ON BOUNDARY VALUE PROBLEMS FOR A CLASS OF SINGULAR INTEGRAL EQUATIONS

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Abstract

This report deals with the solvability of boundary value problems for singular integral equations of the form

$$\Big[\Big(K^n + K^{n-1}K_c\Big)\varphi\Big](t) = f(t), \tag{i}$$

$$\begin{aligned}(F_jK^j\varphi)(t) = \varphi_j(t),\ \ j = 0, ..., n-1,\ \ \varphi_j(t) \in \text{ Ker } K, \quad \text{if } \ æ > 0,\\ (G_0f)(t) = 0,\ (G_jR_{j-1}...R_0)f(t) = 0,\ \ j = 1, ..., n-1 \quad \text{if } \ æ < 0,\end{aligned} \tag{ii}$$

By an algebraic method we reduce the problem (i) - (ii) to a system of linear algebraic equations which gives all solutions in a closed form.

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1 Introduction

The theory of general boundary value problem induced by right invertible operators were investigated by Przeworska - Rolewicz and has been developed by many other mathematicians (c.f. [2], [4]). In this paper, we give an application of this theory to solve the following boundary value problem:

$$\Big[\Big(K^n + K^{n-1}K_c\Big)\varphi\Big](t) = f(t), \tag{i}$$

$$\begin{aligned} (F_j K^j \varphi)(t) &= \varphi_j(t), \;\; j=0,...,n-1, \;\; \varphi_j(t) \in \text{ Ker } K, \quad \text{if } \;\text{æ} > 0, \\ (G_0 f)(t) = 0, \; (G_j R_{j-1}...R_0) f(t) &= 0, \;\; j = 1,...,n-1 \quad \text{if } \;\text{æ} < 0, \end{aligned} \tag{ii}$$

where K_c is an operator of multiplication by the function $c(t)$; F_j, G_j $(j = 0, ..., n-1)$ are initial and co-initial operators, respectively, and

$$\Big(K\varphi\Big)(t) = a(t)\varphi(t) + \frac{b(t)}{\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau, \quad \text{æ} = \text{Ind } K.$$

2 Preliminaries

We recall some notations and results which are used in the sequel (see [2], [4]).

Let Γ be a simple regular closed arc in complex plane and let $X = H^{\mu}(\Gamma) \quad (0 < \mu < 1)$. Denote by D^+ the domain bounded by Γ (assume that $0 \in D^+$) and by D^- -its complement including the point at infinity. The set of all linear operators with domains and ranges contained in X will be denoted by $L(X)$. Write $L_0(X) := \Big\{A \in L(X) : \text{dom}A = X\Big\}$.

Let $R(X)$ be the set of all right invertible operators belonging to $L(X)$. For $D \in R(X)$, we denote by $\mathcal{R}_D$ the set of all its right inverses.

An operator $F \in L(X)$ is said to be an initial operator for an operator $D \in R(X)$ corresponding to a right inverse R of D if

$$F^2 = F, \quad F(\text{dom D}) = \text{ Ker } D, \quad FR = 0.$$

Denote by $\mathcal{F}_D$ the set of all initial operators for $D \in R(X)$.

It is well - known the following fact:

$F \in L(X)$ is an initial operator for $D \in R(X)$ corresponding to $R \in \mathcal{R}_D$ if and only if $F = I - RD$ on dom$D$.

Let $\Lambda(X)$ be the set of all left invertible operators belonging to $L_0(X)$. For $V \in \Lambda(X)$, we denote by $\mathcal{L}_V$ the set of all its left inverser.

If $V \in \Lambda(X)$ and $L \in \mathcal{L}_V$ then the operator

$$G := I - VL$$

is called the co-initial operators for V corresponding to $L \in \mathcal{L}_V$.

Denote by $\mathcal{G}_V$ the set of all co-initial operators for $V \in \Lambda(X)$.

Lemma 1 (see [2]). *Let $A, B \in L(X)$,* Im$A \subset$ dom$B$, Im$B \subset$ dom A. *Then equation $(I-AB)x = y$ has solutions if and only if $(I-BA)u = By$ does and there is one-to-one correspondence between the two sets of solutions, given by*

$$u = Bx \longleftrightarrow x = y + Au.$$

2. The results

Let

$$(K\varphi)(t) := a(t)\varphi(t) + \frac{b(t)}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau,$$

where $a(t), b(t) \in X, a^2(t) - b^2(t) = 1,\ \ 0 \neq æ = \mathrm{Ind}K.$

Denote

$$(R_0\varphi)(t) := a(t)\varphi(t) - \frac{b(t)Z(t)}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)}{Z(\tau)(\tau - t)}d\tau,$$

where

$$Z(t) = e^{\Gamma(t)}t^{-æ/2}, \quad \Gamma(t) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\ln\left(\tau^{-æ}\frac{a(\tau)-b(\tau)}{a(\tau)+b(\tau)}\right)}{\tau - t}d\tau.$$

It is known that (see [2], [4])

i) If $æ > 0$ then K is right invertible and

$$\mathcal{R}_K = \Big\{R = R_0 + (I - R_0K)T : T \in L_0(X)\Big\}.$$

Let $F_0, ..., F_{n-1}$ be given initial operators for K corresponding to $R_0, ..., R_{n-1} \in \mathcal{R}_K$, where

$$R_j = R_0 + (I - R_0K)T_j \quad (j = 1, ..., n-1); T_1, ..., T_{n-1} \in L_0(X). \qquad (1)$$

ii) If $æ < 0$ then K is left inverbible and

$$\mathcal{L}_K = \Big\{R = R_0 + T(I - KR_0) : T \in L(X),\ \ \mathrm{dom}\ T = (I - KR_0)X\Big\}.$$

In this case, let $G_0, ..., G_{n-1}$ be given co-initial operators for K corresponding to $R_0, ..., R_{n-1} \in \mathcal{L}_K$, where

$$R_j = R_0 + T_j(I - KR_0)\ \ (j = 1, ..., n-1);\ \ T_1, ..., T_{n-1} \in L(X). \qquad (2)$$

Lemma 2. *If* $\text{æ} > 0$, *then*

$$(F_0\varphi)(t) = \sum_{k=0}^{\text{æ}-1} u_k(\varphi)\psi_k(t) \quad \text{on } X,$$

where $\psi_k(t) = b(t)Z(t)t^k$ $(k = 0, ..., \text{æ} - 1)$ *and* $u_k(\varphi)$ $(k = 0, ..., \text{æ} - 1)$ *are linear functionals which are defined by*

$$u_k(\varphi) = \frac{1}{2\pi i}\int_\Gamma \frac{\tau^{\text{æ}-1-k}}{e^{\Gamma^-(\tau)}}\Big[\varphi(\tau) - \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau}d\tau_1\Big]d\tau, \qquad (3)$$

where $\Gamma^-(t)$ *is a boundary value of the function* $\Gamma(z)$ *in* D^-.

Proof. We have

$$\begin{aligned}(F_0\varphi)(t) &= \Big[(I - R_0K)\varphi\Big](t) \\ &= \varphi(t) - a^2(t)\varphi(t) - \frac{a(t)b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau \\ &+ \frac{b(t)Z(t)}{\pi i}\int_\Gamma \frac{1}{Z(\tau)(\tau - t)}\Big[a(\tau)\varphi(\tau) + \frac{b(\tau)}{\pi i}\int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau}d\tau_1\Big]d\tau \\ &= \varphi(t) - a^2(t)\varphi(t) - \frac{a(t)b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau \\ &+ b(t)Z(t)\Big[\frac{1}{\pi i}\int_\Gamma \frac{\varphi^+(\tau)d\tau}{X^+(\tau)(\tau - t)} - \frac{1}{\pi i}\int_\Gamma \frac{\varphi^-(\tau)d\tau}{X^-(\tau)(\tau - t)}\Big],\end{aligned}$$

where $\varphi^+(t) = \frac{1}{2}\Big[(I + S)\varphi\Big](t)$, $\varphi^-(t) = \frac{1}{2}\Big[(-I + S)\varphi\Big](t)$, $X^+(t) = e^{\Gamma^+(t)}, X^-(t) = t^{-\text{æ}}e^{\Gamma^-(t)}$ $(\Gamma^+(t), \Gamma^-(t)$ are boundary values of the function $\Gamma(z)$ in D^+, D^-, respectively).

On the other hand

$$\begin{aligned}&\frac{1}{\pi i}\int_\Gamma \frac{\varphi^+(\tau)d\tau}{X^+(\tau)(\tau - t)} - \frac{1}{\pi i}\int_\Gamma \frac{\varphi^-(\tau)d\tau}{X^-(\tau)(\tau - t)} = \frac{\varphi^+(t)}{X^+(t)} - \frac{1}{\pi i}\int_\Gamma \frac{\varphi^-(\tau)\tau^{\text{æ}}d\tau}{e^{\Gamma^-(\tau)}(\tau - t)} \\ &= \frac{\varphi^+(t)}{X^+(t)} - \frac{t^{\text{æ}}}{\pi i}\int_\Gamma \frac{\varphi^-(\tau)d\tau}{e^{\Gamma^-(\tau)}(\tau - t)} - \sum_{k=0}^{\text{æ}-1}\frac{t^k}{\pi i}\int_\Gamma \frac{\tau^{\text{æ}-1-k}\varphi^-(\tau)}{e^{\Gamma^-(\tau)}}d\tau\end{aligned}$$

$$= \frac{\varphi^+(t)}{X^+(t)} + \frac{\varphi^-(t)}{X^-(t)} - \sum_{k=0}^{œ-1} \frac{t^k}{\pi i} \int_\Gamma \frac{\tau^{œ-1-k}\varphi^-(\tau)}{e^{\Gamma^-(\tau)}} d\tau$$

$$= \frac{b(t)}{Z(t)}\varphi(t) + \frac{a(t)}{Z(t)\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau - \sum_{k=0}^{œ-1} \frac{t^k}{\pi i} \int_\Gamma \frac{\tau^{œ-1-k}\varphi^-(\tau)}{e^{\Gamma^-(\tau)}} d\tau.$$

Hence

$$(F_0\varphi)(t) = \sum_{k=0}^{œ-1} \frac{b(t)Z(t)t^k}{2\pi i} \int_\Gamma \frac{\tau^{œ-1-k}}{e^{\Gamma^-(\tau)}} \Big[\varphi(\tau) - \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau} d\tau_1\Big] d\tau.$$

The lemma is proved.

By similar arguments, we obtain the following result:

Lemma 3. *If* $œ < 0$, *then*

$$(G_0\varphi)(t) = \sum_{k=0}^{|œ|-1} v_k(\varphi)\psi_k(t) \qquad on\ X,$$

where $\psi_k(t) = b(t)t^k \quad (k = 0, ..., |œ| - 1)$ *and* $v_k(\varphi) \quad (k = 0, ...|œ| - 1)$ *are linear functionals defined by*

$$v_k(\varphi) = \frac{1}{2\pi i} \int_\Gamma \tau^{|œ|-1-k} e^{\Gamma^-(\tau)} \Big[\frac{\varphi(\tau)}{Z(\tau)} - \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau_1)d\tau_1}{Z(\tau_1)(\tau_1 - \tau)}\Big] d\tau \tag{4}$$

where $\Gamma^-(t)$ *is a boundary value of the function* $\Gamma(z)$ *in* D^-.

In the sequel, for every function $c(t) \in X$, we write

$$(K_c\varphi)(t) = c(t)\varphi(t).$$

Consider singular integral equation of the form

$$\Big[(K^n + K^{n-1}K_c)\varphi\Big](t) = f(t), \tag{5}$$

with mixed boundary conditions

$$\begin{array}{ll} \text{i)} & (F_jK^j\varphi)(t) = \varphi_j(t), \quad \varphi_j(t) \in \text{ Ker } K, \quad j = 0, ..., n-1 \quad \text{if } œ > 0, \\ \text{ii)} & (G_0f)(t) = 0, \ \ (G_jR_0...R_{j-1}f)(t) = 0, \quad j = 1, ..., n-1 \quad \text{if } œ < 0, \end{array} \tag{6}$$

where $f(t), c(t) \in X$; F_j, G_j $(j = 0, ..., n-1)$ are defined by (1) and (2), respectively, $1 < n \in \mathbb{N}$.

Theorem 1. *Suppose that $1 + c(t)a(t) \pm c(t)b(t) \neq 0$ for all $t \in \Gamma$. Then every solution of the problem (5) - (6) can be found in a closed form .*

Proof

Let œ > 0, we have $K \in R(X)$.

Hence, the equation (5) is equivalent to the equation

$$\varphi(t) = -(R_0...R_{n-1}K^{n-1}K_c\varphi)(t) + (R_0...R_{n-1}f)(t) + (R_0...R_{n-2}z_{n-1})(t) \\ + ... + (R_0 z_1)(t) + z_0(t),$$

where $z_0(t), ..., z_{n-1}(t) \in$ Ker K are arbitrary.

Thus, the problem (5)-(6) is equivalent to the equation

$$\varphi(t) = -(R_0...R_{n-1}K^{n-1}K_c\varphi)(t) + (R_0...R_{n-1}f)(t) + (R_0...R_{n-2}\varphi_{n-1})(t) \\ + ... + (R_0\varphi_1((t) + \varphi_0(t),$$

i.e.

$$\Big[(I + R_0...R_{n-1}K^{n-1}K_c)\varphi\Big](t) = f_1(t), \tag{7}$$

where

$$f_1(t) = (R_0...R_{n-1}f)(t) + (R_0...R_{n-2}\varphi_{n-1})(t) + ... + (R_0\varphi_1)(t) + \varphi_0(t).$$

By the Taylor-Gontcharov formula for right invertible operators (see [4]), (7) is iquivalent to the equation

$$\Big[(I + R_0(I - F_1 - \sum_{k=2}^{n-1} R_1...R_{k-1}F_kK^{k-1})K_c)\varphi\Big](t) = f_1(t). \tag{8}$$

By lemma 1, in order to solve the equation (8) it is enough to solve the equation

$$\Big[(I + K_cR_0 - F_1K_cR_0 - \sum_{k=2}^{n-1} R_1...R_{k-1}F_kK^{k-1}K_cR_0)\psi\Big](t) = g(t),$$

where

$$g(t) = \Big[(I - F_1 - \sum_{k=2}^{n-1} R_1...R_{k-1}F_kK^{k-1})K_cf_1\Big](t).$$

Rewrite this equation in the form

$$\Big[(I+K_cR_0-F_1K_cR_0-\sum_{k=2}^{n-1}R_1...R_{k-1}F_kK^{k-1}K_cR_0)K_Z\phi\Big](t)=g(t), \quad (9)$$

where $\phi(t)=\psi(t)/Z(t)$.

From lemma 2, we have

$$\Big(F_kK^{k-1}K_cR_0K_Z\phi\Big)(t)=\sum_{j=0}^{œ-1}u_{jk}(\phi)\psi_j(t), \qquad k=1,...,n-1,$$

where $u_{jk}(\phi)=u_j\Big[(I-T_kK)K^{k-1}K_cR_0K_Z\phi\Big]$; $u_j(\varphi),\psi_j(t)$ $(j=0,...,œ-1)$ are defined by (3).

Hence, (9) is of the form

$$(M\phi)(t)-\sum_{k=1}^{n-1}\sum_{j=0}^{œ-1}u_{jk}(\phi)\psi_{jk}(t)=g(t), \quad (10)$$

where $\psi_{j1}(t):=\psi_j(t)$, $\quad \psi_{jk}(t)=(R_1...R_{k-1}\psi_j)(t) \quad (k=2,...,n-1)$, $\quad j=0,...,œ-1$ and

$$(M\phi)(t):=\Big[1+c(t)a(t)\Big]Z(t)\phi(t)-\frac{c(t)b(t)Z(t)}{\pi i}\int_\Gamma\frac{\phi(\tau)}{\tau-t}d\tau. \quad (11)$$

Write this equation in the form

$$(M\phi)(t)-\sum_{k=1}^{q}\widetilde{u}_k(\phi)\widetilde{\psi}_k(t)=g(t), \quad (12)$$

where $q=œ(n-1),\{\widetilde{u}_1(\phi),...,\widetilde{u}_q(\phi)\}$ is a permutation of $\{u_{jk}(\phi),j=0,...œ-1;k=1,...,n-1\}$ and $\{\widetilde{\psi}_1(t),...,\widetilde{\psi}_q(t)\}$ is obtained by this permutation from the set of functions $\{\psi_{jk}(t),j=0,...,œ-1;k=1,...,n-1\}$.

Denote

$$(N\phi)(t):=\frac{[1+a(t)c(t)]\phi(t)+\frac{c(t)b(t)Z_1(t)}{\pi i}\int_\Gamma\frac{\phi(\tau)}{Z_1(\tau)(\tau-t)}d\tau}{[(1+a(t)c(t))^2-c^2(t)b^2(t)]Z(t)},$$

where

$$
\begin{aligned}
Z_1(t) &= Z(t)e^{\Gamma_1(t)}t^{-\text{\oe}_1/2}[(1+a(t)c(t))^2-b^2(t)c^2(t)]^{1/2},\\
\Gamma_1(t) &= \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\ln(\tau^{-\text{\oe}_1}G(\tau))}{\tau-t}d\tau,\\
G(t) &= \frac{1+a(t)c(t)+b(t)c(t)}{1+a(t)c(t)-b(t)c(t)},\quad \text{\oe}_1=\operatorname{Ind}G(t).
\end{aligned}
$$

If $\text{\oe}_1 = 0$, then M is invertible and $M^{-1} = N$. Hence, the equation (12) is equivalent to the equation

$$
\phi(t)-\sum_{k=1}^{q}\widetilde{u}_k(\phi)(N\widetilde{\psi}_k)(t)=(Ng)(t). \tag{13}
$$

Without loss of generality, we can assume that $\{(N\widetilde{\psi}_k)(t)\}_{k=\overline{1,q}}$ is a linearly independent system. Then every solution of (13) can be found in a closed form by means of the system of linear algebraic equations

$$
\widetilde{u}_j(\phi)-\sum_{k=1}^{q}a_{jk}\widetilde{u}_k(\phi)=\widetilde{u}_k(Ng),\quad j=0,...,q,
$$

where $a_{jk}=\tilde{u}_j(N\tilde{\psi}_k)$; $k,j=1,...,q$.

If $\text{\oe}_1 > 0$, then M is right invertible and N is a right inverse of M. Hence, the equation (12) is equivalent to the equation

$$
\phi(t)-\sum_{k=1}^{q}\tilde{u}_k(\phi)(N\tilde{\psi}_k)(t)=(Ng)(t)+y(t),
$$

where $y(t)\in$ Ker M is arbitrary.

We now can solve this equation by the same method as for the equation (13), i.e. every its solution can be found in a closed form.

If $\text{\oe}_1 < 0$, then M is left invertible and N is a left inverse of M. Hence, the equation (12) is equivalent to the system

$$
\begin{cases}
\phi(t)-\sum\limits_{k=1}^{q}\widetilde{u}_k(\phi)(N\widetilde{\psi}_k)(t)=(Ng)(t),\\
\int\limits_{\Gamma}\dfrac{g(\tau)+\sum\limits_{k=1}^{q}\widetilde{u}_k(\phi)\widetilde{\psi}_k(\tau)}{Z_1(\tau)}\tau^{\nu-1}d\tau=0,\quad \nu=1,...,|\text{\oe}_1|,
\end{cases}
$$

i.e.

$$\begin{cases} \phi(t) - \sum\limits_{k=1}^{q} \widetilde{u}_k(\phi)(N\widetilde{\psi}_k)(t) = (Ng)(t), \\ \sum\limits_{k=1}^{q} b_{\nu k}\widetilde{u}_k(\phi) = f_\nu, \quad \nu = 1, ..., |æ_1|, \end{cases} \tag{14}$$

where

$$f_\nu = -\int\limits_\Gamma \frac{g(\tau)\tau^{\nu-1}d\tau}{Z_1(\tau)}, \quad b_{\nu k} = \int\limits_\Gamma \frac{\widetilde{\psi}_k(\tau)\tau^{\nu-1}}{Z_1(\tau)}d\tau.$$

Without loss of generality, we can assume that $\{(N\widetilde{\psi}_k)(t)\}_{k=\overline{1,q}}$ is a linearly independent system. Every solution of (14) can be found in a closed form by means of the system of linear algebraic equations

$$\begin{cases} \widetilde{u}_j(\phi) - \sum\limits_{k=1}^{q} a_{jk}\widetilde{u}_k(\phi) = \tilde{u}_j(Ng), \quad j = 1, ..., q, \\ \sum\limits_{k=1}^{q} b_{\nu k}\widetilde{u}_k(\phi) = f_\nu, \quad \nu = 1, ..., |æ_1|, \end{cases}$$

where $a_{kj} = \tilde{u}_k(N\widetilde{\psi}_j), k, j = 1, ..., q$.

Thus, every solution of the equation (12) can be found in a closed form.

Due to the result of Lemma 1, every solution of the problem (5)-(6) is defined by the formula

$$\varphi(t) = (R_0 K_Z \phi)(t) + f_1(t),$$

where $\phi(t)$ is a solution of the equation (12), i.e. every solution of the problem (5)-(6) can be found in a closed form.

Let $æ < 0$, we have $K \in \Lambda(X)$.

The Taylor-Gontcharov formula for left invertible operators (see [4]) and (6) together imply

$$[(I - K^n R_{n-1}...R_0)f](t) = (G_0 f)(t) + [(\sum_{k=1}^{n-1} K^k G_k R_{k-1}...R_0)f](t) = 0,$$

i.e.

$$f(t) = \Big[(K^n R_{n-1}...R_0)f\Big](t).$$

Hence, the problem (5)-(6) is equivalent to the equation

$$[(K^n + K^{n-1}K_c)\varphi](t) = (K^n R_{n-1}...R_0 f)(t),$$

i.e.

$$[(K + K_c)\varphi](t) = (KR_{n-1}...R_0 f)(t). \tag{15}$$

If $\varphi(t)$ is a solution of (15) then

$$(G_0 K_c \varphi)(t) = (G_0 K R_{n-1}...R_0 f)(t) - (G_0 K \varphi)(t) = 0,$$

Thus, (15) is equivalent to the system

$$\begin{cases} [(I + R_0 K_c)\varphi](t) & = f_2(t), \\ (G_0 K_c \varphi)(t) & = 0, \end{cases} \tag{16}$$

where $f_2(t) = (R_{n-1}...R_0 f)(t)$.

From Lemma 3, $(G_0 K_c \varphi)(t) = 0$ if and only if

$$v_k(K_c \varphi) = 0, \quad k = 0, ..., |æ| - 1,$$

where $v_k(\varphi)$ $(k = 0, ..., |æ| - 1)$ are defined by (4).

Hence, the system (16) is equivalent to the system

$$\begin{cases} [(I + R_0 K_c)\varphi](t) & = f_2(t), \\ v_k(K_c \varphi) & = 0, \quad k = 0, ..., |æ| - 1. \end{cases}$$

Consider the system of equations

$$\begin{cases} [(I + K_c R_0)\psi](t) & = (K_c f_2)(t), \\ v_k(\psi) & = 0, \quad k = 0, ..., |æ| - 1. \end{cases} \tag{17}$$

It is easy to check that the system (16) has solutions if and only if the system (17) does. Moreover, if $\varphi(t)$ is a solution of (16) then $\psi(t) = c(t)\varphi(t)$ is a solution of (17). Conversely, if $\psi(t)$ is a solution of (17) then

$$\varphi(t) = \frac{b(t)\psi(t) + \frac{b(t)Z(t)}{\pi i} \int\limits_{\Gamma} \frac{\psi(\tau)}{Z(\tau)(\tau - t)} d\tau + f_2(t)}{1 + c(t)a(t) + c(t)b(t)} \tag{18}$$

is a solution of (16).

Hence, in order to solve the system (16), it is enough to solve the system (17).

Rewrite (17) in the form

$$\begin{cases} (M\phi)(t) & = (K_c f_2)(t), \\ \tilde{v}_k(\phi) & = 0, \quad k = 0, ..., |œ| - 1. \end{cases} \tag{19}$$

where $\phi(t) = \psi(t)/Z(t), \tilde{v}_k(\phi) = v_k(K_Z\phi)$ and M is defined by (11).

By the same method as for the equation (12), every solution of this system can be found in a closed form. So every solution of the problem (5)-(6) is defined by the formula

$$\varphi(t) = \frac{b(t)Z(t)\phi(t) + \frac{b(t)Z(t)}{\pi i} \int\limits_{\Gamma} \frac{\phi(\tau)}{(\tau - t)} d\tau + f_2(t)}{1 + c(t)a(t) + c(t)b(t)},$$

where $\phi(t)$ is a solution of the system (19), i.e. every solution of the problem (5)-(6) can be found in a closed form.
The theorem is proved.

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