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Author(s)	Nguyen, Van Ngoc
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ON THE SOLVABILITY OF ATMOSPHERIC POLLUTION PROBLEM

NGUYEN VAN NGOC

ABSTRACT. The investigation of mathematical models of environment pollution problems attracts great attention of mathematicians. This work deals with a mixed initial boundary problem of nonstationary diffusion equation describing an air pollution process. The uniqueness and existence theorems are obtained for the considered problem in three- dimensional case.

1. INTRODUCTION

Let G be a bounded domain in the space R^3 with a sufficiently regular boundary S, $x = (x_1, x_2, x_3) \in G, 0 < t < T < \infty$. Denote by Q_T the cylinder :$G \times (0,T)$, by $\vec{U} = \vec{U}(x) = (u_1, u_2, u_3)$ the wind velocity, and by $\varphi(x,t)$ the concentration of the pollutant. As is known, the process of pollutant transport and diffusion in the atmosphere is described by the following equation [2]

$$\frac{\partial \varphi}{\partial t} + \mathrm{div}(\vec{U}\varphi) + \sigma\varphi - \mathrm{div}(\mu\nabla\varphi) = f, \tag{1.1}$$

where $\sigma(x)$ and $\mu(x)$ are continuous functions in $\overline{G}$, moreover $0 \leqslant \sigma(x) \leqslant \sigma_0, 0 < \mu_0 \leqslant \mu(x) \leqslant \mu_1, \sigma_0, \mu_0$, and μ_1 are positive constants, ∇ is the gradient operator, $f = f(x,t)$ is a given function in a suitable space.

We shall consider the equation (1.1) with the flowing boundary and initial conditions

$$\varphi\mid_{S_1} = 0, \quad \frac{\partial \varphi}{\partial n}\mid_{S_2} = 0, \tag{1.2}$$

$$\varphi\mid_{t=0} = \varphi_0(x), \tag{1.3}$$

where n is outer normal to S, S_1 and S_2 denote two parts of S : $S_1 \cup S_2 = S$, and besides

$$U_n\mid_{S_1} \leqslant 0, \quad U_n\mid_{S_2} = U_n^+ > 0. \tag{1.4}$$

Here U_n is the project of $\vec{U}$ to the normal n, $U_n^+ = U_n^+(x) \in C(\overline{S_2})$.

For the vector $\vec{U}$ we make the following assumption (see [1])

$$\mathrm{div}\vec{U} = 0. \tag{1.5}$$

The uniqueness of solution for the problem (1.1)-(1.4) was devoted formally in [2]. To our knowledge, the existence of solution for this problems has not payed attention appropriately.

The aim of the present work is to obtain existence and uniqueness theorems for the problem (1.1)-(1.4) in some suitable functional spaces.

2. Uniqueness Theorem

In the sequence we assume that the numbers and functions used in this work are real. Let $H^m(G)$ be the Sobolev space (see [1,3]), where m is a non-negative integer number. We shall use the following notations

$$H = H^0(G) = L^2(G)\,, \vec{H} = H \times H \times H$$

Definition 1. Denote by $\widetilde{H}^1(G)$ the space defined as the closure of the set $C^1(\overline{G})$ of differentiable functions satisfying the condition

$$\varphi\, |_{S_1} = 0, \tag{2.1}$$

whith respect to the norm of the space $H^1(G)$. The norm in is defined by the same way as in $H^1(G)$.

Le us define the following bilinear form

$$\phi(\varphi, \varphi^*) = (\sigma\varphi, \varphi^*)_H + (\mu\nabla\varphi, \nabla\varphi^*)_{\vec{H}}, \tag{2.2}$$

where $(,)_X$ denotes the scalar product in the Hilbert space X. By virtue of the conditions on σ, μ and (2.1) the bilinear form ϕ satisfies the following inequalities (see [1,3])

$$\alpha||\varphi||_{H^1(G)} \leqslant \phi(\varphi, \varphi) \leqslant \beta||\varphi||_{H^1(G)}, \tag{2.3}$$

where α and β are certain positive constants .

Let X be a Banach space. Denote by $L^2(0, T; X)$ the space of functions v such that (see[3])

$$v : (0, T) \longrightarrow X,$$

$$||v||_{L^2(0,T;X)} := \Big(\int_0^T ||v||_X^2 dt\Big)^{1/2} < \infty.$$

We assume that

$$f(x, t) \in L^2(0, T; H), \quad \varphi_0(x) \in H. \tag{2.4}$$

Definition 2. The function $\varphi \in L^2(0, T; \widetilde{H}^1(G))$ satisfying the condition (1.3) is called the generalized solution in Q_T of the problem (1.1)-(1.4) if

$$J(\varphi, \varphi^*) := \int_0^T [(\frac{\partial\varphi}{\partial t}, \varphi^*)_H + \phi(\varphi, \varphi^*) + \int_{S_2} U_n^+ \varphi\varphi^* ds$$
$$- (\varphi, \operatorname{div}(\varphi^*\vec{U})) - (f, \varphi^*)_H]dt = 0$$

for any $\varphi^* \in L_2(0, T; \widetilde{H}^1(G))$.

It is not difficult to show that the equality (2.5) is equivalent to the following equality

$$\int_{Q_T} [\frac{\partial\varphi}{\partial t} + \operatorname{div}(\vec{U}\varphi) + \sigma\varphi - \operatorname{div}(\mu\nabla\varphi) - f]\varphi^* dQ_T = 0, \tag{2.5}$$

where $\varphi, \varphi^* \in L_2(0,T;\widetilde{H}^1(G))$ and $\vec{U}$ satisfies the condition (1.5).

Theorem 2.1. *Under the conditions (1.5) and (2.4) the problem (1.1)-(1.4) has at most one solution in* $L^2(0,T;\widetilde{H}^1(G))$.

Proof. Let $f=0$ and $\varphi_0=0$. We shall show that this problem has only the trivial solution. Indeed, taking $\varphi^*=\varphi$ in (2.5) we have

$$J(\varphi,\varphi)=\int_0^T\Big[\frac{1}{2}\frac{d}{dt}||\varphi||_H^2+\phi(\varphi,\varphi)+\int_{S_2}U_n^+\varphi^2ds-(\varphi,\operatorname{div}(\varphi\vec{U}))_H\Big]dt=0. \tag{2.6}$$

Using the Green formula and condfitions (1.5), (2.1) we get

$$\int_0^T(\varphi,\operatorname{div}(\varphi\vec{U}))_H dt=\frac{1}{2}\int_0^T dt\int_{S_2}U_n^+\varphi^2ds. \tag{2.7}$$

In virtue of the condition $\varphi(x,0)=0$, from (2.7) and (2.8) it follows

$$\frac{1}{2}||\varphi(T)||_H^2+\int_0^T\phi(\varphi,\varphi)dt+\frac{1}{2}\int_0^T dt\int_{S_2}U_n^+\varphi^2ds=0.$$

Due to (1.4) and (2.3), from the last equality we have $\varphi(x,t)\equiv 0$ in Q_T. □

3. Existence Theorem

Assume that $\{\omega_j(x)\}(j=1,2,...)$ is an orthonormal basic in the space $\widetilde{H}^1(G)$. We shall find an approximative solution of the equation (2.5) in the form

$$\varphi_m(x,t)=\sum_{j=1}^m p_{jm}(t)\omega_j(x), \tag{3.1}$$

where $p_{jm}(t)$ are determined from conditions

$$J(\varphi_m,a(t)\omega_j)=0, \tag{3.2}$$

$$p_{jm}(0)=(\varphi_0,\omega_j), \tag{3.3}$$

$a(t)$ is an arbitrary function in $C[0,T]$.

Now substituting (3.1) into (3.2), after some transformations we get

$$\int_0^T\{\sum_{i=1}^m\frac{dp_{jm}(t)}{dt}(\omega_i,\omega_j)_H+\sum_{i=1}^m p_{im}(t)[\phi(\omega_i,\omega_j)$$
$$+\int_{S_2}U_n^+\omega_i\omega_j ds-(\omega_i,\operatorname{div}(\omega_j\vec{U}))]\}a(t)dt=\int_0^T(f,\omega_j)_H a(t)dt.$$
$$j=1,2,...,m.$$

The last equality is true for any arbitrary function $a(t) \in C[0,T]$. From this it follows

$$\sum_{i=1}^{m} \frac{dp_{im(t)}}{dt}(\omega_i, \omega_j)_H + \sum_{i=1}^{m} p_{im}(t)[\phi(\omega_i, \omega_j) + \int_{S_2} U_n^+ \omega_i \omega_j ds - (\omega_i, \operatorname{div}(\omega_j \vec{U}))] = (f, \omega_j)_H,$$
$$j = 1, 2, ..., m.$$

Sinse the system functions $\{\omega_j\}$ is linear independent, then for any $m \geqslant 1$ the determinant of the matrix with elements $(\omega_i, \omega_j)_H; i, j = 1, 2, ..., m$ is not equal to zero (see [3], p.299). Therefore, the system (3.4) may be reduced to the form

$$P'_m(t) = AP_m(t) + F_m(t), P_m(0) = \phi_{0m}, \tag{3.4}$$

The system of differential equations (3.5) has a unique solution belonging to $H^1(0,T)$ (see [3]).

Now multiplying by $p_{jm}(t)$ both parts (3.4), summing the obtained equalities over j from 1 to m and taking into account of (2.8), (3.1) we get

$$\frac{1}{2}\frac{d}{dt}||\varphi_m(t)||_H^2 + \phi(\varphi_m, \varphi_m) + \frac{1}{2}\int_{S_2} U_n^+ \varphi_n^2 ds = (f, \varphi_m)H. \tag{3.5}$$

Integrating (3.5) with respect to t from 0 to $\xi \in (0,T)$ we have

$$\frac{1}{2}||\varphi_m(\xi)||_H^2 - \frac{1}{2}||\varphi_m(0)||_H^2 + \int_0^\xi \phi(\varphi_m, \varphi_m) dt + \frac{1}{2}\int_0^\xi dt \int_{S_2} U_n^+ \varphi_m^2 ds = \int_0^\xi (f, \varphi_m)_H dt.$$

Obviously, from the last equality we get

$$\int_0^\xi \phi(\varphi_m, \varphi_m) dt + \frac{1}{2}\int_0^\xi dt \int_{S_2} U_n^+ \varphi_m^2 ds \leqslant \frac{1}{2}||\varphi_m(0)||_H^2 + \int_0^\xi (f, \varphi_m)_H dt$$
$$\leqslant \frac{1}{2}||\varphi_0||_H^2 + \frac{\varepsilon}{2}\int_0^\xi ||\varphi_m||_H^2 dt + \frac{1}{2\varepsilon}\int_0^\xi ||f||_H^2 dt, \tag{3.6}$$

where ε is a positive constant. Choosing ε sufficienly small and taking account of the equivalence of $||\varphi_m||_{H^1(G)}^2$ to the following

$$\phi(\varphi_m, \varphi_m) + \frac{1}{2}\int_{S_2} U_n^+ \varphi_m^2 ds$$

(see [3], p. 149) from (3.7) we obtain the estimate

$$||\varphi_m||_{L^2(0,T;\widetilde{H}^1(G))} \leqslant C, \tag{3.7}$$

where C is positive independent on m constant.

In view of the estimate (3.8) we can conclude that $\varphi_m \to \varphi (m \to \infty)$ weakly in $L^2(0,T;\widetilde{H}^1(G))$. It is not difficult to show that the functional

J defined by the formula (2.5) is continuous on $L^2(0,T;\widetilde{H}^1(G))$. Therefre, letting in (3.2) $m \to \infty$ we obtain

$$J(\varphi, a\omega) = 0; \quad j = 1,2,3,... \tag{3.8}$$

Sine the set of all linear combinations of functions $a(t)\omega_j(x)$, where $j = 1,2,...$ and $a(t)$ is arbitrary function from $C[0,T$ is dense in $L^2(0,T;\widetilde{H}^1(G))$ (see [3], p.301) then from (3.9) it follows that the equality (2.5) is true for an arbitrary $\varphi^* \in L^2(0,T;\widetilde{H}^1(G))$. By the same way as in [1,3] we can show that the condition (1.3) is fulfilled.

Thus, one has the following

Theorem 3.1. *Under the above mentioned assumptions $\vec{U}, \sigma, \mu$ and U_n^+, for every f and φ_0 satisfying the condition (2.4) the problem (1.1)-(1.4) has an unique solution $\varphi \in L_2(0,T;\widetilde{H}^1(G))$.*

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Nguyen Van Ngoc
Hanoi Institute of Mathematics
18 Hoang Quoc Viet Road
P.O. Box 10307, BoHo, Hanoi, Vietnam
E-mail address: nvngoc@math.ac.vn