ON BOUNDARY VALUE PROBLEMS FOR A CLASS OF SINGULAR INTEGRAL EQUATIONS

Nguyen, Van Mau; Nguyen, Tan Hoa

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Osaka University
ON BOUNDARY VALUE PROBLEMS
FOR A CLASS OF SINGULAR INTEGRAL
EQUATIONS

Nguyen Van Mau, Nguyen Tan Hoa
Hanoi University of Science, VNUH

Abstract

This report deals with the solvability of boundary value problems
for singular integral equations of the form

\[
\left( K^n + K^{n-1}K_c \right) \varphi(t) = f(t),
\]  

(i)

\[
F_j^j \varphi(t) = \varphi_j(t), \quad j = 0, ..., n - 1, \quad \varphi_j(t) \in \text{Ker } K, \quad \text{if } \alpha > 0,
\]

(FjKjφ)(t) = φj(t), j = 0, ..., n - 1, φj(t) ∈ Ker K, if α > 0,

(ii)

\[
(G_0f)(t) = 0, \quad (G_jR_{j-1}...R_0)f(t) = 0, \quad j = 1, ..., n - 1 \quad \text{if } \alpha < 0.
\]

By an algebraic method we reduce the problem (i) - (ii) to a system
of linear algebraic equations which gives all solutions in a closed form.

Key words and phrases: initial and co-initial operators, singular integral
equations, boundary value problem.
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1 Introduction

The theory of general boundary value problem induced by right invertible
operators were investigated by Przeworska - Rolewicz and has been developed
by many other mathematicians (c.f. [2], [4]). In this paper, we give an
application of this theory to solve the following boundary value problem:

\[
\left( K^n + K^{n-1}K_c \right) \varphi(t) = f(t),
\]  

(i)
\[(F_j K^j \varphi)(t) = \varphi_j(t), \quad j = 0, ..., n - 1, \quad \varphi_j(t) \in \text{Ker } K, \quad \text{if } \omega > 0, \]
\[(G_0 f)(t) = 0, \quad (G_j R_{j-1} \ldots R_0) f(t) = 0, \quad j = 1, ..., n - 1 \quad \text{if } \omega < 0, \quad (\text{ii})\]

where \(K_c\) is an operator of multiplication by the function \(c(t)\); \(F_j, G_j\) \((j = 0, ..., n - 1)\) are initial and co-initial operators, respectively, and

\[
(K \varphi)(t) = a(t) \varphi(t) + \frac{b(t)}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau, \quad \omega = \text{Ind } K.
\]

2 Preliminaries

We recall some notations and results which are used in the sequel (see [2], [4]).

Let \(\Gamma\) be a simple regular closed arc in complex plane and let \(X = H^{\mu}(\Gamma)\) \((0 < \mu < 1)\). Denote by \(D^+\) the domain bounded by \(\Gamma\) (assume that \(0 \in D^+\)) and by \(D^-\) its complement including the point at infinity. The set of all linear operators with domains and ranges contained in \(X\) will be denoted by \(L(X)\). Write \(L_0(X) := \{A \in L(X) : \text{dom } A = X\}\).

Let \(R(X)\) be the set of all right invertible operators belonging to \(L(X)\). For \(D \in R(X)\), we denote by \(R_D\) the set of all its right inverses.

An operator \(F \in L(X)\) is said to be an initial operator for an operator \(D \in R(X)\) corresponding to a right inverse \(R\) of \(D\) if

\[F^2 = F, \quad F(\text{dom } D) = \text{Ker } D, \quad FR = 0.\]

Denote by \(F_D\) the set of all initial operators for \(D \in R(X)\).

It is well-known the following fact:

\(F \in L(X)\) is an initial operator for \(D \in R(X)\) corresponding to \(R \in R_D\) if and only if \(F = I - RD\) on \(\text{dom } D\).

Let \(\Lambda(X)\) be the set of all left invertible operators belonging to \(L_0(X)\). For \(V \in \Lambda(X)\), we denote by \(L_V\) the set of all its left inverses.

If \(V \in \Lambda(X)\) and \(L \in L_V\) then the operator

\[G := I - VL\]

is called the co-initial operators for \(V\) corresponding to \(L \in L_V\).

Denote by \(G_V\) the set of all co-initial operators for \(V \in \Lambda(X)\).
Lemma 1 (see [2]). Let $A, B \in L(X) \cap \text{dom} B$, $\text{Im} B \subset \text{dom} A$. Then equation $(I - AB)x = y$ has solutions if and only if $(I - BA)u = By$ does and there is one-to-one correspondence between the two sets of solutions, given by

$$u = Bx \iff x = y + Au.$$  

2. The results

Let

$$(K\varphi)(t) := a(t)\varphi(t) + \frac{b(t)}{\pi i} \int \frac{\varphi(\tau)}{\tau - t} d\tau,$$

where $a(t), b(t) \in X, a^2(t) - b^2(t) = 1, \quad 0 \neq c_0 = \text{Ind} K$.

Denote

$$(R_0\varphi)(t) := a(t)\varphi(t) - \frac{b(t)}{\pi i} \int \frac{\varphi(\tau)}{Z(\tau)(\tau - t)} d\tau,$$

where

$$Z(t) = e^{\Gamma(t)}t^{-c_0/2}, \quad \Gamma(t) = \frac{1}{2\pi i} \int \frac{\ln \left( \frac{\tau c_0 a(\tau) - b(\tau)}{a(\tau) + b(\tau)} \right)}{\tau - t} d\tau.$$

It is known that (see [2], [4])

i) If $c_0 > 0$ then $K$ is right invertible and

$$\mathcal{R}_K = \left\{ R = R_0 + (I - R_0 K)T : T \in L_0(X) \right\}.$$

Let $F_0, \ldots, F_{n-1}$ be given initial operators for $K$ corresponding to $R_0, \ldots, R_{n-1} \in \mathcal{R}_K$, where

$$R_j = R_0 + (I - R_0 K)T_j \quad (j = 1, \ldots, n - 1); T_1, \ldots, T_{n-1} \in L_0(X). \quad (1)$$

ii) If $c_0 < 0$ then $K$ is left invertible and

$$\mathcal{L}_K = \left\{ R = R_0 + T(I - KR_0) : T \in L(X), \quad \text{dom} T = (I - KR_0)X \right\}.$$

In this case, let $G_0, \ldots, G_{n-1}$ be given co-initial operators for $K$ corresponding to $R_0, \ldots, R_{n-1} \in \mathcal{L}_K$, where

$$R_j = R_0 + T_j(I - KR_0) \quad (j = 1, \ldots, n - 1); \quad T_1, \ldots, T_{n-1} \in L(X). \quad (2)$$
Lemma 2. If $\alpha > 0$, then
\[
(F_0\varphi)(t) = \sum_{k=0}^{\alpha-1} u_k(\varphi)\psi_k(t) \text{ on } X,
\]
where $\psi_k(t) = b(t)Z(t)t^k$ ($k = 0, \ldots, \alpha - 1$) and $u_k(\varphi)$ ($k = 0, \ldots, \alpha - 1$) are linear functionals which are defined by
\[
u_k(\varphi) = \frac{1}{2\pi i} \int_\Gamma \frac{\tau^\alpha - 1 - k}{e^{\Gamma^-}(t)} \left[ \varphi(\tau) - \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau} d\tau_1 \right] d\tau,
\]
where $\Gamma^-(t)$ is a boundary value of the function $\Gamma(z)$ in $D^-$.

Proof. We have
\[
(F_0\varphi)(t) = \left[ (I - R_0K)\varphi \right](t)
= \varphi(t) - a^2(t)\varphi(t) - \frac{a(t)b(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau
+ \frac{b(t)Z(t)}{\pi i} \int_\Gamma \frac{1}{Z(\tau)(\tau - t)} \left[ a(\tau)\varphi(\tau) + \frac{b(\tau)}{\pi i} \int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau} d\tau_1 \right] d\tau
= \varphi(t) - a^2(t)\varphi(t) - \frac{a(t)b(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau
+ \frac{b(t)Z(t)}{\pi i} \left[ \frac{1}{\pi i} \int_\Gamma \frac{\varphi^+(\tau)d\tau}{X^+(\tau)(\tau - t)} - \frac{1}{\pi i} \int_\Gamma \frac{\varphi^-(\tau)d\tau}{X^-(\tau)(\tau - t)} \right],
\]
where $\varphi^+(t) = \frac{1}{2} \left[ (I + S)\varphi \right](t)$, $\varphi^-(t) = \frac{1}{2} \left[ (-I + S)\varphi \right](t)$, $X^+(t) = e^{\Gamma^+(t)}$, $X^-(t) = e^{-\alpha e^{\Gamma^-(t)}}$ ($\Gamma^+(t), \Gamma^-(t)$ are boundary values of the function $\Gamma(z)$ in $D^+, D^-$, respectively).

On the other hand
\[
\frac{1}{\pi i} \int_\Gamma \frac{\varphi^+(\tau)d\tau}{X^+(\tau)(\tau - t)} - \frac{1}{\pi i} \int_\Gamma \frac{\varphi^-(\tau)d\tau}{X^-(\tau)(\tau - t)} = \frac{\varphi^+(t)}{X^+(t)} - \frac{1}{\pi i} \int_\Gamma \frac{\varphi^-(t)\tau^\alpha d\tau}{e^{\Gamma^-(\tau)(\tau - t)}}
= \frac{\varphi^+(t)}{X^+(t)} - \frac{t^\alpha}{\pi i} \int_\Gamma \frac{\varphi^-(\tau)d\tau}{e^{\Gamma^-(\tau)(\tau - t)}} - \sum_{k=0}^{\alpha-1} \frac{t^k}{\pi i} \int_\Gamma \frac{\tau_{\alpha-1-k} \varphi^-(\tau)}{e^{\Gamma^-(\tau)(\tau - t)}} d\tau.
\]
The lemma is proved.

By similar arguments, we obtain the following result:

**Lemma 3.** If $\alpha < 0$, then

$$(G_0\varphi)(t) = \sum_{k=0}^{1+\alpha} v_k(\varphi)\psi_k(t) \quad \text{on } X,$$

where $\psi_k(t) = b(t)t^k$ ($k = 0, \ldots, 1+\alpha$) and $v_k(\varphi)$ ($k = 0, \ldots, 1+\alpha$) are linear functionals defined by

$$v_k(\varphi) = \frac{1}{2\pi i} \int_\Gamma \tau^{1+\alpha-k} e^{\Gamma^{-1}(\tau)} \frac{\varphi(\tau) - \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau_1) d\tau_1}{\tau_1 - \tau}}{Z(\tau)} d\tau$$

where $\Gamma^{-1}(t)$ is a boundary value of the function $\Gamma(z)$ in $D^-$. In the sequel, for every function $c(t) \in X$, we write

$$(K_c\varphi)(t) = c(t)\varphi(t).$$

Consider singular integral equation of the form

$$[(K^n + K^{n-1}K_c)\varphi](t) = f(t),$$

with mixed boundary conditions

i) $(F_jK^j\varphi)(t) = \varphi_j(t)$, $\varphi_j(t) \in \text{Ker } K$, $j = 0, \ldots, n-1$ if $\alpha > 0$,

ii) $(G_0f)(t) = 0$, $(G_jR_0\ldots R_{j-1}f)(t) = 0$, $j = 1, \ldots, n-1$ if $\alpha < 0$,
where \( f(t), c(t) \in X; \; F_j, G_j \) (\( j = 0, \ldots, n - 1 \)) are defined by (1) and (2), respectively, \( 1 < n \in \mathbb{N} \).

**Theorem 1.** Suppose that \( 1 + c(t)a(t) \pm c(t)b(t) \neq 0 \) for all \( t \in \Gamma \). Then every solution of the problem (5) - (6) can be found in a closed form.

**Proof**

Let \( \alpha > 0 \), we have \( K \in R(X) \).

Hence, the equation (5) is equivalent to the equation

\[
\varphi(t) = -(R_0 \ldots R_{n-1}K^{n-1}K_c\varphi)(t) + (R_0 \ldots R_{n-1}f)(t) + (R_0 \ldots R_{n-2}z_{n-1})(t) + \ldots + (R_0z_1)(t) + z_0(t),
\]

where \( z_0(t), \ldots, z_{n-1}(t) \in \text{Ker} \; K \) are arbitrary.

Thus, the problem (5)-(6) is equivalent to the equation

\[
\varphi(t) = -(R_0 \ldots R_{n-1}K^{n-1}K_c\varphi)(t) + (R_0 \ldots R_{n-1}f)(t) + (R_0 \ldots R_{n-2}\varphi_{n-1})(t) + \ldots + (R_0\varphi_1)((t) + \varphi_0(t),
\]

i.e.

\[
\left[(I + R_0 \ldots R_{n-1}K^{n-1}K_c)\varphi\right](t) = f_1(t),
\]

where

\[
f_1(t) = (R_0 \ldots R_{n-1}f)(t) + (R_0 \ldots R_{n-2}\varphi_{n-1})(t) + \ldots + (R_0\varphi_1)(t) + \varphi_0(t).
\]

By the Taylor-Gontcharov formula for right invertible operators (see [4]), (7) is equivalent to the equation

\[
\left[(I + R_0(I - F_1 - \sum_{k=2}^{n-1} R_1 \ldots R_{k-1} F_k K^{k-1})K_c)\varphi\right](t) = f_1(t).
\]  

By lemma 1, in order to solve the equation (8) it is enough to solve the equation

\[
\left[(I + K_c R_0 - F_1 K_c R_0 - \sum_{k=2}^{n-1} R_1 \ldots R_{k-1} F_k K^{k-1} K_c R_0)\psi\right](t) = g(t),
\]

where

\[
g(t) = \left[(I - F_1 - \sum_{k=2}^{n-1} R_1 \ldots R_{k-1} F_k K^{k-1})K_c f_1\right](t).
\]
Rewrite this equation in the form

\[
(I + K_c R_0 - F_1 K_c R_0 - \sum_{k=2}^{n-1} R_1 ... R_{k-1} F_k K^{k-1} K_c R_0) K Z \phi(t) = g(t), \quad (9)
\]

where \( \phi(t) = \psi(t)/Z(t) \).

From lemma 2, we have

\[
\left( F_k K^{k-1} K_c R_0 K Z \phi \right)(t) = \sum_{j=0}^{\infty-1} u_{jk}(\phi) \psi_j(t), \quad k = 1, ..., n - 1,
\]

where \( u_{jk}(\phi) = u_j \left( (I - T_k K) K^{k-1} K_c R_0 K Z \phi \right) \); \( u_j(\varphi), \psi_j(t) \) \( j = 0, ..., \infty - 1 \)

are defined by (3).

Hence, (9) is of the form

\[
(M \phi)(t) - \sum_{k=1}^{\infty-1} \sum_{j=0}^{n-1} u_{jk}(\phi) \psi_{jk}(t) = g(t), \quad (10)
\]

where \( \psi_{j1}(t) := \psi_j(t), \quad \psi_{jk}(t) = (R_1 ... R_{k-1} \psi_j)(t) \) \( k = 2, ..., n - 1, \quad j = 0, ..., \infty - 1 \) and

\[
(M \phi)(t) := \left[ 1 + c(t)a(t) \right] Z(t) \phi(t) - \frac{c(t)b(t)Z(t)}{\pi i} \int_{\gamma} \frac{\phi(\tau)}{\tau - t} d\tau. \quad (11)
\]

Write this equation in the form

\[
(M \phi)(t) - \sum_{k=1}^{q} \tilde{u}_k(\phi) \tilde{\psi}_k(t) = g(t), \quad (12)
\]

where \( q = \infty(n - 1) \), \{\tilde{u}_1(\phi), ..., \tilde{u}_q(\phi)\} is a permutation of \{u_{jk}(\phi), j = 0, ..., \infty - 1; k = 1, ..., n - 1\} and \{\tilde{\psi}_1(t), ..., \tilde{\psi}_q(t)\} is obtained by this permutation from the set of functions \{\psi_{jk}(t), j = 0, ..., \infty - 1; k = 1, ..., n - 1\}.

Denote

\[
(N \phi)(t) := \frac{[1 + a(t)c(t)]\phi(t) + c(t)b(t)Z_1(t)}{\pi i} \int_{\gamma} \frac{\phi(\tau)}{Z_1(\tau)(\tau - t)} d\tau \frac{1}{[(1 + a(t)c(t))^2 - c^2(t)b^2(t)]Z(t)},
\]
where

\[ Z_1(t) = Z(t)e^{\Gamma_1(t)t}t^{-\alpha_1/2}[(1 + a(t)c(t))^2 - b^2(t)c^2(t)]^{1/2}, \]
\[ \Gamma_1(t) = \frac{1}{2\pi i} \int_{\Gamma} \ln\left(\frac{\tau^{-\alpha_1}G(\tau)}{\tau - t}\right) d\tau, \]
\[ G(t) = \frac{1 + a(t)c(t) + b(t)c(t)}{1 + a(t)c(t) - b(t)c(t)}, \quad \alpha_1 = \text{Ind } G(t). \]

If \( \alpha_1 = 0 \), then \( M \) is invertible and \( M^{-1} = N \). Hence, the equation (12) is equivalent to the equation

\[ \phi(t) - \sum_{k=1}^{q} \tilde{u}_k(\phi)(N\tilde{\psi}_k)(t) = (Ng)(t). \]  

Without loss of generality, we can assume that \( \{(N\tilde{\psi}_k(t))\}_{k=1}^{q} \) is a linearly independent system. Then every solution of (13) can be found in a closed form by means of the system of linear algebraic equations

\[ \tilde{u}_j(\phi) - \sum_{k=1}^{q} a_{jk}\tilde{u}_k(\phi) = \tilde{u}_k(Ng), \quad j = 0, \ldots, q, \]

where \( a_{jk} = \tilde{u}_j(N\tilde{\psi}_k); \; k, j = 1, \ldots, q. \)

If \( \alpha_1 > 0 \), then \( M \) is right invertible and \( N \) is a right inverse of \( M \). Hence, the equation (12) is equivalent to the equation

\[ \phi(t) - \sum_{k=1}^{q} \tilde{u}_k(\phi)(N\tilde{\psi}_k)(t) = (Ng)(t) + y(t), \]

where \( y(t) \in \text{Ker } M \) is arbitrary.

We now can solve this equation by the same method as for the equation (13), i.e. every its solution can be found in a closed form.

If \( \alpha_1 < 0 \), then \( M \) is left invertible and \( N \) is a left inverse of \( M \). Hence, the equation (12) is equivalent to the system

\[ \begin{cases} 
\phi(t) - \sum_{k=1}^{q} \tilde{u}_k(\phi)(N\tilde{\psi}_k)(t) = (Ng)(t), \\
\int_{\Gamma} g(\tau) + \sum_{k=1}^{q} \tilde{u}_k(\phi)\tilde{\psi}_k(\tau) \frac{1}{Z_1(\tau)}\tau^{\nu-1}d\tau = 0, \; \nu = 1, \ldots, |\alpha_1|,
\end{cases} \]
i.e.

\[
\begin{align*}
\phi(t) - \sum_{k=1}^{q} \tilde{u}_k(\phi)(N\tilde{\psi}_k)(t) &= (Ng)(t), \\
\sum_{k=1}^{q} b_{\nu k} \tilde{u}_k(\phi) &= f_{\nu}, \quad \nu = 1, \ldots, |\omega_1|,
\end{align*}
\]  

(14)

where

\[
f_{\nu} = - \int_{\Gamma} \frac{g(\tau)\tau^{\nu-1}d\tau}{Z_1(\tau)}, \quad b_{\nu k} = \int_{\Gamma} \tilde{\psi}_k(\tau)\tau^{\nu-1} \frac{d\tau}{Z_1(\tau)}.
\]

Without loss of generality, we can assume that \(\{(N\tilde{\psi}_k)(t)\}_{k=1}^{q}\) is a linearly independent system. Every solution of (14) can be found in a closed form by means of the system of linear algebraic equations

\[
\begin{align*}
\tilde{u}_j(\phi) - \sum_{k=1}^{q} \alpha_{jk} \tilde{u}_k(\phi) &= \tilde{u}_j(Ng), \quad j = 1, \ldots, q, \\
\sum_{k=1}^{q} b_{\nu k} \tilde{u}_k(\phi) &= f_{\nu}, \quad \nu = 1, \ldots, |\omega_1|,
\end{align*}
\]

where \(\alpha_{kj} = \tilde{u}_k(N\tilde{\psi}_j), k, j = 1, \ldots, q\).

Thus, every solution of the equation (12) can be found in a closed form.

Due to the result of Lemma 1, every solution of the problem (5)-(6) is defined by the formula

\[
\varphi(t) = (R_0K_2\phi)(t) + f_1(t),
\]

where \(\phi(t)\) is a solution of the equation (12), i.e. every solution of the problem (5)-(6) can be found in a closed form.

Let \(\omega < 0\), we have \(K \in \Lambda(X)\).

The Taylor-Gontcharov formula for left invertible operators (see [4]) and (6) together imply

\[
[(I - K^nR_{n-1} \ldots R_0)f](t) = (G_0f)(t) + \left[ \sum_{k=1}^{n-1} K^k G_k R_{k-1} \ldots R_0 f \right](t) = 0,
\]

i.e.

\[
f(t) = \left[ (K^nR_{n-1} \ldots R_0)f \right](t).
\]
Hence, the problem (5)-(6) is equivalent to the equation
\[ ([K^n + K^{n-1}K_c] \varphi)(t) = (K^nR_{n-1}...R_0f)(t), \]
i.e.
\[ ([K + K_c] \varphi)(t) = (KR_{n-1}...R_0f)(t). \] (15)

If \( \varphi(t) \) is a solution of (15) then
\[ (G_0K_c \varphi)(t) = (G_0KR_{n-1}...R_0f)(t) - (G_0K \varphi)(t) = 0, \]

Thus, (15) is equivalent to the system
\[
\begin{cases}
  [(I + R_0K_c) \varphi](t) = f_2(t), \\
  (G_0K_c \varphi)(t) = 0,
\end{cases}
\] (16)

where \( f_2(t) = (R_{n-1}...R_0f)(t) \).

From Lemma 3, \( (G_0K_c \varphi)(t) = 0 \) if and only if
\[ v_k(K_c \varphi) = 0, \quad k = 0, ..., |\alpha| - 1, \]

where \( v_k(\varphi) \) \( (k = 0, ..., |\alpha| - 1) \) are defined by (4).

Hence, the system (16) is equivalent to the system
\[
\begin{cases}
  [(I + R_0K_c) \varphi](t) = f_2(t), \\
  v_k(K_c \varphi) = 0, \quad k = 0, ..., |\alpha| - 1.
\end{cases}
\]

Consider the system of equations
\[
\begin{cases}
  [(I + K_cR_0) \psi](t) = (K_c f_2)(t), \\
  v_k(\psi) = 0, \quad k = 0, ..., |\alpha| - 1.
\end{cases}
\] (17)

It is easy to check that the system (16) has solutions if and only if the system (17) does. Moreover, if \( \varphi(t) \) is a solution of (16) then \( \psi(t) = c(t) \varphi(t) \) is a solution of (17). Conversely, if \( \psi(t) \) is a solution of (17) then
\[
\varphi(t) = \frac{b(t)\psi(t) + b(t)Z(t) \int_1^t \frac{\psi(\tau)}{Z(\tau)(\tau-t)} d\tau + f_2(t)}{1 + c(t)a(t) + c(t)b(t)} \] (18)
is a solution of (16).

Hence, in order to solve the system (16), it is enough to solve the system (17).

Rewrite (17) in the form

\[
\begin{cases}
(M\phi)(t) = (K_c f_2)(t), \\
\tilde{v}_k(\phi) = 0, \quad k = 0, ..., |\alpha| - 1.
\end{cases}
\] (19)

where \(\phi(t) = \psi(t)/Z(t), \tilde{v}_k(\phi) = v_k(K_Z\phi)\) and \(M\) is defined by (11).

By the same method as for the equation (12), every solution of this system can be found in a closed form. So every solution of the problem (5)-(6) is defined by the formula

\[
\varphi(t) = \frac{b(t)Z(t)\phi(t) + \frac{b(t)Z(t)}{\pi i} \int \frac{\phi(\tau)}{\tau - t} d\tau + f_2(t)}{1 + c(t)a(t) + c(t)b(t)},
\]

where \(\phi(t)\) is a solution of the system (19), i.e. every solution of the problem (5)-(6) can be found in a closed form.

The theorem is proved.

References


