SOME DEFINITIONS FOR CONVOLUTIONS AND THE CONVOLUTIONS FOR THE FOURIER TRANSFORMS WITH GEOMETRIC VARIABLES

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ABSTRACT. This paper gives some general definitions of convolutions with or without weight-element for the linear operators, and constructs some convolutions with and without weight-function for the Fourier transform with geometric variables. A new generalized convolution with the weight-function for the Fourier-cosine, Fourier-sine transforms is also constructed.

1. INTRODUCTION

The theory of the convolutions of integral transforms has been studied for a long time ago, and it has many applications (see Bochner [1], Fox [5], Hömander [7], Tichmarsh [11] and references whereas). One knows that there are several relations, explicit or implicit, between the integral transforms of Cauchy, Fourier, Hankel, Laplace, Melin (see [11]). In recent years, many papers devoted to those transforms are given the convolutions, generalized convolutions, polyconvolutions and theirs applications (see Britvina [2], [3], Tuan [12] and references therein). On the other hand, a constructed convolution can be regarded as a new integral transform. In our view, the integral transforms of Fourier type, in addition, deserve the interest.

In this paper, we present some general definitions for convolutions with, and without weight-element for the linear operators from the linear space to the commutative algebra, and to give some available convolutions for the Fourier transform with the geometric variables: shift, similitude and inverter. A usual, there exists some different convolutions for the certain transform, and conversely, the transform can be the convolution for some different transforms. A new convolution with weight-function, a generalized convolution are constructed in Subsection 3.4, and the Fourier transform with linear-fractional shift is posed at the end of Section 4.

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2. The general definitions for convolutions

Let U be the linear space and let V be the commutative algebra on the field $\mathcal{K}$. Let $T \in L(U, V)$ be the linear operator from U to V.

Definition 2.1. A bilinear map $*$ $U \times U :\longrightarrow U$ is called the convolution for T, if $T(*(f,g)) = T(f)T(g)$ for any $f, g \in U$. The image $*(f,g)$ is denoted by $f \underset{T}{*} g$.

Let γ be the element in algebra V.

Definition 2.2. A bilinear map $*$ $U \times U :\longrightarrow U$ is called the convolution with the weight-element γ for T, if $T(*(f,g)) = \gamma T(f)T(g)$ for any $f, g \in U$. The image $*(f,g)$ is denoted by $f \overset{\gamma}{\underset{T}{*}} g$.

Each of the identities in Definitions 2.1, 2.2 is called the factorization identity (see Britvina [2]). In [2], [8], the authors have dealt with the generalized convolution for two integral transforms and constructed some convolutions for the well-known integral transforms. Let U_1, U_2, U_3 be the linear spaces on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are the linear operators from U_1, U_2, U_3 to V respectively.

Definition 2.3. A bilinear map $*$ $U_1 \times U_2 :\longrightarrow U_3$ is called the convolution with the weight-element γ for the operators K_3, K_1, K_2, if $K_3(*(f,g)) = \gamma K_1(f)K_2(g)$ for any $f \in U_1, g \in U_2$. The image $*(f,g)$ is denoted by $f \overset{\gamma}{\underset{K_3,K_1,K_2}{*}} g$. If γ is the unit of V, we say briefly the convolution for K_3, K_1, K_2.

Remark 2.4. From Definition 2.3 it follows that if the operator K_3 is injective, the convolution $f \overset{\gamma}{\underset{K_3,K_1,K_2}{*}} g$ is formal dertermined uniquely, because

$f \overset{\gamma}{\underset{K_3,K_1,K_2}{*}} g = K_3^{-1}(\gamma K_1(f)K_2(g))$ for any $f \in U_1, g \in U_2$.

In next sections, we only consider $U = U_1 = U_2 = U_3 = L_1(\mathbb{R}^n)$ with the integral by Lebesgue's mean, V the algebra of all functions (real or complex) defined on $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$, let $<x, y>$ denote the scalar product, and $|x|^2 = <x, x>$.

3. Some convolutions for the transforms of Fourier type

This section gives some convolutions for the transforms of Fourier type. Namely, two convolutions for each of the Fourier transforms with geometric variabels are given, and a convolution with weight-function, a generalized convolution for the Fourier-cosine and Fourier-sine transforms on entire $\mathbb{R}^n$ are obtained.

3.1. **Convolutions for the Fourier transform with shift.** Let $h \in \mathbb{R}^n$ be fixed. Denote by F the Fourier transform. The Fourier transform with shift, denoted by F_h, is defined by

$$(F_h f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i<x+h,y>} f(y)dy.$$

If $h = 0 \in \mathbb{R}^n$, we admit $F_h = F$. Using the factorization identity of the Fourier convolution, it is easy to prove (see [11, p. 59], or [7, p. 163])

Theorem 3.1. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \underset{F_h}{*} g)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x-y)g(y)dy \tag{3.1}$$

defines the convolution for F_h, the factorization identity as follows

$$F_h(f \underset{F_h}{*} g)(x) = (F_h f)(x)(F_h g)(x).$$

Theorem 3.2. *Put $\gamma_1(x) = e^{-\frac{1}{2}|x|^2}$. If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \overset{\gamma_1}{\underset{F_h}{*}} g)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|x-u-v|^2 + i<h,x-u-v>} dudv \tag{3.2}$$

defines the convolution with the weight-function γ_1 for F_h.

Proof. We have

$$\int_{\mathbb{R}^n} \left|(f \overset{\gamma_1}{\underset{F_h}{*}} g)(x)\right| dx \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(u)||g(v)||e^{-\frac{1}{2}|x-u-v|^2 + i<h,x-u-v>}| dudvdx$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(u)||g(v)|e^{-\frac{1}{2}|x-u-v|^2} dxdvdu < +\infty.$$

We prove the factorization identity. The following formula holds (see [11, p. 81], or Rudin [10, Lemma 7.6])

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i<x,y-u-v> - \frac{1}{2}|y-u-v|^2} dy = e^{-\frac{|x|^2}{2}}. \tag{*}$$

We then have

$$\gamma_1(x)(F_h f)(x)(F_h g)(x) =$$
$$e^{-\frac{|x|^2}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(u)e^{-i<x+h,u>} du \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(v)e^{-i<x+h,v>} dv$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)e^{-i<x+h,u>} e^{-i<x+h,v>} e^{-\frac{x^2}{2}} dudv$$
$$= \frac{1}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v) \int_{\mathbb{R}^n} e^{-i<x,y-u-v> - \frac{1}{2}|y-u-v|^2} dy e^{-i<x+h,u+v>} dudv$$

$$= \frac{1}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|y-u-v|^2 - i<x,y> + i<h,-u-v>} dudvdy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i<x+h,y>} \Big[\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|y-u-v|^2 + i<h,y-u-v>} dudv \Big] dy$$

$$= F_h(f \overset{\gamma_1}{\underset{F_h}{*}} g)(x).$$

The proof is complete. □

3.2. **The convolutions for the Fourier transform with similitude.** Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ ($\alpha_i \neq 0\ \forall i = 1, \dots, n$) be fixed. For any $x \in \mathbb{R}^n$, we write $\alpha \cdot x = (\alpha_1 x_1, \dots, \alpha_n x_n)$. The Fourier transform with similitude, denoted by F_α, is defined by

$$(F_\alpha f)(x) = \frac{|\alpha|}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i<\alpha \cdot x, y>} f(y) dy.$$

Similarly to Theorem 3.1, we can prove

Theorem 3.3. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \underset{F_\alpha}{*} g)(x) = \frac{|\alpha|}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x-y)g(y)dy \tag{3.3}$$

defines the convolution for F_α, and the factorization identity is

$$F_\alpha(f \underset{F_\alpha}{*} g)(x) = (F_\alpha f)(x)(F_\alpha g)(x).$$

Theorem 3.4. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \overset{\gamma_1}{\underset{F_\alpha}{*}} g)(x) = \frac{(\prod_{j=1}^{n} \alpha_j)|\alpha|}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{|\alpha \cdot (x-u-v)|^2}{2}} dudv \tag{3.4}$$

defines the convolution with the weight-function γ_1 for F_α.

Proof. We have

$$\int_{\mathbb{R}^n} \big|(f \overset{\gamma_1}{\underset{F_\alpha}{*}} g)(x)\big| dx = \frac{|\prod_{j=1}^{n} \alpha_j||\alpha|}{(2\pi)^n} \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u)g(v)e^{-|\alpha \cdot (x-u-v)|^2} dudv \Big| dx$$

$$\leq \frac{|\prod_{j=1}^{n} \alpha_j||\alpha|}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(u)||g(v)|e^{-|\alpha \cdot (x-u-v)|^2} dudvdx < +\infty.$$

We now prove the factorization identity. From the formula $(*)$ it follows

$$(**)\qquad \frac{(\prod_{j=1}^{n}\alpha_j)}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i<x,\alpha\cdot(y-u-v)>-\frac{1}{2}|\alpha\cdot(y-u-v)|^2}dy = e^{-\frac{|x|^2}{2}}.$$

Then

$$\gamma_1(x)(F_\alpha f)(x)(F_\alpha g)(x) = \frac{e^{-\frac{|x|^2}{2}}|\alpha|^2}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)e^{-i<\alpha\cdot x,u+v>}dudv$$

$$= \frac{|\alpha|^2}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{|x|^2}{2}}e^{-i<\alpha\cdot x,u>}e^{-i<\alpha\cdot x,v>}dudv$$

$$= \frac{|\alpha|^2\prod_{j=1}^{n}\alpha_j}{(2\pi)^{\frac{3n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)\Big[\int_{\mathbb{R}^n} e^{-i<x,\alpha\cdot(y-u-v)>-\frac{1}{2}|\alpha\cdot(y-u-v)|^2}dy\Big]\times$$

$$e^{-i<\alpha\cdot x,u>-i<\alpha\cdot x,v>}dudv = \frac{|\alpha|^2(\prod_{j=1}^{n}\alpha_j)}{(2\pi)^{\frac{3n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|\alpha\cdot(y-u-v)|^2}\times$$

$$e^{-i<\alpha\cdot x,y>}dudvdy = \frac{|\alpha|}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i<\alpha\cdot x,y>}\Big[\frac{|\alpha|(\prod_{j=1}^{n}\alpha_j)}{(2\pi)^n}\times$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|\alpha\cdot(y-u-v)|^2}dudv\Big]dy = F_\alpha(f \overset{\gamma_1}{\underset{F_\alpha}{*}} g)(x).$$

The theorem is proved. □

Comment. We do not rest satisfied with the assumption $\alpha_1\dots\alpha_n \neq 0$ at the begining of the subsection. So, the construction convolutions with weight-function for F_α, in special cases of $\alpha_1\dots\alpha_n = 0$, is the open problem.

3.3. **The convolution for the Fourier transform with inverter.** For any $x\in\mathbb{R}^n$, $(x_i \neq 0, \forall i = 1,\dots,n)$, let us write $\frac{1}{x} = (\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n})$. The Fourier transform with the inverter, denoted by F_v, is defined by the following

$$(F_v f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i<y,\frac{1}{x}>}f(y)dy \text{ if } x_i\neq 0\ \forall i=1,\dots,n, \text{ zero if } x_i = 0.$$

Without difficulty we can prove

Theorem 3.5. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \underset{F_v}{*} g)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} f(x-y)g(y)dy \tag{3.5}$$

defines the convolution for the Fourier transform with inverter, the factorization identity as follows

$$F_v(f \underset{F_v}{*} g)(x) = (F_v f)(x)(F_v g)(x).$$

Consider the function:

$$\gamma_2(x) = e^{-\frac{1}{2}|\frac{1}{x}|^2} \text{ if } x_i \neq 0 \; \forall i = 1, \dots, n, \text{ zero if } \; x_i = 0.$$

Theorem 3.6. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \overset{\gamma_2}{\underset{F_v}{*}} g)(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|x-u-v|^2} dudv \tag{3.6}$$

defines the convolution with the weight-function γ_2 for F_v, and the factorization identity is

$$F_v(f \overset{\gamma_2}{\underset{F_v}{*}} g)(x) = \gamma_2(x)(F_v f)(x)(F_v g)(x). \tag{3.7}$$

Proof. Obviously, if at least one of the x_i is zero ($\exists i = 1, 2, \dots, n$ such that $x_i = 0$), (3.7) holds. Consider $x_i \neq 0, \;\; \forall i = 1, 2, \dots, n$. We have

$$\int\limits_{\mathbb{R}^n} \Big|(f \overset{\gamma_2}{\underset{F_v}{*}} g)(x)\Big| dx \leq \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |f(u)||g(v)|e^{-\frac{1}{2}|x-u-v|^2} dudvdx < +\infty.$$

The formula $(*)$ gives

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{-i<y-u-v,\frac{1}{x}>-\frac{1}{2}|y-u-v|^2} dy = e^{-\frac{1}{2}|\frac{1}{x}|^2}. \tag{$***$}$$

Then

$$\gamma_2(x)(F_v f)(x)(F_v g)(x) = \frac{e^{-\frac{1}{2}|\frac{1}{x}|^2}}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} f(u)e^{-i<u,\frac{1}{x}>} du \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} g(v)e^{-i<v,\frac{1}{x}>} dv$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|\frac{1}{x}|^2} e^{-i<u,\frac{1}{x}>} e^{-i<v,\frac{1}{x}>} dudv$$

$$= \frac{1}{(2\pi)^{\frac{3n}{2}}} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)\Big[\int\limits_{\mathbb{R}^n} e^{-i<y-u-v,\frac{1}{x}>-\frac{1}{2}|y-u-v|^2} dy\Big] e^{-i<u,\frac{1}{x}>-i<v,\frac{1}{x}>} dudv$$

$$= \frac{1}{(2\pi)^{\frac{3n}{2}}} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|y-u-v|^2-i<y,\frac{1}{x}>} dudvdy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{-i<y,\frac{1}{x}>} \Big[\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{1}{2}|y-u-v|^2} dudv\Big] dy = F_v(f \overset{\gamma_2}{\underset{F_v}{*}} g)(x).$$

The theorem is completely proved. $\square$

Remark 3.7. The transform defined by (3.1) can be regarded as the convolution for the transforms: F, F_h, F_v, and for F_α if $|\alpha| = 1$.

3.4. **Two convolutions for the Fourier-cosine and Fourier-sine transforms on $\mathbb{R}^n$.** This subsection offers a convolution with weight-function, and a new generalized convolution for the Fourier-cosine and Fourier-sine transforms on entire $\mathbb{R}^n$, just for the realization of our Definition 2.3. Another generalized convolutions and polyconvolutions will be addressed in another papers.

For any $x, y, z \in \mathbb{R}^n$, write $\cos xy, \sin xy$ instead of $\cos<x, y>$, $\sin<x, y>$, and $\cos x(y \pm z), \sin x(y \pm z)$ instead of $\cos < x, y \pm z >, \sin < x, y \pm z >$ respectively.

It is well-known that the transforms

$$(T_c f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \cos xy f(y) dy,$$

$$(T_s f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \sin xy f(y) dy,$$

are the Fourier-cosine, Fourier-sine transforms respectively on entire $\mathbb{R}^n$.

Theorem 3.8. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(3.8) \quad (f \overset{\gamma_1}{\underset{T_c}{*}} g)(x) = \frac{1}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u) g(v) \big[e^{-\frac{|x+u+v|^2}{2}} + e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}} \big] dudv.$$

defines the convolution with the weight-function γ_1 for the integral transform T_c. The factorization identity is

$$T_c(f \overset{\gamma_1}{\underset{T_c}{*}} g)(x) = \gamma_1(x)(T_c f)(x)(T_c g)(x).$$

Proof. Choosing $x = 0$ in the formula $(*)$, we also get the identities

$$(3.9) \qquad \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}|y \pm u \pm v|^2} dy = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}|y|^2} dy = 1,$$

(or see [10, Lemma 7.6]). Then

$$\int\limits_{\mathbb{R}^n} |(f \overset{\gamma_1}{\underset{T_c}{*}} g)|(x) dx \le \frac{1}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} |f(u)||g(v)| e^{-\frac{|x+u+v|^2}{2}} dudvdx + \frac{1}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} |f(u)||g(v)| e^{-\frac{|x+u-v|^2}{2}} dudvdx$$

$$+\frac{1}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(u)||g(v)|e^{-\frac{|x-u+v|^2}{2}}dudvdx$$
$$+\frac{1}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(u)||g(v)|e^{-\frac{|x-u-v|^2}{2}}dudvdx$$
$$=\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(u)||g(v)|dudv<+\infty,$$

Moreover

$$\gamma_1(x)(T_cf)(x)(T_cg)(x)=\frac{\gamma_1(x)}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos xu\cos xvdudv$$
$$=\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos x(u+v)dudv$$
$$+\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos x(u-v)dudv$$
$$+\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos x(u-v)dudv$$
$$+\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos x(u+v)dudv.$$

By the formula $(*)$, we receive

$$\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos x(u+v)dudv$$
$$=\frac{\gamma_1(x)}{8(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)(e^{i<x,u+v>}+e^{-i<x,u+v>})dudv$$
$$=\frac{1}{8(2\pi)^{3n/2}}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\int\limits_{\mathbb{R}^n}e^{-i<y,x>}e^{-\frac{|y+u+v|^2}{2}}dydudv$$
$$+\frac{1}{8(2\pi)^{3n/2}}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\int\limits_{\mathbb{R}^n}e^{i<y,x>}e^{-\frac{|y+u+v|^2}{2}}dydudv$$
$$=\frac{1}{4(2\pi)^{3n/2}}\int\limits_{\mathbb{R}^n}\cos xy\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)e^{-\frac{|y+u+v|^2}{2}}dydudv. \tag{3.10}$$

Similarly,

$$\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(u)g(v)\cos x(u-v)dudv$$

$$= \frac{1}{4(2\pi)^{3n/2}} \int\limits_{\mathbb{R}^n} \cos xy \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{|y+u-v|^2}{2}} dydudv, \tag{3.11}$$

$$\frac{\gamma_1(x)}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v) \cos x(u-v) dudv$$

$$= \frac{1}{4(2\pi)^{3n/2}} \int\limits_{\mathbb{R}^n} \cos xy \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{|y-u+v|^2}{2}} dydudv, \tag{3.12}$$

$$\frac{\gamma_1(x)}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v) \cos x(u+v) dudv$$

$$= \frac{1}{4(2\pi)^{3n/2}} \int\limits_{\mathbb{R}^n} \cos xy \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v)e^{-\frac{|y-u-v|^2}{2}} dydudv. \tag{3.13}$$

Thus

$$\gamma_1(x)(T_c f)(x)(T_c g)(x) = \frac{1}{4(2\pi)^{3n/2}} \int\limits_{\mathbb{R}^n} \cos xy \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v)\big[e^{-\frac{|y+u+v|^2}{2}}$$
$$+ e^{-\frac{|y+u-v|^2}{2}} + e^{-\frac{|y-u+v|^2}{2}} + e^{-\frac{|y-u-v|^2}{2}}\big]dudvdy = T_c(f \overset{\gamma_1}{\underset{T_c}{*}} g)(x).$$

□

Theorem 3.9. *If for any $f, g \in L_1(\mathbb{R}^n)$, then*

$$(f \overset{\gamma_1}{\underset{T_c,T_s,T_s}{*}} g)(x) = \frac{1}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v)\big[-e^{-\frac{|x+u+v|^2}{2}} + e^{-\frac{|x+u-v|^2}{2}}$$
$$+ e^{-\frac{|x-u+v|^2}{2}} - e^{-\frac{|x-u-v|^2}{2}}\big] dudv \tag{3.14}$$

defines the convolution with the weight-function γ_1 for the transforms T_c, T_s, T_s. The factorization identity is

$$T_c(f \overset{\gamma_1}{\underset{T_c,T_s,T_s}{*}} g)(x) = \gamma_1(x)(T_s f)(x)(T_s g)(x).$$

Proof. It is similar the proof of Theorem 3.8, we can prove $f \overset{\gamma_1}{\underset{T_c,T_s,T_s}{*}} g \in L_1(\mathbb{R}^n)$ for any $f, g \in L_1(\mathbb{R}^n)$. Suffice it to prove the factorization identity. Using the formula (3.9) and the identities (3.10), (3.11), (3.12), (3.13) we get

$$\gamma_1(x)(T_s f)(x)(T_s g)(x) = \frac{\gamma_1(x)}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v) \sin xu \sin xv dudv$$
$$= \frac{-\gamma_1(x)}{4(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} f(u)g(v) \cos x(u+v) dudv$$

$$+\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)\cos x(u-v)dudv$$
$$+\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)\cos x(u-v)dudv$$
$$-\frac{\gamma_1(x)}{4(2\pi)^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)\cos x(u+v)dudv$$
$$=\frac{1}{4(2\pi)^{3n/2}}\int\limits_{\mathbb{R}^n}\cos xy\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(u)g(v)\big[-e^{-\frac{|y+u+v|^2}{2}}+e^{-\frac{|y+u-v|^2}{2}}$$
$$+e^{-\frac{|y-u+v|^2}{2}}-e^{-\frac{|y-u-v|^2}{2}}\big]dudvdy=T_c(f\underset{T_c,T_s,T_s}{\overset{\gamma_1}{*}}g)(x).$$

The proof is complete. ☐

3.5. Some normed rings on $L_1(\mathbb{R}^n)$.

Definition 3.10. (see Naimark [9]) A vector space V with a ring structure and a vector norm is called the normed ring if $\|vw\| \le \|v\|\|w\|$, for all $v, w \in V$.
If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Let X denote the linear space $L_1(\mathbb{R}^n)$.
Now we define norms for $f \in X$. For the convolutions defined by (3.1), (3.2), (3.3), (3.5), (3.6), (3.8) for F_h, F_α, F_v, T_c the norm is choosed normally as follows

$$\|f\| = \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}|f(x)|dx.$$

For the convolution defined by (3.4), the norm is

$$\|f\| = \frac{|\alpha|}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}|f(x)|dx.$$

Conclusion 3.11. X, *equipped with each of seven above mentioned convolution multiplications, becomes the commutative normed ring having no unit.*

We prove the conclusion. For briefness of our proof (including Conclusion 3.12 below), let us use the common symbols $\mathcal{H}$, and $*$ for the transforms F_h, F_α, F_v, T_c and for the above convolutions respectively. It is clearly X has a ring structure commutative with the convolution multiplication. First, we prove the multiplicative inequality. We now prove for the convolution (3.2), the proof for the others is similar, and easier. By the formula (3.9), we get

$$\int_{\mathbb{R}^n} |f*g|(x)dx \leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(u)||g(v)||e^{-\frac{|x-u-v|^2}{2}+i<h,x-u-v>}|dudvdx$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(u)||g(v)|\int_{\mathbb{R}^n} |e^{-\frac{|x-u-v|^2}{2}+i<h,x-u-v>}|dxdudv$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(u)||g(v)|dudv\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-u-v|^2}{2}}dx$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(u)||g(v)|dudv.$$

Hence,

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |f*g|(x)dx \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |f(u)|du.\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |g(v)|dv.$$

Thus

$$\|f*g\| \leq \|f\|\|g\|.$$

Now it suffices to prove X has no unit element. Suppose that there exists an $e \in X$ such that $f*e = e*f = f$, for any $f \in X$. The factorization identities imply $\gamma_0 \mathcal{H}f\mathcal{H}e = \mathcal{H}f$, where $\gamma_0 = 1$ corresponding to the convolutions (3.1), (3.3), (3.5), and $\gamma_0 = \gamma_1, \gamma_1, \gamma_2, \gamma_1$ if the convolution is of (3.2), (3.4), (3.6), (3.8) respectively. We then have $\mathcal{H}f(\gamma_0\mathcal{H}e - 1) = 0$. Choosing $f = \gamma_0$ and using the formulae $(*)$, $(**)$, $(***)$ in the respective case, we conclude $\gamma_0(x)(\mathcal{H}e)(x) = 1$ for almost every $x \in \mathbb{R}^n$. On the other side,

$$\lim_{x\to\infty}\gamma_1(x) = 0, \qquad \lim_{x_1,\dots,x_n\to\infty}\gamma_2(x) = 1, \quad \lim_{x\to\infty}(\mathcal{H}e)(x) = 0$$

(see [11, Theorem 1], or [10, Theorem 7.5]), which contradict to the last identity.

Conclusion 3.12. X, *equipped with the convolution multiplication* (3.14), *becomes the commutative normed ring having divisor of zero and no unit.*

We prove the conclusion. For $f \in X$, the norm is

$$\|f\| = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |f(x)|dx.$$

By the convolution multiplication, X has a commutative ring structure. It is clearly that if $f(x) = f(-x)$, and $g(x) = -g(-x)$ for all $x \in \mathbb{R}^n$, then $f*g = 0$.

Now we prove X has no unit. Suppose that there exists an $e \in X$ such that $f*e = e*f = f$, for any $f \in X$. The factorization identity implies $\gamma_1(T_sf)(T_se) = T_cf$. We choose $f_0(x) = e^{-\frac{|x|^2}{2}} \in X$. Obviously, $T_sf_0 = 0$, i.e. the left-side of the last identity is zero-function. On the other hand, from formula $(*)$ it follows $f_0 = Ff_0 = T_c f_0 + iT_s f_0 = T_c f_0$. It follows that $T_c f$ is exactly the nonzero-function f_0, which contradicts to the last identity.

4. The Fourier transform with linear-fractional shift on $\mathbb{R}$

In this subsection, we discuss a transform of Fourier type, namely, it is the Fourier transform with linear-fractional shift on $\mathbb{R}$. In the sequel, the functions dealt with here are defined on $\mathbb{R}$.
It is well-known that the linear-fractional function of the form

$$\omega(x) = \frac{ax+b}{cx+d}$$

caries out an one-to-one mapping of the extend real axis $\overline{\mathbb{R}}$ into itself. Moreover, the set of all linear-fractional functions is a group. In the next, this group will be denoted by $\mathcal{V}$. Group $\mathcal{V}$, in general, is infinite.

Two linear-fractional functions

$$L_1(x) = \frac{a_1x+b_1}{c_1x+d_1}, \qquad \text{and} \qquad L_2(x) = \frac{a_2x+b_2}{c_2x+d_2}$$

will be considered identical in group $\mathcal{V}$ if and only if $L_1(x) = L_2(x)$ for all values of x in $\overline{\mathbb{R}}$. For this to be so it is necessry and sufficient that the corresponding coefficients be proportional to each other, i.e. $a_2 = \lambda a_1,\ \ b_2 = \lambda b_1,\ \ c_2 = \lambda c_1,\ \ d_2 = \lambda d_1,\ \ \lambda \neq 0$. Since, one usually assumes $ad - bc = 1$. Denote by I the unit element of group $\mathcal{V}$. For any $\omega \in \mathcal{V}$, let us write $\omega^k(x) = \omega(\omega^{k-1}(x))$, $k = 1, 2, \dots$, where $\omega^0 = I$.

Definition 4.1. A linear-fractional function $\omega \in \mathcal{V}$ is said to be involution of n-order, if $\omega^n \equiv I, \omega^m \not\equiv I, m = 1, 2, \dots, n-1$.

The following is a necessary and suffucient condition for a linear-fractional function to be involution of n-order

Theorem 4.2. ([4], or [6, p. 496]) *Let*

$$\omega(x) = \frac{ax+b}{cx+d}$$

be given and let $n \in \mathbb{N}$, $n \geq 2$ be fixed. Then ω is involution of n-order if and only if

$$\begin{cases} a + d = 2\cos\dfrac{k\pi}{n}, & \textit{for some } \ k \in \{1, \dots, n-1\},\ (n,k) = 1, \\ ad - bc = 1. \end{cases}$$

By the theorem, for any $n \in \mathbb{N}$ fixed, there exists an infinite set of linear-fractional functions which are involution of n-order. Then, each of the n-order involution functions is the generator of a n-terms cyclic group.

Example 4.3. Consider $n = 2, 3, 4, 5, 6$. It easy to check that a n-order involution linear-fractional function is of the following form

- $n = 2$:
$$-x+a,\ \frac{a}{x},\ \text{or}\ \frac{ax+b}{cx-a}.$$
- $n = 3$:
$$\frac{abx+b^2}{(-a^2+a-1)x+b(1-a)},\ \text{or}\ \frac{abx+b^2}{(-a^2-a-1)x-b(1+a)}.$$
- $n = 4$:
$$\frac{abx+b^2}{(-a^2+\sqrt{2}a-1)x+b(\sqrt{2}-a)},$$
or
$$\frac{abx+b^2}{(-a^2-\sqrt{2}a-1)x-b(\sqrt{2}+a)}.$$
- $n = 5$:
$$\frac{2abx+2b^2}{(-2a^2+(\sqrt{5}+1)a-2)x+b(\sqrt{5}+1-2a)},$$
$$\frac{2abx+2b^2}{(-2a^2+(\sqrt{5}-1)a-2)x+b(\sqrt{5}-1-2a)},$$
$$\frac{2abx+2b^2}{(-2a^2+(1-\sqrt{5})a-2)x+b(1-\sqrt{5}-2a)},$$
or
$$\frac{2abx+2b^2}{(-2a^2-(\sqrt{5}+1)a-2)x-b(1+\sqrt{5}+2a)}.$$
- $n = 6$:
$$\frac{abx+b^2}{(-a^2+\sqrt{3}a-1)x+b(\sqrt{3}-a)},$$
or
$$\frac{abx+b^2}{(-a^2-\sqrt{3}a-1)x-b(\sqrt{3}+a)}.$$

Let $\omega(x) = \dfrac{ax+b}{cx+d}$ be given. Consider the transform
$$(Wf)(x) = \begin{cases} f(\omega(x)), \text{ if } x \neq \frac{-d}{c}, \\ 0, \text{ if } x = \frac{-d}{c}. \end{cases}$$
Note that if $f \in L_1(\mathbb{R})$, in general, it is possible $Wf \notin L_1(\mathbb{R})$. Now the Fourier transform with linear-fractional shift is defined as follows
$$(\mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\frac{ax+b}{cx+d}t} f(t)dt \text{ if } x \neq \frac{-d}{c}, \text{zero if } x = \frac{-d}{c}. \tag{4.1}$$
Clearly, $(\mathcal{F}f)(x) := (WFf)(x)$.

Remark 4.4. (i) If $f \in L_1(\mathbb{R})$, then $(\mathcal{F}f)(x)$ is determined for every $x \in \mathbb{R}$.
(ii) The Fourier transform with linear-fractional shift (4.1) can be considered compositions of the transforms F_h, F_α, F_v.
(iii) The phase function of transform (4.1) is $\omega(x,t) = \frac{ax+b}{cx+d}t$ (see [7, p. 236]).
(iv) The asymptotic behaviour of the function $(\mathcal{F}f)(x)$ as follows:

If f is continuous and belongs to $L_1(\mathbb{R})$, then $\lim\limits_{x\to\infty}(\mathcal{F}f)(x) = (Ff)(\frac{a}{c})$, and $\lim\limits_{x\to -d/c}(\mathcal{F}f)(x) = 0$ (see [10, Theorem 7.5]).

We would like to discuss the transforms W, FW, and $\mathcal{F}$. First, we construct the functions $f \in L_1(\mathbb{R})$ such that $Wf \in L_1(\mathbb{R})$.

Example 4.5. Let $m > 0$ be fixed. Consider the linear-fractional function

$$\omega(x) = \frac{mx - m^2}{x+m}, \text{ and the function } f(x) = \begin{cases} e^{\frac{-m^2}{m^2-x^2}}, & \text{if } |x| < m, \\ 0, & \text{if } x \geq m. \end{cases}$$

It is easy to check $|\omega(x)| < m$ if and only if $x > 0$, and $f \in C_0^\infty(\mathbb{R})$ (see [13, p. 8]). Denote by $\mathcal{S}$ the space of rapidly decreasing functions (see [10]). We then have $f \in \mathcal{S}$. Moreover,

$$f(\omega(x)) = \begin{cases} e^{\frac{-(x+m)^2}{4mx}} & \text{if } \ x > 0, \\ 0, & \text{if } \ x \leq 0. \end{cases}$$

Hence, $Wf = f(\omega) \in \mathcal{S}$, and $Wf = f(\omega) \in L_1(\mathbb{R})$. By Theorem 7.7 in [10], we have $FWf \in \mathcal{S}$. On the other hand, again by Theorem 7.7 in [10], there exists a function $\varphi \in \mathcal{S}$ such that $f = F\varphi$. It implies $\mathcal{F}\varphi = WF\varphi = Wf \in \mathcal{S}$. Thus, there are the infinite sets of liner-fractional function $\omega \in \mathcal{V}$, and of functions $f, \varphi \in \mathcal{S}$ and such that the functions $\mathcal{F}\varphi, FWf \in \mathcal{S}$.

Comment. The investigation for the transform (4.1) remains open.

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