ON THE POLYCONVOLUTION FOR THE FOURIER, FOURIER SINE AND FOURIER COSINE TRANSFORMS

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ABSTRACT. A polyconvolution for the Fourier, Fourier sine and Fourier cosine integral transforms is introduced, its properties and applications to integral equations and systems of integral equations are considered.

1. INTRODUCTION

In 1941, R.V. Churchill introduced the convolution for the Fourier sine and Fourier cosine transforms (see [3])

$$\left(f \underset{1}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y) \left[g\left(|x-y|\right) - g(x+y)\right] dy, \ x > 0 \tag{1.1}$$

which satisfies the factorization equality

$$F_S\left(f \underset{1}{*} g\right)(y) = \left(F_S f\right)(y) \left(F_C g\right)(y), \ \forall y > 0. \tag{1.2}$$

Note that (1.2) contains two integral transforms: Fourier sine and Fourier cosine. This is quite different from previous convolutions such as Fourier convolution, Laplace convolution, Mellin convolution, Fourier cosine convolution (see [19]), Fourier sine convolution, Hilbert convolution and Hankel convolution (see [4]). In the factorization equalities of these convolutions only one integral transform is involved. For example, the convolution of the functions f and g for the Fourier integral transforms is (see [19])

$$\left(f \underset{F}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y) g(y) dy, \ x \in \mathbb{R} \tag{1.3}$$

which satisfies the following property

$$F\left(f \underset{F}{*} g\right)(y) = \left(F f\right)(y) \left(F g\right)(y), \ \forall y \in \mathbb{R}. \tag{1.4}$$

The convolution of two functions f and g for the Laplace integral transform has the form (see [19])

$$\left(f \underset{L}{*} g\right)(x) = \int_0^{x} f(x-y) g(y) dy, \ x > 0 \tag{1.5}$$

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and the factorization property holds

$$L\left(f \underset{L}{*} g\right)(y) = (Lf)(y)\,(Lg)(y), \ \forall y \in \mathbb{C}. \tag{1.6}$$

The convolution of two functions f and g for the Fourier cosine integral transforms is given by the integral (see [19])

$$\left(f \underset{F_C}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\left[g\left(|x-y|\right) + g\left(x+y\right)\right] dy, \ x > 0 \tag{1.7}$$

which satisfies

$$F_C\left(f \underset{F_C}{*} g\right)(y) = (F_C f)(y)\,(F_C g)(y), \ \forall y > 0. \tag{1.8}$$

The convolution with the weight-function $\gamma_1(x) = \sin x$ for the Fourier sine integral transforms is defined as follows (see [4, 10])

$$\left(f \overset{\gamma_1}{\underset{F_S}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)\big[\mathrm{sign}(x+y-1)g(|x+y-1|) - g(x+y+1) +$$

$$+\mathrm{sign}(x-y-1)g(|x-y-1|) - \mathrm{sign}(x-y+1)g(|x-y+1|)\big]dy, \ x > 0 \tag{1.9}$$

with

$$F_S\left(f \overset{\gamma_1}{\underset{F_S}{*}} g\right)(y) = \sin y\,(F_S f)(y)\,(F_S g)(y), \ \forall y > 0. \tag{1.10}$$

Afterwards, S.B. Yakubovich and co-authors published a series of papers devoted to the generalized convolutions of several index integral transforms, such as integral transforms of Mellin type (see [21]), integral transforms of Kontorovich-Lebedev type (see [22]) and the G transforms (see [18]).

In 1998, V.A. Kakichev and Nguyen Xuan Thao proposed a constructive method of defining the generalized convolution for any integral transforms K_1, K_2 and K_3 with the weight-function $\gamma(x)$ of functions f and g, for which we have the factorization property (see [6])

$$K_1\left(f \overset{\gamma}{*} g\right)(y) = \gamma(y)\,(K_2 f)(y)\,(K_3 g)(y).$$

Subsequently, there have been some papers published on the generalized convolution for the Stieltjes, Hilbert and Fourier cosine-sine transforms (see [8]), the H-transforms (see [7]), the I-transforms (see [16]), the Fourier, Fourier cosine and sine transforms (see [14]), the Fourier cosine and sine transforms (see [11]) and the Kontorovich-Lebedev, Fourier sine and cosine transforms (see [17])... For example, the generalized convolution for the Fourier cosine and Fourier sine has been defined (see [11]) by the formula

$$\left(f \underset{2}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\left[\mathrm{sign}\,(y-x)\,g\left(|x-y|\right) + g(x+y)\right]dy, \ x > 0 \tag{1.11}$$

which satisfies the factorization equality

$$F_C\left(f \underset{2}{*} g\right)(y) = (F_S f)(y)(F_S g)(y), \ \forall y > 0. \tag{1.12}$$

The convolution with the weight-function $\gamma_2(x) = \cos x$ for the Fourier cosine integral transform is defined as (see [13])

$$\left(f \overset{\gamma_2}{\underset{F_C}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)\big[g(|x+y-1|) + g(x+y+1) +$$

$$+ g(|x-y-1|) + g(|x-y+1|)\big]dy, \ x > 0 \tag{1.13}$$

which satisfies

$$F_C\left(f \overset{\gamma_2}{\underset{F_C}{*}} g\right)(y) = \cos y (F_C f)(y)(F_C g)(y), \ \forall y > 0. \tag{1.14}$$

The generalized convolution with the weigth-function $\gamma_1(x) = \sin x$ for the Fourier cosine and sine transforms has the form (see [12])

$$\left(f \overset{\gamma_1}{\underset{1}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)\big[g(|x+y-1|) - g(x+y+1) -$$

$$- g(|x-y-1|) + g(|x-y+1|)\big]dy, \ x > 0 \tag{1.15}$$

and the factorization property holds

$$F_C\left(f \overset{\gamma_1}{\underset{1}{*}} g\right)(y) = \sin y (F_S f)(y)(F_C g)(y), \ \forall y > 0. \tag{1.16}$$

The generalized convolution with a weight-function $\gamma_1(x) = \sin x$ for the Fourier sine and Fourier cosine transforms of the functions f and g is defined by (see [15])

$$\left(f \overset{\gamma_1}{\underset{2}{*}} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)\big[g(|x+y-1|) - g(x+y+1) +$$

$$+ g(|x-y-1|) - g(|x-y+1|)\big]dy, \ x > 0 \tag{1.17}$$

which satisfies the factorization property

$$F_S\left(f \overset{\gamma_1}{\underset{2}{*}} g\right)(y) = \sin y (F_C f)(y)(F_C g)(y), \ \forall y > 0. \tag{1.18}$$

In 1997, V.A. Kakichev proposed a constructive method of defining the polyconvolution for $n+1$ integral transforms K, K_1, K_2, ..., K_n with the weight-function $\gamma(x)$ of functions f_1, f_2,..., f_n for which we have the factorization property (see [5])

$$K\left[*\left(f_1, f_2, ..., f_n\right)\right](y) = \gamma(y)(K_1 f_1)(y)(K_2 f_2)(y)...(K_n f_n)(y).$$

The polyconvolution for the Hilbert, Stieltjes and Fourier cosine transforms was introduced by Nguyen Xuan Thao in 1999 (see [9]).
In this paper we define the polyconvolution of the Fourier, Fourier sine and Fourier cosine integral transforms, of which some properties are proved as well as some

relationships pointed out to several well-known convolutions and generalized convolutions. We also show that it does not exist aliquot of zero. Finally, we apply this notion for solving some integral equations and systems of integral equations.

2. Polyconvolution for the Fourier, Fourier sine and Fourier cosine transforms

Definition 2.1. *Polyconvolution for the Fourier, Fourier sine and Fourier cosine integral transforms with the weight-function $\gamma(x) = e^{-x}$ of the functions f, g and h is defined by*

$$\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty}\theta(x,u,v,w)f(u)g(v)h(w)dudvdw,\ x>0 \tag{2.1}$$

where

$$\theta(x,u,v,w) = \frac{1}{2\pi^2}\Big[\frac{iu+1}{(iu+1)^2+(x+v-w)^2} - \frac{iu+1}{(iu+1)^2+(x+v+w)^2} + \frac{iu+1}{(iu+1)^2+(x-v-w)^2} - \frac{iu+1}{(iu+1)^2+(x-v+w)^2}\Big]$$

Theorem 2.1. *Let f be a function in $L(\mathbb{R})$, g and h be the functions in $L(\mathbb{R}_+)$, then the polyconvolution* (2.1) *for the Fourier, Fourier sine and Fourier cosine transforms with the weight-function $\gamma(x) = e^{-x}$ of the functions f, g and h has a meaning and belongs to $L(\mathbb{R}_+)$ and the factorization property holds*

$$F_S\big[\overset{\gamma}{*}(f,g,h)\big](y) = e^{-y}(Ff)(y)(F_Cg)(y)(F_Sh)(y),\ \forall y>0 \tag{2.2}$$

Proof. We first prove that $\overset{\gamma}{*}(f,g,h)(x)$ has a meaning and belongs to $L(\mathbb{R}_+)$. We observe that $\forall c_1, c_2 \in \mathbb{C}$:

$$|c_1+c_2| \geqslant \frac{1}{2}\big(|c_1|+|c_2|\big)\Big|\frac{c_1}{|c_1|}+\frac{c_2}{|c_2|}\Big|$$

from which

$$\big|(iu+1)^2+(x+v-w)^2\big| \geqslant \frac{1}{2}\big[|(iu+1)^2|+(x+v-w)^2\big]\Big|\frac{(iu+1)^2}{|(iu+1)^2|}+1\Big|$$

$$\geqslant \frac{1}{2}\big[1+u^2+(x+v-w)^2\big]$$

thus

$$\Big|\frac{iu+1}{(iu+1)^2+(x+v-w)^2}\Big| \leqslant \frac{2\sqrt{1+u^2}}{1+u^2+(x+v-w)^2} \leqslant 2$$

similarly

$$\Big|\frac{iu+1}{(iu+1)^2+(x-v+w)^2}\Big| \leqslant \frac{2\sqrt{1+u^2}}{1+u^2+(x-v+w)^2} \leqslant 2,$$

$$\Big|\frac{iu+1}{(iu+1)^2+(x+v+w)^2}\Big| \leqslant \frac{2\sqrt{1+u^2}}{1+u^2+(x+v+w)^2} \leqslant 2$$

and

$$\left|\frac{iu+1}{(iu+1)^2+(x-v-w)^2}\right| \leqslant \frac{2\sqrt{1+u^2}}{1+u^2+(x-v-w)^2} \leqslant 2$$

we obtain

$$\left|\overset{\gamma}{*}(f,g,h)(x)\right| \leqslant \frac{4}{\pi^2}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}|f(u)g(v)h(w)|dudvdw$$

$$= \frac{4}{\pi^2}\int\limits_{-\infty}^{+\infty}|f(u)|du\int\limits_{0}^{+\infty}|g(v)|dv\int\limits_{0}^{+\infty}|h(w)|dw \leqslant +\infty$$

On the other hand,

$$\int\limits_{0}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}\left[\left|\frac{iu+1}{(iu+1)^2+(x+v-w)^2}\right|+\left|\frac{iu+1}{(iu+1)^2+(x-v+w)^2}\right|\right]\times$$

$$\times|f(u)g(v)h(w)|dxdudvdw \leqslant$$

$$\leqslant \int\limits_{0}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}\left[\frac{2\sqrt{1+u^2}}{1+u^2+(x+v-w)^2}+\frac{2\sqrt{1+u^2}}{1+u^2+(x-v+w)^2}\right]\times$$

$$\times|f(u)g(v)h(w)|dxdudvdw$$

we have

$$\int\limits_{0}^{+\infty}\left[\frac{2\sqrt{1+u^2}}{1+u^2+(x+v-w)^2}+\frac{2\sqrt{1+u^2}}{1+u^2+(x-v+w)^2}\right]dx =$$

$$= 2\Big(\arctan\frac{x+v-w}{\sqrt{1+u^2}}+\arctan\frac{x-v+w}{\sqrt{1+u^2}}\Big)\Big|_{x=0}^{+\infty} = 2\pi$$

thus

$$\int\limits_{0}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}\left[\left|\frac{iu+1}{(iu+1)^2+(x+v-w)^2}\right|+\left|\frac{iu+1}{(iu+1)^2+(x-v+w)^2}\right|\right]\times$$

$$\times|f(u)g(v)h(w)|dxdudvdw \leqslant$$

$$\leqslant 2\pi\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}|f(u)g(v)h(w)|dxdudvdw \leqslant +\infty$$

Similarly, we get

$$\int\limits_{0}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}\left[\left|\frac{iu+1}{(iu+1)^2+(x+v+w)^2}\right|+\left|\frac{iu+1}{(iu+1)^2+(x-v-w)^2}\right|\right]\times$$

$$\times|f(u)g(v)h(w)|dxdudvdw \leqslant +\infty$$

Therefore, we have $\overset{\gamma}{*}(f,g,h)(x) \in L(\mathbb{R}_+)$. We now prove that the polyconvolution (2.1) satisfies the factorization equality (2.2). Indeed,

$$e^{-y}(Ff)(y)(F_Cg)(y)(F_Sh)(y) =$$

$$= \frac{2}{\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} \sin(wy)\cos(vy)e^{-(iu+1)y} f(u)g(v)h(w)dudvdw$$

$$= \frac{1}{\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} [\sin(w+v)y + \sin(w-v)y]e^{-(iu+1)y} f(u)g(v)h(w)dudvdw$$

On the other hand, applying formula 2.2.16, p.65, V.1 in [2] we obtain

$$\sin[(w+v)y]e^{-(iu+1)y} + \sin[(w-v)y]e^{-(iu+1)y} =$$

$$= \frac{1}{\pi}\int_{0}^{+\infty} \Big[\frac{iu+1}{(iu+1)^2+(w+v-x)^2} - \frac{iu+1}{(iu+1)^2+(w+v+x)^2}\Big]\sin(xy)dx +$$

$$+ \frac{1}{\pi}\int_{0}^{+\infty} \Big[\frac{iu+1}{(iu+1)^2+(w-v-x)^2} - \frac{iu+1}{(iu+1)^2+(w-v+x)^2}\Big]\sin(xy)dx$$

it follows that

$$e^{-y}(Ff)(y)(F_C g)(y)(F_S h)(y) = \sqrt{\frac{2}{\pi}}\int_{0}^{+\infty} \overset{\gamma}{*}(f,g,h)(x)\sin(xy)dx$$

$$= F_S\big[\overset{\gamma}{*}(f,g,h)\big](y)$$

The proof is complete. □

Theorem 2.2 (Titchmarsch-type Theorem). *Let f be a function in $L(e^{-x}, \mathbb{R})$, g and h be the functions in $L(e^{x}, \mathbb{R}_+)$. If $\overset{\gamma}{*}(f,g,h) \equiv 0$ then either $f \equiv 0$ or $g \equiv 0$ or $h \equiv 0$.*

Proof. We apply the Fourier sine transform to both sides of $\overset{\gamma}{*}(f,g,h) \equiv 0$, an application of Theorem 2.1 yields

$$e^{-y}(Ff)(y)(F_C g)(y)(F_S h)(y) \equiv 0, \ \forall y > 0$$

therefore

$$(Ff)(y)(F_C g)(y)(F_S h)(y) \equiv 0, \ \forall y > 0 \tag{2.3}$$

As $(Ff)(y)$, $(F_C g)(y)$ and $(F_S h)(y)$ are analytic, from (2.3) it may be concluded that $Ff \equiv 0$ or $F_S g \equiv 0$ or $F_C h \equiv 0$ and so $f \equiv 0$ or $g \equiv 0$ or $h \equiv 0$. □

Theorem 2.3. *Let f be a function in $L(\mathbb{R})$, g and h be the functions in $L(\mathbb{R}_+)$. The polyconvolution for the Fourier, Fourier sine and Fourier cosine integral transforms relates to the known convolutions as follows*

a) $\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \Big\{\big[k(u,v) \underset{F}{*} g(|v|)\big](w) \underset{F}{*} \big[\text{sign}w h(|w|)\big]\Big\}(x) f(u)du$

b) $\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \Big(h(w) \underset{1}{*} \big(k(u,v) \underset{F_C}{*} g(v)\big)(w)\Big)(x) f(u)du$

c) $\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\Big(\big(h(w)\underset{1}{*}k(u,w)\big)(v)\underset{F_C}{*}g(v)\Big)(x)f(u)du$

here,

$$k(u,v) = \frac{iu+1}{(iu+1)^2+v^2}$$

Proof. We have

$$\int\limits_0^{+\infty}\int\limits_0^{+\infty}\Big[\frac{iu+1}{(iu+1)^2+(x+v-w)^2}+\frac{iu+1}{(iu+1)^2+(x-v-w)^2}\Big]g(v)h(w)dvdw =$$

$$= \int\limits_0^{+\infty}h(w)dw\int\limits_0^{-\infty}\frac{iu+1}{(iu+1)^2+(x-v-w)^2}g(|v|)d(-v)+$$

$$+\int\limits_0^{+\infty}h(w)dw\int\limits_0^{+\infty}\frac{iu+1}{(iu+1)^2+(x-v-w)^2}g(|v|)dv =$$

$$= \int\limits_0^{+\infty}h(w)dw\int\limits_{-\infty}^{+\infty}\frac{iu+1}{(iu+1)^2+(x-v-w)^2}g(|v|)dv$$

$$= \sqrt{2\pi}\int\limits_0^{+\infty}\big[k(u,v)\underset{F}{*}g(|v|)\big](x-w)h(w)dw$$

Similarly, we get

$$\int\limits_0^{+\infty}\int\limits_0^{+\infty}\Big[\frac{iu+1}{(iu+1)^2+(x+v+w)^2}+\frac{iu+1}{(iu+1)^2+(x-v+w)^2}\Big]g(v)h(w)dvdw =$$

$$= \sqrt{2\pi}\int\limits_0^{+\infty}\big[k(u,v)\underset{F}{*}g(|v|)\big](x+w)h(w)dw$$

$$= -\sqrt{2\pi}\int\limits_{-\infty}^{0}\big[k(u,v)\underset{F}{*}g(|v|)\big](x-w)\mathrm{sign}w h(|w|)dw$$

it follows that

$$\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{\pi\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\big[k(u,v)\underset{F}{*}g(|v|)\big](x-w)\mathrm{sign}w h(|w|)f(u)dudw$$

$$= \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\Big\{\big[k(u,v)\underset{F}{*}g(|v|)\big](w)\underset{F}{*}\big[\mathrm{sign}w h(|w|)\big]\Big\}(x)f(u)du$$

The equality **a)** is proved. On the other hand

$$\int_0^{+\infty}\Big[\frac{iu+1}{(iu+1)^2+(x+v-w)^2}+\frac{iu+1}{(iu+1)^2+(x-v-w)^2}\Big]g(v)dv =$$
$$=\sqrt{2\pi}\big(k(u,v)\underset{F_C}{*}g(v)\big)(|x-w|)$$

and

$$\int_0^{+\infty}\Big[\frac{iu+1}{(iu+1)^2+(x+v+w)^2}+\frac{iu+1}{(iu+1)^2+(x-v+w)^2}\Big]g(v)dv =$$
$$=\sqrt{2\pi}\big(k(u,v)\underset{F_C}{*}g(v)\big)(x+w)$$

thus

$$\overset{\gamma}{*}(f,g,h)(x)=$$
$$=\frac{1}{\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(u)du\int_0^{+\infty}\Big[\big(k(u,v)\underset{F_C}{*}g(v)\big)(|x-w|)-\big(k(u,v)\underset{F_C}{*}g(v)\big)(x+w)\Big]h(w)dw$$
$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\Big(h(w)\underset{1}{*}\big(k(u,v)\underset{F_C}{*}g(v)\big)(w)\Big)(x)f(u)du$$

The equality **b)** is proved. In the same maner we can obtain the equality **c)**. The proof is complete. □

Theorem 2.4. *Let f and φ be the functions in $L(\mathbb{R})$, g, h, k and ψ be the functions in $L(\mathbb{R}_+)$. The polyconvolution for the Fourier, Fourier sine and Fourier cosine integral transforms is neither commutative nor associative, and the following formulas holds*

a) $\overset{\gamma}{*}\big(f,g,\overset{\gamma}{*}(\varphi,\psi,h)\big)=\overset{\gamma}{*}\big(\varphi,\psi,\overset{\gamma}{*}(f,g,h)\big)$

b) $\overset{\gamma}{*}\big(f,g,(h\underset{1}{*}k)\big)=\overset{\gamma}{*}\big(f,k,(h\underset{1}{*}g)\big)$

c) $\big(\overset{\gamma}{*}(f,g,h)\underset{1}{*}k\big)=\overset{\gamma}{*}\big(f,(g\underset{F_C}{*}k),h\big)$

d) $\overset{\gamma}{*}\big(f,(g\underset{2}{*}h),k\big)=\overset{\gamma}{*}\big(f,(g\underset{2}{*}k),h\big)$

e) $\overset{\gamma}{*}\big(f,(g\overset{\gamma_1}{\underset{1}{*}}h),k\big)=\overset{\gamma}{*}\big(f,(k\overset{\gamma_1}{\underset{1}{*}}h),g\big)=\overset{\gamma}{*}\big(f,h,(g\overset{\gamma_1}{\underset{F_S}{*}}k)\big)$

f) $\overset{\gamma}{*}\big(f,g,(h\overset{\gamma_1}{\underset{2}{*}}k)\big)=\overset{\gamma}{*}\big(f,k,(h\overset{\gamma_1}{\underset{2}{*}}g)\big)$

Proof. The proof follows easily from formulas (1.2), (1.8), (1.10), (1.12), (1.16), (1.18) and (2.2). For example, we have

$$F_S\Big[\overset{\gamma}{*}\big(f,g,\overset{\gamma}{*}(\varphi,\psi,h)\big)\Big]=\gamma^2 FfF_CgF\varphi F_C\psi F_Sh$$
$$=F_S\Big[\overset{\gamma}{*}\big(\varphi,\psi,\overset{\gamma}{*}(f,g,h)\big)\Big]$$

Therefore, $\overset{\gamma}{*}\big(f, g, \overset{\gamma}{*}(\varphi, \psi, h)\big) = \overset{\gamma}{*}\big(\varphi, \psi, \overset{\gamma}{*}(f, g, h)\big)$, and formula **a)** is proved. By the same way, one can verify the other parts, too. □

3. Applications to integral equations and systems of integral equations

Let

$$\theta_f^1(x,u) = \frac{1}{\sqrt{2\pi}}\big[f(|x-u|) - f(x+u)\big]$$

$$\theta_f^2(x,u) = \frac{1}{2\sqrt{2\pi}}\big[\operatorname{sign}(x+u-1)f(|x+u-1|) - f(x+u+1) + \\ + \operatorname{sign}(x-u-1)f(|x-u-1|) - \operatorname{sign}(x-u+1)f(|x-u+1|)\big]$$

$$\theta_f^3(x,u) = \frac{1}{\sqrt{2\pi}}\big[\operatorname{sign}(u-x)f(|x-u|) + f(x+u)\big]$$

$$\theta_f^4(x,u) = \frac{1}{2\sqrt{2\pi}}\big[f(|x+u-1|) - f(x+u+1) - f(|x-u-1|) + f(|x-u+1|)\big]$$

Theorem 3.1. *Consider the system of integral equations*

$$(3.1)\quad \begin{aligned} & f(x) + \lambda_1 \int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \theta(x,u,v,w)\varphi(u)\psi(v)g(w)\,du\,dv\,dw = q_1(x) \\ & \int_0^{+\infty} \big[\lambda_2\theta^1_{\varphi_1}(x,u) + \lambda_3\theta^2_{\varphi_2}(x,u) + \lambda_4\theta^1_{\varphi_3}(x,u)\big]f(u)\,du + g(x) = q_2(x), \ \ x>0 \end{aligned}$$

where $\varphi_1 = (\xi_1 \underset{2}{*} \xi_2)$, $\varphi_2 = (\xi_3 \underset{1}{*} \xi_4)$, $\varphi_3 = (\xi_6 \overset{\gamma_1}{\underset{1}{*}} \xi_5)$; φ, ψ, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, q_1 *and* q_2 *are the functions of* $L(\mathbb{R}_+)$, λ_1, λ_2, λ_3 *and* λ_4 *are the complex constants,* f *and* g *are the unknown functions. With the condition*

$$1 - \lambda_1 F_C\phi(x) \neq 0, \forall x > 0$$

here $\phi = \lambda_2\big(\overset{\gamma}{*}(\varphi,\psi,\xi_1) \underset{2}{*} \xi_2\big) + \lambda_3\big(\overset{\gamma}{*}(\varphi,\psi,\xi_3) \overset{\gamma_1}{\underset{1}{*}} \xi_4\big) + \lambda_4\big(\overset{\gamma}{*}(\varphi,\psi,\xi_5) \overset{\gamma_1}{\underset{1}{*}} \xi_6\big)\big)$
there exists a solution in $\mathrm{L}(\mathbb{R}_+)$ *of* (3.1) *which is defined by*

$$f = q_1 + \Big(q_1 \underset{1}{*} l\Big) - \lambda_1[\overset{\gamma}{*}(\varphi,\psi,q_2)] - \lambda_1\Big[\overset{\gamma}{*}(\varphi,\psi,q_2) \underset{1}{*} l\Big] \in L(\mathbb{R}_+)$$

$$g = q_2 - \lambda_2(q_1 \underset{1}{*} \varphi_1) - \lambda_3(q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) - \lambda_4\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1)\big) +$$

$$+ (q_2 \underset{1}{*} l) - \lambda_2\big((q_1 \underset{1}{*} \varphi_1) \underset{1}{*} l\big) - \lambda_3\big((q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) \underset{1}{*} l\big) - \lambda_4\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1) \underset{1}{*} l\big) \in L(\mathbb{R}_+)$$

with $l \in L(\mathbb{R}_+)$ *and is defined by*

$$F_C l = \frac{\lambda_1 F_C\phi}{1 - \lambda_1 F_C\phi}.$$

Proof. Applying the Fourier sine transform to the both sides of the equations of the system (3.1) and using (1.1), (1.2), (1.9), (1.10), (1.17), (1.18), (1.11), (1.12) and Theorem 2.1, we obtain the linear system

$$F_S f + \lambda_1 \gamma (F\varphi)(F_C\psi)(F_S g) = F_S q_1$$
$$\big[\lambda_2 (F_S\xi_1)(F_S\xi_2) + \lambda_3\gamma_1 (F_S\xi_3)(F_C\xi_4) + \lambda_4\gamma_1(F_C\xi_5)(F_S\xi_6)\big] F_S f + F_S g = F_S q_2$$

We have

$$\Delta = \begin{vmatrix} 1 & \lambda_1\gamma(F\varphi)(F_C\psi) \\ \lambda_2(F_S\xi_1)(F_S\xi_2) + \lambda_3\gamma_1(F_S\xi_3)(F_C\xi_4) + \lambda_4\gamma_1(F_C\xi_5)(F_S\xi_6) & 1 \end{vmatrix}$$
$$= 1 - \lambda_1 F_C\Big[\lambda_2\big(\overset{\gamma}{*}(\varphi,\psi,\xi_1) \underset{2}{*} \xi_2\big) + \lambda_3\big(\overset{\gamma}{*}(\varphi,\psi,\xi_3) \overset{\gamma_1}{\underset{1}{*}} \xi_4\big) + \lambda_4\big(\overset{\gamma}{*}(\varphi,\psi,\xi_5) \overset{\gamma_1}{\underset{1}{*}} \xi_6\big)\Big]$$
$$= 1 - \lambda_1 F_C\phi$$

so that

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1 F_C\phi}{1-\lambda_1 F_C\phi}.$$

Furthermore, we get

$$\Delta_f = \begin{vmatrix} F_S q_1 & \lambda_1\gamma(F\varphi)(F_C\psi) \\ F_S q_2 & 1 \end{vmatrix} = F_S q_1 - \lambda_1 F_S[\overset{\gamma}{*}(\varphi,\psi,q_2)].$$

Moreover

$$\Delta_g = \begin{vmatrix} 1 & F_S q_1 \\ \lambda_2(F_S\xi_1)(F_S\xi_2) + \lambda_3\gamma_1(F_S\xi_3)(F_C\xi_4) + \lambda_4\gamma_1(F_C\xi_5)(F_S\xi_6) & F_S q_2 \end{vmatrix}$$
$$= F_S q_2 - \lambda_2 F_S(q_1 \underset{1}{*} \varphi_1) - \lambda_3 F_S(q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) - \lambda_4 F_S\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1)\big).$$

Due to Wiener-Levi's theorem (see [1]), there exists a function $l \in L(\mathbb{R}_+)$ such that

$$F_C l = \frac{\lambda_1 F_C\phi}{1-\lambda_1 F_C\phi}.$$

Hence

$$F_S f = (1 + F_C l)\Big\{F_S q_1 - \lambda_1 F_S[\overset{\gamma}{*}(\varphi,\psi,q_2)]\Big\}$$
$$= F_S q_1 + F_S\Big(q_1 \underset{1}{*} l\Big) - \lambda_1 F_S[\overset{\gamma}{*}(\varphi,\psi,q_2)] - \lambda_1 F_S\Big[\overset{\gamma}{*}(\varphi,\psi,q_2) \underset{1}{*} l\Big]$$

and consequently

$$f = q_1 + \Big(q_1 \underset{1}{*} l\Big) - \lambda_1[\overset{\gamma}{*}(\varphi,\psi,q_2)] - \lambda_1\Big[\overset{\gamma}{*}(\varphi,\psi,q_2) \underset{1}{*} l\Big] \in L(\mathbb{R}_+)$$

Similarly, we show that

$$F_S g = (1 + F_C l)\Big[F_S q_2 - \lambda_2 F_S(q_1 \underset{1}{*} \varphi_1) - \lambda_3 F_S(q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) - \lambda_4 F_S\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1)\big)\Big]$$
$$= F_S q_2 - \lambda_2 F_S(q_1 \underset{1}{*} \varphi_1) - \lambda_3 F_S(q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) - \lambda_4 F_S\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1)\big) +$$
$$+ F_S(q_2 \underset{1}{*} l) - \lambda_2 F_S\big((q_1 \underset{1}{*} \varphi_1) \underset{1}{*} l\big) - \lambda_3 F_S\big((q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) \underset{1}{*} l\big) - \lambda_4 F_S\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1) \underset{1}{*} l\big)$$

namely,

$$g = q_2 - \lambda_2(q_1 \underset{1}{*} \varphi_1) - \lambda_3(q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) - \lambda_4\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1)\big) +$$

$$+(q_2 \underset{1}{*} l) - \lambda_2\big((q_1 \underset{1}{*} \varphi_1) \underset{1}{*} l\big) - \lambda_3\big((q_1 \overset{\gamma_1}{\underset{F_S}{*}} \varphi_2) \underset{1}{*} l\big) - \lambda_4\big(\xi_5 \overset{\gamma_1}{\underset{2}{*}} (\xi_6 \underset{2}{*} q_1) \underset{1}{*} l\big) \in L(\mathbb{R}_+).$$

□

Theorem 3.2. *Consider the system of integral equations*

$$(3.2)\quad \begin{aligned} &f(x) + \lambda_1 \int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \theta(x,u,v,w)\varphi(u)\psi(w)g(v)\,du\,dv\,dw = q_1(x) \\ &\int_0^{+\infty} \big[\lambda_2\theta^3_{\varphi_1}(x,u) + \lambda_3\theta^4_{\varphi_2}(x,u) + \lambda_4\theta^3_{\varphi_3}(x,u)\big] f(u)\,du + g(x) = q_2(x), \ x > 0 \end{aligned}$$

where $\varphi_1 = (\xi_1 \underset{2}{*} \xi_2)$, $\varphi_2 = (\xi_3 \underset{F_C}{*} \xi_4)$, $\varphi_3 = \xi_5 \underset{1}{*} (\xi_6 \overset{\gamma_2}{\underset{1}{*}} \xi_7)$; φ, ψ, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, ξ_7, q_1 *and* q_2 *are the functions of* $L(\mathbb{R}_+)$, λ_1, λ_2, λ_3, λ_4 *and* λ_5 *are the complex constants,* f *and* g *are the unknown functions. With the condition*

$$1 - \lambda_1 F_C\phi(x) \neq 0, \forall x > 0$$

here $\phi = \lambda_2\big(\overset{\gamma}{*}(\varphi,\xi_2,\psi) \underset{2}{*} \xi_1\big) + \lambda_3\big(\overset{\gamma}{*}(\varphi,\xi_3,\psi) \overset{\gamma_1}{\underset{1}{*}} \xi_4\big) + \lambda_4\Big(\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \big(\overset{\gamma}{*}(\varphi,\xi_6,\psi) \underset{2}{*} \xi_5\big)\Big)$ *there exists a solution in* $L(\mathbb{R}_+)$ *of* (3.2) *which is defined by*

$$f = q_1 + \Big(q_1 \underset{1}{*} l\Big) - \lambda_1[\overset{\gamma}{*}(\varphi,q_2,\psi)] - \lambda_1\Big[\overset{\gamma}{*}(\varphi,q_2,\psi) \underset{1}{*} l\Big] \in L(\mathbb{R}_+)$$

$$\begin{aligned} g = q_2 &- \lambda_2\big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2\big) - \lambda_3\big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1)\big) - \\ &-\lambda_4\big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1)\big) + (q_2 \underset{F_C}{*} l) - \lambda_2 F_C\Big(\big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2\big) \underset{F_C}{*} l\Big) - \\ &-\lambda_3\big[\big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1)\big) \underset{F_C}{*} l\big] - \lambda_4\big[\big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1)\big) \underset{F_C}{*} l\big] \in L(\mathbb{R}_+) \end{aligned}$$

with $l \in L(\mathbb{R}_+)$ *and is defined by*

$$F_C l = \frac{\lambda_1 F_C\phi}{1 - \lambda_1 F_C\phi}.$$

Proof. Taking the Fourier sine transform of the both sides of the first equation, the Fourier cosine transform of the both sides of the second equation and according to (1.11), (1.12), (1.15), (1.16), (1.7), (1.8), (1.13), (1.14) and Theorem 2.1, we give the linear system

$$F_S f + \lambda_1\gamma(F\varphi)(F_S\psi)(F_C g) = F_S q_1$$

$$\big[\lambda_2(F_S\xi_1)(F_C\xi_2) + \lambda_3\gamma_1(F_C\xi_3)(F_C\xi_4) + \lambda_4\gamma_2(F_S\xi_5)(F_C\xi_6)(F_C\xi_7)\big] F_S f + F_C g = F_C q_2.$$

We get

$$\Delta = \begin{vmatrix} 1 & \lambda_1\gamma(F\varphi)(F_S\psi) \\ \lambda_2(F_S\xi_1)(F_C\xi_2) + \lambda_3\gamma_1(F_C\xi_3)(F_C\xi_4) + \lambda_4\gamma_2(F_S\xi_5)(F_C\xi_6)(F_C\xi_7) & 1 \end{vmatrix}$$

$$= 1 - \lambda_1 F_C\Big[\lambda_2\big(\overset{\gamma}{*}(\varphi,\xi_2,\psi) \underset{2}{*} \xi_1\big) + \lambda_3\big(\overset{\gamma}{*}(\varphi,\xi_3,\psi) \overset{\gamma_1}{\underset{1}{*}} \xi_4\big) + \lambda_4\Big(\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \big(\overset{\gamma}{*}(\varphi,\xi_6,\psi) \underset{2}{*} \xi_5\big)\Big)\Big]$$

$$= 1 - \lambda_1 F_C\phi$$

which is

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1 F_C \phi}{1 - \lambda_1 F_C \phi}$$

On the other hand

$$\Delta_f = \begin{vmatrix} F_S q_1 & \lambda_1 \gamma (F\varphi)(F_S \psi) \\ F_C q_2 & 1 \end{vmatrix} = F_S q_1 - \lambda_1 F_S [\overset{\gamma}{*}(\varphi, q_2, \psi)]$$

and

$$\Delta_g = \begin{vmatrix} 1 & F_S q_1 \\ \lambda_2 (F_S \xi_1)(F_C \xi_2) + \lambda_3 \gamma_1 (F_C \xi_3)(F_C \xi_4) + \lambda_4 \gamma_2 (F_C \xi_7)(F_C \xi_6)(F_S \xi_5) & F_C q_2 \end{vmatrix}$$

$$= F_C q_2 - \lambda_2 F_C \big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2 \big) - \lambda_3 F_C \big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1) \big) - \lambda_4 F_C \big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1) \big)$$

By the Wiener-Levi theorem (see [1]), there exists a function $l \in L(\mathbb{R}_+)$ such that

$$F_C l = \frac{\lambda_1 F_C \phi}{1 - \lambda_1 F_C \phi}.$$

This implies that

$$\begin{aligned} F_S f &= (1 + F_C l) \left\{ F_S q_1 - \lambda_1 F_S [\overset{\gamma}{*}(\varphi, q_2, \psi)] \right\} \\ &= F_S q_1 + F_S \left(q_1 \underset{1}{*} l \right) - \lambda_1 F_S [\overset{\gamma}{*}(\varphi, q_2, \psi)] - \lambda_1 F_S \left[\overset{\gamma}{*}(\varphi, q_2, \psi) \underset{1}{*} l \right] \end{aligned}$$

thus

$$f = q_1 + \left(q_1 \underset{1}{*} l \right) - \lambda_1 [\overset{\gamma}{*}(\varphi, q_2, \psi)] - \lambda_1 \left[\overset{\gamma}{*}(\varphi, q_2, \psi) \underset{1}{*} l \right] \in L(\mathbb{R}_+)$$

Likewise, we obtain

$$F_C g = (1 + F_C l) \Big[F_C q_2 - \lambda_2 F_C \big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2 \big) -$$

$$- \lambda_3 F_C \big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1) \big) - \lambda_4 F_C \big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1) \big) \Big] =$$

$$= F_C q_2 - \lambda_2 F_C \big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2 \big) - \lambda_3 F_C \big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1) \big) -$$

$$- \lambda_4 F_C \big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1) \big) + F_C (q_2 \underset{F_C}{*} l) - \lambda_2 F_C \Big(\big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2 \big) \underset{F_C}{*} l \Big) -$$

$$- \lambda_3 F_C \big[\big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1) \big) \underset{F_C}{*} l \big] - \lambda_4 F_C \big[\big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1) \big) \underset{F_C}{*} l \big].$$

Therefore,

$$g = q_2 - \lambda_2 \big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2 \big) - \lambda_3 \big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1) \big) -$$

$$- \lambda_4 \big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1) \big) + (q_2 \underset{F_C}{*} l) - \lambda_2 F_C \Big(\big((q_1 \underset{2}{*} \xi_1) \underset{F_C}{*} \xi_2 \big) \underset{F_C}{*} l \Big) -$$

$$- \lambda_3 \big[\big((\xi_3 \overset{\gamma_1}{\underset{2}{*}} \xi_4) \underset{2}{*} q_1) \big) \underset{F_C}{*} l \big] - \lambda_4 \big[\big((\xi_7 \overset{\gamma_2}{\underset{F_C}{*}} \xi_6) \underset{F_C}{*} (\xi_5 \underset{2}{*} q_1) \big) \underset{F_C}{*} l \big] \in L(\mathbb{R}_+).$$

□

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