$$

$$=\frac{1}{2\pi^2 t}\Gamma(-c)\Big\{sh\frac{\pi t}{2} - sh[t.\text{artan }(-v)] + sh\frac{\pi t}{2} - sh[t.\text{artan } v]\Big\}$$
$$=\frac{1}{\pi^2 t}\Gamma(-c)\sinh\frac{\pi t}{2}.$$

If $t = Imu = 0$ then

$$\frac{1}{2\pi^2}\Gamma(-c)\Big\{\int_0^{+\infty} ch[t.\text{arctan }(x-v)]d(arctan(x-v))+$$

$$+\int_0^{+\infty} \cosh[t\text{arctan }(x+v)]d(\text{arctan }(x+v))\Big\} =$$

$$= \frac{1}{2\pi^2}\Gamma(-c)\left\{ \int_0^{+\infty} d(arctan(x-v)) + \int_0^{+\infty} d(arctan(x+v)) \right\}$$
$$= \frac{1}{2\pi^2}\Gamma(-c)\{arctan(x-v)\Big|_0^{+\infty} + arctan(x+v)\Big|_0^{+\infty}\}$$
$$= \frac{1}{2\pi^2}\Gamma(-c)\{\frac{\pi}{2} - arctan(-v) + \frac{\pi}{2} - arctanv\}$$
$$= \frac{1}{2\pi}\Gamma(-c).$$

So $\int_0^{+\infty} |\theta_1(x,u,v)|dx < +\infty.$

On the other hand, from the formula 7 ([2] p.277) we get,

$$\big(F_C(\frac{1}{\pi^2 i}e^{-y}y^{-u-1}\sin(vy))\big)(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\pi^2 i}\int\limits_0^{+\infty} e^{-y}y^{-u-1}\sin(vy)\cos(xy)dy$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\pi^2 i}\int\limits_0^{+\infty} e^{-y}y^{-u-1}\Big[\sin\big(y(x-v)\big) + \sin\big(y(x+v)\big)\Big]dy = \sqrt{\frac{2}{\pi}}\theta_1(x,u,v).$$

It follows that

$$\Big(F_C\big(\theta_1(x,u,v)\big)\Big)(y) = \sqrt{\frac{\pi}{2}}\frac{1}{\pi^2 i}e^{-y}y^{-u-1}\sin(vy).$$

The last equality yield

$$\gamma(y)(M^{-1}f)(y)(F_S g)(y) = e^{-y}y^{-1}\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} f(u)y^{-u}du\sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty} g(v)\sin(vy)dv$$

$$= \int\limits_{c-i\infty}^{c+i\infty}\int\limits_0^{+\infty}\sqrt{\frac{\pi}{2}}\frac{1}{\pi^2 i}e^{-y}y^{-u-1}\sin(vy)f(u)g(v)dudv$$

$$= \int\limits_{c-i\infty}^{c+i\infty}\int\limits_0^{+\infty}\Big(\sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\theta_1(x,u,v)\cos(xy)dx\Big)f(u)g(v)dudv$$

$$= F_C\big(f \overset{\gamma}{*} g\big)(y).$$

Theorem 1 is thus proved.□

Theorem 2. *If* $f(u) \in L\big(ch\frac{\pi Imu}{2}, \mathbb{R}\big)$ $(c < -2)$ *and* $g \in L(\mathbb{R}_+)$, *then*

$$\big(f\overset{\gamma}{*}g\big)(x) = \frac{1}{i\sqrt{2\pi^3}}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(-u)f(u)\big[(1+v^2)^{\frac{u}{2}}\sin(-u.arctanv)\underset{T}{*}g(v)\big](x)du, \quad \forall x > 0.$$

Proof.

$$\big(f \overset{\gamma}{*} g\big)(x) =$$

$$= \frac{1}{i\sqrt{2\pi^3}}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(-u)f(u)\Big(\frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty}\Big\{sign(v-x)\big[1+|x-v|^2\big]^{\frac{u}{2}}\sin\big(-u.arctan|x-v|\big)+$$

$$+\big[1+(x+v)^2\big]^{\frac{u}{2}}\sin\big(-u.arctan(x+v)\big)\Big\}g(v)dv\Big)du$$

$$= \frac{1}{i\sqrt{2\pi^3}}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(-u)f(u)\Big[(1+v^2)^{\frac{u}{2}}\sin(-u.arctanv)\underset{T}{*}g(v)\Big](x)du, \quad \forall x > 0. \quad \Box$$

It is easy to find out that the generalized convolution (8) is neither commutative nor associative. But we have

Theorem 3. *Let $g, h \in L(\mathbb{R}_+)$ and $f \in L\Big(ch\frac{\pi Imu}{2}, \mathbb{R}\Big)$ $(c < -2)$, then the following equalities holds*

a) $f \overset{\gamma}{*} \Big(g{*}h\Big) = \Big(f \overset{\gamma}{*} g\Big) \underset{F_C}{*} h.$

b) $f{*}\Big(g \overset{\gamma}{*} h\Big) = g{*}\Big(f \overset{\gamma}{*} h\Big).$

c) $f \underset{1}{\overset{\beta}{*}} \Big(g \overset{\gamma}{*} h\Big) = g \underset{1}{\overset{\beta}{*}} \Big(f \overset{\gamma}{*} h\Big).$

d) $f \overset{\gamma}{*} \Big(g \underset{F_S}{\overset{\beta}{*}} h\Big) = h \underset{1}{\overset{\beta}{*}} \Big(f \overset{\gamma}{*} g\Big).$

Proof. We prove a). Base on (1),(9), we have

$$\begin{aligned} F_C\Big(f \overset{\gamma}{*} \Big(g{*}h\Big)\Big)(y) &= \gamma(y)(M^{-1}f)(y)F_S(g*h)(y) \\ &= \gamma(y)(M^{-1}f)(y)(F_S g)(y)(F_C h)(y) \\ &= F_C\big(f \overset{\gamma}{*} g\big)(y)(F_C h)(y) = F_C\Big(\big(f \overset{\gamma}{*} g\big) \underset{F_C}{*} h\Big)(y), \quad \forall y > 0. \end{aligned}$$

Hence $f \overset{\gamma}{*} \Big(g{*}h\Big) = \Big(f \overset{\gamma}{*} g\Big) \underset{F_C}{*} h$. The proof for b), c), d) are similar to those of a). The theorem is proved.

3 Applications to integral equations

Consider the system of integral equations

$$\begin{aligned} f(x) + \lambda_1 \int_{c-i\infty}^{c+i\infty} \int_0^{+\infty} \theta_1(x,u,v)\varphi(u)g(v)dudv + \lambda_2 \int_0^{+\infty} \theta_2(x,u)g(u)du + \\ +\lambda_3 \int_0^{+\infty} \theta_3(x,u)g(u)du = h(x) \end{aligned} \tag{16}$$

$$\lambda_4 \int_0^{+\infty} \theta_4(x,u)f(u)du + g(x) = k(x),$$

where $x > 0$, and $c, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote complex parameters with $c < -2$, kernels: $\theta_1(x,u,v)$ is defined in (7),

$$\theta_2(x,u) = \frac{1}{\sqrt{2\pi}}\big[\text{sign } (u-x)\xi(|x-u|) + \xi(x+u)\big],$$

$$\theta_3(x,u) = \frac{1}{2\sqrt{2\pi}}\big[\zeta(|x+u-1|) + \zeta(|x-u+1|) - \zeta(x+u+1) - \zeta(|x-u-1|)\big],$$

$$\theta_4(x,u) = \frac{1}{\sqrt{2\pi}}\big[\text{sign } (x-u)\psi(|x-u|) + \psi(x+u)\big],$$

$\varphi, \xi, \zeta, \psi, h, k$ are given functions and f, g are unknown functions.

Theorem 4. *Suppose that* $\varphi \in L\left(ch\frac{\pi Imu}{2}, \mathbb{R}\right)$ $(Reu < -2)$ *and* $\psi, \xi, \zeta, h, k \in L(\mathbb{R}_+)$, $\delta(x) = \lambda_1(\varphi \overset{\gamma}{*} \psi)(x) + \lambda_2(\xi \underset{T}{*} \psi)(x) + \lambda_3(\psi \underset{1}{\overset{\beta}{*}} \zeta)(x)$. *Then, with the condition*

$$1 - \lambda_4(F_C\delta)(y) \neq 0, \quad \forall y > 0,$$

the system (16) *have a solution*

$$f(x) = h(x) + (h \underset{F_c}{*} q)(x) - \lambda_1\left(\varphi \overset{\gamma}{*} k\right)(x) -$$

$$-\lambda_1\left((\varphi \overset{\gamma}{*} k) \underset{F_c}{*} q\right)(x) - \lambda_2\left(\xi \underset{T}{*} k\right)(x) - \lambda_2\left((\xi \underset{T}{*} k) \underset{F_c}{*} q\right)(x) -$$

$$-\lambda_3(k \underset{1}{\overset{\beta}{*}} \zeta)(x) - \lambda_3((k \underset{1}{\overset{\beta}{*}} \zeta) \underset{F_c}{*} q)(x) \in L(\mathbb{R}_+)$$

$$g(x) = k(x) + (k * q)(x) - \lambda_4(\psi * h)(x) - \lambda_4((\psi * h) * q)(x) \in L(\mathbb{R}_+)$$

with $q(x) \in L(\mathbb{R}_+)$ *satisfying*

$$\frac{\lambda_4(F_C\delta)(y)}{1 - \lambda_4(F_C\delta)(y)} = (F_Cq)(y).$$

Proof.

Using (2), (4), (5), (9) we obtain the linear system

$$(F_Cf)(y) + \lambda_1\gamma(y)(M^{-1}\varphi)(y)(F_Sg)(y) + \lambda_2(F_S\xi)(y)(F_Sg)(y) +$$

$$+\lambda_3 siny(F_C\zeta)(y)(F_Sg)(y) = (F_Ch)(y)$$

$$\lambda_4(F_S\psi)(y)(F_Cf)(y) + (F_Sg)(y) = (F_Sk)(y)$$

On the other hand

$$\Delta = \begin{vmatrix} 1 & \lambda_1\gamma(y)(M^{-1}\varphi)(y) + \lambda_2(F_S\xi)(y) + \lambda_3 siny(F_C\zeta)(y) \\ \lambda_2(F_S\psi)(y) & 1 \end{vmatrix}$$

$$= 1 - \lambda_4F_C(\lambda_1(\varphi \overset{\gamma}{*} \psi) + \lambda_2(\xi \underset{T}{*} \psi) + \lambda_3(\psi \underset{1}{\overset{\beta}{*}} \zeta))(y) = 1 - \lambda_4(F_C\delta)(y) \neq 0$$

$$\Delta_1 = \begin{vmatrix} (F_Ch)(y) & \lambda_1\gamma(y)(M^{-1}\varphi)(y) + \lambda_2(F_S\xi)(y) + \lambda_3 siny(F_C\zeta)(y) \\ (F_Sk)(y) & 1 \end{vmatrix}$$

$$= (F_Ch)(y) - \lambda_1F_C(\varphi \overset{\gamma}{*} k)(y) - \lambda_2F_C(\xi \underset{T}{*} k)(y) - \lambda_3F_C(k \underset{1}{\overset{\beta}{*}} \zeta)(y),$$

$$\Delta_2 = \begin{vmatrix} 1 & (F_Sh)(y) \\ \lambda_4(F_S\psi)(y) & (F_Sk)(y) \end{vmatrix}$$

$$= (F_Sk)(y) - \lambda_4F_S(\psi * h)(y).$$

Hence

$$(F_Cf)(y) = \frac{1}{\Delta}[(F_Ch)(y) - \lambda_1F_C(\varphi \overset{\gamma}{*} k)(y) - \lambda_2F_C(\xi \underset{T}{*} k)(y) - \lambda_3(k \underset{1}{\overset{\beta}{*}} \zeta)(y)] \tag{17}$$

In virtue of Wiener - Levi's theorem [1] there is a function $q(x) \in L(\mathbb{R}_+)$ such that

$$\frac{\lambda_4(F_C\delta)(y)}{1-\lambda_4(F_C\delta)(y)} = (F_Cq)(y). \tag{18}$$

From (1), (17), (18) we get

$$\begin{aligned}(F_Cf)(y) &= [1+(F_Cq)(y)][(F_Ch)(y) - \lambda_1 F_C(\varphi \overset{\gamma}{*} k)(y) - \lambda_2 F_C(\xi \underset{T}{*} k)(y) - \lambda_3 F_C(k \overset{\beta}{\underset{1}{*}} \zeta)(y)]\\
&= (F_Ch)(y) + (F_Cq)(y)(F_Ch)(y) - \lambda_1 F_C\Big(\varphi \overset{\gamma}{*} k\Big)(y) -\\
&- \lambda_1 F_C\Big(\varphi \overset{\gamma}{*} k\Big)(y)(F_Cq)(y) - \lambda_2 F_C\Big(\xi \underset{T}{*} k\Big)(y) - \lambda_2 F_C\Big(\xi \underset{T}{*} k\Big)(y)(F_Cq)(y) -\\
&- \lambda_3 F_C(k \overset{\beta}{\underset{1}{*}} \zeta)(y) - \lambda_3 F_C(k \overset{\beta}{\underset{1}{*}} \zeta)(y)(F_Cq)(y)\\
&= (F_Ch)(y) + F_C(h \underset{F_C}{*} q)(y) - \lambda_1 F_C\Big(\varphi \overset{\gamma}{*} k\Big)(y) -\\
&- \lambda_1 F_C\Big((\varphi \overset{\gamma}{*} k) \underset{F_C}{*} q\Big)(y) - \lambda_2 F_C\Big(\xi \underset{T}{*} k\Big)(y) - \lambda_2 F_C\Big((\xi \underset{T}{*} k) \underset{F_C}{*} q\Big)(y) -\\
&- \lambda_3 F_C(k \overset{\beta}{\underset{1}{*}} \zeta)(y) - \lambda_3 F_C((k \overset{\beta}{\underset{1}{*}} \zeta) \underset{F_C}{*} q)(y).\end{aligned}$$

From last equation, we obtain

$$f(x) = h(x) + (h \underset{F_C}{*} q)(x) - \lambda_1\Big(\varphi \overset{\gamma}{*} k\Big)(x) -$$
$$-\lambda_1\Big((\varphi \overset{\gamma}{*} k) \underset{F_C}{*} q\Big)(x) - \lambda_2\Big(\xi \underset{T}{*} k\Big)(x) - \lambda_2\Big((\xi \underset{T}{*} k) \underset{F_C}{*} q\Big)(x) -$$
$$-\lambda_3(k \overset{\beta}{\underset{1}{*}} \zeta)(x) - \lambda_3((k \overset{\beta}{\underset{1}{*}} \zeta) \underset{F_C}{*} q)(x) \in L(R_+).$$

Similarly,

$(F_Sg)(y) = [1+(F_Cq)(y)][(F_Sk)(y) - \lambda_4 F_s(\psi * h)(y)]$.

From (2) we have

$$(F_Sg)(y) = (F_Sk)(y) + F_S(k * q)(y) - \lambda_4 F_S(\psi * h)(y) - \lambda_4 F_S((\psi * h) * q)(y).$$

Hence

$$g(x) = k(x) + (k * q)(x) - \lambda_4(\psi * h)(x) - \lambda_4((\psi * h) * q)(x) \in L(R_+).$$

The proof is complete.

Consider the system of integral equations

$$\begin{cases} f(x) + \lambda_1 \displaystyle\int_{c-i\infty}^{c+i\infty}\int_0^{+\infty} \theta_1(x,u,v)\varphi(u)g(v)dudv = h(x) \\ \lambda_2 \displaystyle\int_0^{+\infty} \theta_4(x,u)f(u)du + \lambda_3 \int_0^{+\infty} \theta_5(x,u)f(u)du + g(x) = k(x), x > 0 \end{cases} \tag{19}$$

here, $c, \lambda_1, \lambda_2, \lambda_3$ denote complex parameters with $c < -2$, kernels:
$\theta_1(x,u,v)$ is defined in (7),

$$\theta_4(x,u) = \frac{1}{\sqrt{2\pi}}\big[\text{sign }(x-u)\psi(|x-u|) + \psi(x+u)\big],$$

$$\theta_5(x,u) = \frac{1}{2\sqrt{2\pi}}\big[\xi(|x+u-1|) + \xi(|x-u-1|) - \xi(x+u+1) - \xi(|x-u+1|)\big],$$

with

$$\xi(x) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} \xi_1(u)\big[\text{sign }(u-x)\xi_2(|u-x|) + \xi_2(u+x)\big]du,$$

where $\varphi, \psi, \xi_1, \xi_2, h, k$ are given functions. f, g are unknown functions

Theorem 5. *Suppose that* $\varphi \in L\Big(ch^{\frac{\pi Imu}{2}}, \mathbb{R}\Big)$ $(Reu < -2)$ *and* $\psi, \xi_1, \xi_2, h, k \in L(\mathbb{R}_+)$, $l(x) = \lambda_2(\varphi \overset{\gamma}{*} \psi)(x) + \lambda_3(\xi_2 \underset{1}{\overset{\beta}{*}}(\varphi \overset{\gamma}{*} \xi_1))(x)$ *. Then, with the condition*

$$1 - \lambda_3(F_C l)(y) \neq 0, \quad \forall y > 0,$$

the system (19) *have a solution*

$$f(x) = h(x) - \lambda_1(\varphi \overset{\gamma}{*} k)(x) + (q \underset{F_C}{*} h)(x) - \lambda_1(q \underset{F_C}{*} (\varphi \overset{\gamma}{*} k))(x) \in L(R_+)$$

$$g(x) = k(x) - \lambda_2(\psi * h)(x) - \lambda_3(\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h))(x)$$

$$+(k * q)(x) - \lambda_2((\psi * h) * q)(x) - \lambda_3((\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h)) * q)(x) \in L(R_+).$$

with $q(x) \in L(\mathbb{R}_+)$ *satisfying*

$$\frac{\lambda_3(F_C l)(y)}{1 - \lambda_3(F_C l)(y)} = (F_C q)(y).$$

Proof.

Using (2), (4), (6), (9) we obtain the linear system

$$(F_C f)(y) + \lambda_1\gamma(y)(M^{-1}\varphi)(y)(F_S g)(y) = (F_C h)(y)$$

$$\lambda_2(F_S\psi)(y)(F_C f)(y) + \lambda_3 siny(F_S\xi_1)(y)(F_S\xi_2)(y)(F_C f)(y) + (F_S g)(y) = (F_S k)(y)$$

On the other hand

$$\Delta = \begin{vmatrix} 1 & \lambda_1\gamma(y)(M^{-1}\varphi)(y) \\ \lambda_2(F_S\psi)(y) + \lambda_3 siny(F_S\xi_1)(y)(F_S\xi_2)(y) & 1 \end{vmatrix}$$

$$= 1 - \lambda_1 F_C(\lambda_2(\varphi \overset{\gamma}{*} \psi) + \lambda_3(\xi_2 \underset{1}{\overset{\beta}{*}}(\varphi \overset{\gamma}{*} \xi_1)))(y) = 1 - \lambda_3(F_C l)(y)$$

$$\Delta_1 = \begin{vmatrix} (F_C h)(y) & \lambda_1\gamma(y)(M^{-1}\varphi)(y) \\ (F_S k)(y) & 1 \end{vmatrix}$$

$$= (F_C h)(y) - \lambda_1 F_C(\varphi \overset{\gamma}{*} k)(y),$$

$$\Delta_2 = \begin{vmatrix} 1 & (F_C h)(y) \\ \lambda_2 (F_S \psi)(y) + \lambda_3 siny (F_S \xi_1)(y)(F_S \xi_2)(y) & (F_S k)(y) \end{vmatrix}$$
$$= (F_S k)(y) - \lambda_2 F_S(\psi * h)(y) - \lambda_3 F_S(\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h))(y).$$

Hence

(20) $$(F_C f)(y) = \frac{1}{\Delta}[(F_C h)(y) - \lambda_1 F_C(\varphi \overset{\gamma}{*} k)(y)].$$

In virtue of Wiener - Levi's theorem [1] there is a function $q(x) \in L(\mathbb{R}_+)$ such that

(21) $$\frac{\lambda_3 (F_C l)(y)}{1 - \lambda_3 (F_C l)(y)} = (F_C q)(y).$$

From $(1), (20), (21)$ we get

$$(F_C f)(y) = [1 + (F_C q)(y)][(F_C h)(y) - \lambda_1 F_C(\varphi \overset{\gamma}{*} k)(y)]$$
$$= (F_C h)(y) - \lambda_1 F_C(\varphi \overset{\gamma}{*} k)(y) + F_C(q \underset{F_C}{*} h)(y) - \lambda_1 F_C(q \underset{F_C}{*} (\varphi \overset{\gamma}{*} k))(y).$$

From last equation, we obtain

$$f(x) = h(x) - \lambda_1(\varphi \overset{\gamma}{*} k)(x) + (q \underset{F_C}{*} h)(x) - \lambda_1(q \underset{F_C}{*} (\varphi \overset{\gamma}{*} k))(x) \in L(R_+).$$

Similarly,

$$(F_S g)(y) = [1 + (F_C q)(y)][(F_S k)(y) - \lambda_2 F_S(\psi * h)(y) - \lambda_3 F_S(\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h))(y)].$$

From (2) we have

$$(F_S g)(y) = (F_S k)(y) - \lambda_2 F_S(\psi * h)(y) - \lambda_3 F_S(\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h))(y)$$
$$+ F_S(k * q)(y) - \lambda_2 F_S((\psi * h) * q)(y) - \lambda_3 F_S((\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h)) * q)(y).$$

Hence

$$g(x) = k(x) - \lambda_2(\psi * h)(x) - \lambda_3(\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h))(x)$$
$$+ (k * q)(x) - \lambda_2((\psi * h) * q)(x) - \lambda_3((\xi_1 \underset{F_S}{\overset{\beta}{*}} (\xi_2 * h)) * q)(x) \in L(R_+).$$

The proof is complete.

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