SHIFTS ON THE HYPERFINITE FACTOR OF TYPE II_1

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Dedicated to my parents

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Introduction

This thesis is devoted to the study of shifts on the hyperfinite factor R of type II_1. This is an extended version of papers [9],[10],[11],[12].

On the structure of isometries, we have the next fundamental result which will be referred to as the Wold decomposition: Every isometry is a direct sum of a unitary operator and some copies of the unilateral shift. The co-rank of a shift (also called its multiplicity) constitutes a complete set of unitary invariants for it. Unitary operators correspond with

*-automorphisms ,so to say,and then how do isometries and *-endomorphisms correspond? Indeed, the main question which R.T.Powers threw out in his paper [19],[21] was " what about *-endomorphisms versus *-automorphisms of C*-algebras." He considered that, since there is an index theory for isometries, perhaps there is a corresponding index theory for *-endomorphisms. Powers [19],[21] called α a shift of a unital C*-algebra A if α is a *-endomorphism of A so that $\alpha(I) = I$ and $\bigcap_{n=1}^{\infty}\alpha^n(A) = \{\lambda I\}$. Following after A.Connes [8], Powers [19],[21] defined that two *-endomorphisms α and β of a unital C*-algebra A are conjugate if there is a *-automorphism γ of A so that $\alpha(a) = \gamma(\beta(\gamma^{-1}(a)))$ for $a \in A$. He also defined two *-endomorphisms α and β as outer conjugate if there is a *-automorphism γ of A and a unitary $u \in A$ so that $\alpha(uau^{-1}) = \gamma(\beta(\gamma^{-1}(a)))$ for all $a \in A$.

Let B(H) be the algebra of all bounded operators on a separable Hilbert space H. Suppose α is a shift of B(H). Let $N_1 = \alpha(B(H))'$. Then N_1 is a factor of type I_n with $n = 2,3,\ldots,\infty$. Powers [19],[21] called this number n the multiplicity of α. He showed also the following theorems about conjugacy and outer conjugacy of shifts of B(H).

Theorem 1. Suppose α and β are shifts of B(H) and there is a pure normal state ω_0 of B(H) which is invariant under α. Then α and β are conjugate if and only if there is a pure normal state ω_1 of B(H) which is invariant under β and α and β have the same multiplicity.

Theorem 2. Suppose α and β are shifts of B(H). Then α and β are outer conjugate if and only if they have the same multiplicity.

Thus conjugacy and outer conjugacy of shifts of B(H) are determined by their multiplicities. He then concentrated his attention to shifts of factors of type II_1 and continuous semigroups of shifts of B(H). For the index of continuous semigroups of shifts of B(H), there are works of Powers [19],[20],[21], Powers and Robinson [22],Arveson [2],[3].

In the case of II_1-factors, Powers defined the index of shifts σ on a hyperfinite II_1-factor R by using the famous Jones index $[R:\sigma(R)]$(Jones[13]). He made discussions on conjugacy classes and outer conjugacy classes of binary shifts. A shift σ of R is called a binary shift if there is a unitary $u \in R$ with $u^2 = I$ such that $R = \{u,\sigma(u),\sigma^2(u),\ldots\}''$ and $u\sigma^k(u) = \pm\sigma^k(u)u$ for $k \in \mathbb{N}$. There are uncountably many nonconjugate (at least countably many non outer conjugate) binary shifts on R([19]). M.Choda[7] generalized this Powers' result first for a shift with such a unitary u as $u^m = I$ ($m \in \mathbb{N}$), and then, considering certain shifts which come from a family of Jones projections[6], showed the existence of a countably infinite number of outer conjugacy classes of shifts on R with a given index ($\in \{4\cos^2(\pi/n);\ n = 3,4,\ldots\} \cup [4,\infty)$). G.L.Price[23] ingeniously succeeded on constructing an example of a shift σ on R of index two which is not a binary shift and ,in [24], he further generalized unitary shifts considered by M.Choda[7].

This topic is also treated in D.Bures and H.s.Yin[5]. Other generalizations of shifts are considered in Price[25].

In this thesis, we shall consider a general shift σ on a group von Neumann algebra $R_m(G)$ on a group G twisted by $m \in Z^2(G,T)$ such that the shift σ is induced from a shift on a group G. This formulation is also taken up in [5] independently. Under this formulation, concepts are simplified and, notions and proofs become clearer. All examples of shifts depending on unitaries which are obtained by Powers[19],Choda[7] and Price[19],[24] can be reduced to shifts induced from groups.

In 1, we shall gather up elementary facts about shifts on von Neumann algebras induced from shifts on groups.

In 2, we shall treat shifts with index two on a hyperfinite factor of type II_1.

In 2.1, we shall introduce signature sequences due to [11],[12], and discuss the triviality of relative commutant algebras. In 2.2, inspired by the construction of Price's non-binary shifts with index two on R, we shall construct uncountably many non-binary shifts on R of index two. Price's construction of a non-binary shift comes from viewing R as the completion of an inductive limit of binary shift algebras (cf.Bratteli[4]).

In 2.3, we shall be concerned with a Powers' problem on outer conjugacy of binary shifts on the hyperfinite II_1-factor.

Let σ be a binary shift on R and $q(\sigma)$ be the number

$\min\{k\in\mathbb{N}\ ;\ \sigma^k(R)'\cap R \neq \mathbb{C}I\}$. Then the number $q(\sigma)$ is an outer conjugacy invariant for a binary shift σ. Powers [11] raised the following problem. " Is the number $q(\sigma)$ the complete outer conjugacy invariant for a binary shift σ." In this section we shall give a negative answer to this Powers' problem. In order to do this, we shall use the sequence of relative commutant algebras $\{\sigma^k(R)'\cap R\ ;\ k = 0,1,2,\ldots\}$. Obviously the set of the (isomorphism classes of) relative commutant algebras $\{\sigma^k(R)'\cap R\ ;\ k = 0,1,2,\ldots\}$ is an outer conjugacy invariant for binary shifts.

In 2.4, we shall consider to represent multipliers of 2.2 as multipliers on the rational function field over a finite field.

In 3, we shall consider shifts with an integral index on a hyperfinite factor of type II_1.

In 3.1, when G is an ES group (cf. Definition 3.1.5), we shall show that the conjugacy classes of shifts coming from shifts on groups are classified by Powers invariants.

In 3.2., we shall generalize results in 2.2 using a different method to any index case.

1. Elementary facts (Preliminaries).

(Shifts on von Neumann algebras induced from shifts on groups.)

Let G be a countable discrete group. A multiplier m on G is a map of $G\times G$ into $\mathbb{T} = \{z\in\mathbb{C};\ |z| = 1\}$ such that

$$m(x,1) = m(1,x) = 1 \quad \text{and} \quad m(x,y)m(xy,z) = m(x,yz)m(y,z)$$

for $x,y,z\in G$, so that $m\in Z^2(G,T)$.

Definition 1.1. Let G be a countable discrete group. A projective representation u of G with a multiplier m is a mapping from G into the unitary group U(B(H)) of B(H) on a separable Hilbert space H and $u(g)u(h) = m(g,h)u(gh)$ for any g and h in G.

For $x\in G$ and $m\in Z^2(G,T)$, define a unitary operator $\lambda_m(x)$ on $\ell^2(G)$ by

$$(\lambda_m(x)\xi)(y) = m(x,x^{-1}y)\xi(x^{-1}y) \quad \text{for } \xi\in\ell^2(G).$$

Then $\lambda_m: G\longrightarrow U(B(\ell^2(G)))$ is a projective representation with m. Let $R_m(G)$ be the von Neumann algebra generated by $\{\lambda_m(x);\ x\in G\}$. We call $R_m(G)$ the (twisted) group von Neumann algebra. Let $\delta_x\in\ell^2(G)$ be the Dirac's delta function defined by

$$\delta_x(y) = 1 \text{ if } x = y \quad \text{and} \quad \delta_x(y) = 0 \text{ if } x \neq y, \text{ for } x,y\in G.$$

We can characterize $R_m(G)$ as follows. The following proposition may be known but we note here for completeness.

Lemma 1.2. Let M be a von Neumann algebra. Suppose that

there exists a projective representation u with m, $u:G \longrightarrow M$. Assume that M is generated by $\{u(g);g\in G\}$. If there exists a faithful normal state ϕ on M such that $\phi(u(y)^*u(x)) = \delta_{x,y}$ for $x,y \in G$, then there is a *-isomorphism $\theta:M \longrightarrow R_m(G)$ such that $\theta(u(x)) = \lambda_m(x)$.

Proof. Take the GNS representation π_ϕ of M induced by ϕ and let ξ be the embedding of M into $L^2(M,\phi)$. Put $U(\delta_x) = \xi(u(x))$ for $x\in G$.

$$\langle U(\delta_x)|U(\delta_y)\rangle = \langle \xi(u(x))|\xi(u(y))\rangle$$
$$= \phi(u(y)^*u(x)) = \delta_{x,y} = \langle \delta_x|\delta_y\rangle.$$

Thus U can be extended to a surjective isometry of $\ell^2(G)$ onto $L^2(M,\phi)$. We have that

$$(U\lambda_m(x)U^*)(\xi(u(y))) = U\lambda_m(x)\delta_y = Um(x,y)\delta_{xy} = m(x,y)U\delta_{xy}$$
$$= m(x,y)\xi(u(xy))$$
$$= \pi_\phi(u(x))\xi(u(y))$$

Hence $R_m(G)$is isomorphic to $\pi_\phi(M)$. Since ϕ is faithful, $\pi_\phi(M)$ is isomorphic to M. Therefore $R_m(G)$ is isomorphic to M. Q.E.D.

Now we can construct shifts on $R_m(G)$ as follows .
An injective homomorphism $\sigma:G \longrightarrow G$ is called a shift (on a group G) if $\bigcap_{k=0}^{\infty}\sigma^k(G) = \{1\}$ and we say that σ preserves the multiplier $m\in Z^2(G,T)$ if $m(\sigma(x),\sigma(y)) = m(x,y)$ for $x,y\in G$.

Lemm 1.3. Let σ be a shift on a group G. Suppose that σ preserves the multiplier $m \in Z^2(G,T)$. Then σ induces a shift σ_m on the (twisted) group von Neumann algebra $R_m(G)$ such that

$\sigma_m(\lambda_m(x)) = \lambda_m(\sigma(x))$ for $x \in G$.

Furthermore, supposing that $R_m(G)$ is a factor,

$[R_m(G):\sigma_m(R_m(G))] = [G:\sigma(G)]$.

Proof. Put $u(x) = \lambda_m(\sigma(x))$ for $x\varepsilon G$ and let M be the von Neumann algebra generated by $\{u(x);x\varepsilon G\}$. Put $\phi = \omega_{\delta_1}$ (= the vector state determined by δ_1). Then those $\{u(x);x \in G\}$ and ϕ satisfy the assumption of Lemma 1.2. Hence $R_m(G)$ is isomorphic to $M \subseteq R_m(G)$. This extended isomorphism σ_m from $R_m(G)$ onto M satisfies $\sigma_m(\lambda_m(x)) = \lambda_m(\sigma(x))$ for $x\varepsilon G$. Thus σ_m is a *endomorphism from $R_m(G)$ into $R_m(G)$ and $\sigma_m(I) = I$. Next we shall show that $\bigcap_{k=1}^{\infty}\sigma_m^k(R_m(G)) = \mathbb{C}I$. Take the GNS representation π_ϕ of $R_m(G)$ induced by ϕ. Let ξ be the embedding of $R_m(G)$ into $L^2(R_m(G),\phi)$. Then $\{\xi(\lambda_m(x));x\varepsilon G\}$ is a CONS in $L^2(R_m(G),\phi)$. Fix any $x(\neq 1) \in G$. Since $\bigcap_{k=1}^{\infty}\sigma^k(G) = \{1\}$, there exists an integer k such that $x \notin \sigma^k(G)$. As $\xi(\sigma_m^k(R_m(G)))$ is contained in the closed subspace spanned by $\xi(\lambda_m(\sigma^k(G)))$, $\xi(\lambda_m(x)) \perp \xi(\sigma_m^k(R_m(G)))$. Hence $\xi(\lambda_m(x)) \perp \xi(\bigcap_{k=1}^{\infty}\sigma_m^k(R_m(G)))$. On the other hand

$\xi(I) = \xi(\lambda_m(1)) \in \xi(\bigcap_{k=1}^{\infty}\sigma_m^k(R_m(G)))$, thus $\xi(\bigcap_{k=1}^{\infty}\sigma_m^k(R_m(G))) = \mathbb{C}\xi(I)$. Therefore $\bigcap_{k=1}^{\infty}\sigma_m^k(R_m(G)) = \mathbb{C}I$. See Jones[13,Example 2.3.2] for $[R_m(G),\sigma_m(R_m(G))] = [G:\sigma(G)]$. Q.E.D.

Summing up above considerations, we put

Definition 1.4. Let σ be a shift on a group G. Suppose that σ preserves the multiplier $m \in Z^2(G,T)$. Then we say that the shift σ_m on $R_m(G)$ comes from a shift on a group.

2. Shifts with index two on a hyperfinite factor of type II_1.

2.1. Fundamental facts and examples.

In this section we shall look at examples of shifts from group theoretic view point. Let G be a countable discrete abelian group and m be a multiplier on G. Define $\omega_m: G \times G \longrightarrow \mathbf{T}$ by $\omega_m(x,y) = m(x,y)\overline{m(y,x)}$. Then ω_m turns out to be an anti-symmetric bicharacter on G (cf.[14]). It is known that if ω_m is non-degenerate, that is, $\omega_m(x,G) = \{1\}$ implies that $x = 1$, then $R_m(G)$ becomes a hyperfinite II_1-factor(cf.Slawny[26]). We put

$X = \prod_{i=0}^{\infty} G_i$, where $G_i \tilde{=} \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z} = \{0,1\}$. A sequence $a: \mathbf{Z} \longrightarrow \{0,1\}$ with $a(0) = 0$ and $a(n) = a(-n)$ is called a signature sequence. A signature sequence $a: \mathbf{Z} \longrightarrow \{0,1\}$ is periodic if there exists an $n \in \mathbf{Z}$ such that $a(j+n) = a(j)$ for any $j \in \mathbf{Z}$. For $x = (x(i))$ and $y = (y(j))$ in X, let us define a multiplier $m_a \in Z^2(X,\mathbf{T})$ by

(2.1.1)

$$m_a(x,y) = (-1)^{\sum_{i>j} a(i-j)x(i)y(j)}.$$

Then m_a is a bicharacter, that is , $m_a(x+y,z) = m_a(x,z)m_a(y,z)$ and $m_a(x,y+z) = m_a(x,y)m_a(x,z)$. Price[23] showed that the group von Neumann algebra $R_{m_a}(X)$ is a factor if and only if the signature sequence a is non-periodic. The following proposition

is a slight refinement of this Price's result.

Proposition 2.1.1. Let $X = \amalg_{i=0}^{\infty} G_i$, $G_i \tilde{=} \mathbf{Z}_2$. Let a be a signature sequence on $\mathbf{Z}$ and m_a be the corresponding multiplier by (2.1.1).

Then the following statements are all equivalent.

(1) the group von Neumann algebra $R_{m_a}(X)$ is a factor

(2) the anti-symmetric bicharactor ω_{m_a} is non-degenerate

(3) the signature sequence a is non-periodic.

Proof. (2) implies (1) by Slawny [26]. (1) implies (3) by Price[23,Theorem 2.3]. (3) implies (2) as in the below. We put

$$A = \begin{pmatrix} a(0) & a(1) & a(2) & a(3) & \cdots \\ a(1) & a(0) & a(1) & a(2) & \cdots \\ a(2) & a(1) & a(0) & a(1) & \cdots \\ a(3) & a(2) & a(1) & a(0) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \quad \text{and} \quad x,y \in X.$$

Then $\omega_{m_a}(x,y) = (-1)^{(Ax|y)}$. Therefore if ω_{m_a} is degenerate, there exists a non-zero $x \in X$ such that $Ax = 0$. If we take the non-zero solution x of minimal length, the x is unique. By using this fact and the proof in Price [23,Theorem 2.3], we can show that (3) implies (2). Q.E.D.

Example 2.1.2.(Powers' binary shifts). Let α be a binary

shift on R with a unitary generator u. Put

$S = \{ k \in \mathbb{N};\ u\alpha^k(u) = -\alpha^k(u)u \}$.

Define the sequence $a:\mathbb{Z} \longrightarrow \{0,1\}$ by $a(n) = 1$ if $|n| \in S$ and $a(n) = 0$ if $|n| \notin S$. Suppose that a is not periodic. Let m_a be as in (2.1). For $x = (x(0),\ldots,x(n),0,0,\ldots) \in X = \amalg_{i=0}^{\infty} G_i$, $G_i \cong \mathbb{Z}_2$, we put $u(x) = u^{x(0)}\alpha(u)^{x(1)}\alpha^2(u)^{x(2)}\ldots\alpha^n(u)^{x(n)}$. Then by lemma 1.2, there exists an isomorphism $\theta:R \longrightarrow R_{m_a}(X)$ such that

$\theta(u(x)) = \lambda_{m_a}(x)$. Define the canonical shift σ on the group X by $(\sigma(x))(j) = x(j-1)$ for $j \geq 1$ and $(\sigma(x))(0) = 0$. Since $m_a(\sigma(x),\sigma(y)) = m_a(x,y)$, by lemma 1.3, σ induces a shift σ_{m_a} on a von Neumann algebra $R_{m_a}(X)$. Then, by the above isomorphism θ, $\theta\alpha\theta^{-1} = \sigma_{m_a}$. Thus the binary shift α is exactly σ_{m_a} under the isomorphism θ.

Remark 2.1.3. Let S be a subset of the positive integers. The binary shift algebra B(S) over S(Powers[19,Definition 3.8]) is the *-algebra generated by elements u_i for $i = 1,2,3,\ldots$ such that $u_i^* = u_i$, $u_i^2 = I$ and $u_iu_j = \sigma(i,j)u_ju_i$ where $\sigma(i,j) = -1$ if $|i-j| \in S$ and $\sigma(i,j) = 1$ if $|i-j| \notin S$. Powers[19] proved that if the signature sequence is non-periodic, then the binary shift algebra B(S) and its C*-completion A(S) are simple and do not depend on the choice of generators $\{u_1,u_2,\ldots\}$. By proposition 2.1.1, this fact turns out to be a corollary of the general theorem by Slawny[26,Theorem 3,7].

Example 2.1.4. Let G be an arbitrary countable discrete

group. Put $X = \prod_{i=0}^{\infty} G_i$, where $G_i \tilde{=} G$. Then we can generalize the above example 2.1.2. In chapter 3 , we shall investigate this case. If $G = \mathbf{Z}_2$, then we have Powers' binary shift. If $G = \mathbf{Z}_n$, then we have shifts considered by M.Choda[7] and G.Price[23].

Example 2.1.5. Let M_2 be the algebra of two by two matrices. Put $M = M_2 \otimes M_2 \otimes \ldots$ be the infinite tensor product with respect to the trace. Then M is a hyperfinite factor of type II_1. We have the canonical shift α such that

$$\alpha(x_1 \otimes x_2 \otimes \ldots x_n \otimes I \otimes I \ldots)$$
$$= I \otimes x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes I \otimes I \otimes \ldots .$$

The shift α also comes from a shift on a group. Recall that $M_2 \tilde{=} R_m(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ for a suitable multiplier $m \in Z^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{T})$. For example m is given by

$$m((i_1,i_2),(j_1,j_2)) = (-1)^{i_2 j_1}$$

Put $G_i = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $X = \prod_{i=1}^{\infty} G_i$. The multiplier $\bar{m} \in Z^2(X,T)$ will be given by

$$\bar{m}(x,y) = \prod_{i=1}^{\infty} m(x_i,y_i).$$

Then $M = M_2 \otimes M_2 \otimes \ldots$ is isomorphic to $R_{\bar{m}}(X)$ and the canonical shift α with index 4 on M corresponds to the induced shift $\sigma_{\bar{m}}$ on $R_{\bar{m}}(X)$.

Next We shall look at a relation between certain signature sequences and relatve commutant algebras associated with them.

Definition 2.1.6. Let a be a non-periodic signature sequence on $\mathbf{Z}$. The sequence a is called <u>essentially periodic</u> if there exist non negative integers k and p such that, for any $n \geq k$, $a(n+p) = a(n)$.

Theorem 2.1.7. <u>Let a be a non-periodic signature sequence and</u> σ_a <u>be the associated shift of the hyperfinite</u> II_1<u>-factor</u> R. <u>The sequence a is essentially periodic if and only if there exists a non negative integer</u> r <u>such that</u>

$$C_r(\sigma_a) = \sigma_a^r(R)' \bigcap R \neq \mathbb{C}I.$$

Proof. At first we shall assume that there exists a non negative integer r such that $C_r(\sigma_a) \neq \mathbb{C}I$. Then there exists a word $g(\neq 1) \in G = \amalg_{i=0}^{\infty} G_i, G_i \cong \mathbf{Z}/2\mathbf{Z}$, such that $m(g,e_n) = m(e_n,g)$ for $n \geq r$, where $e_n(n)=1$ and $e_n(i)=0$ for $i \neq n$. For this g, g is expressed as $g = \Sigma_{i=0}^{d} y(i)e_i$ for some $d \geq 0$ and $y(i) \in \{0,1\}$.

Then $\Sigma_{i=0}^{d} a(n-i)y(i) = 0$ for $n \geq r$. We put

$$j = \min\{ i; y(i) \neq 0\}.$$

Furthermore we put, for $n \geq r$,

$$a_0(n) = a(n-d), a_1(n) = a(n-d+1), \ldots, a_{d-j-1}(n) = a(n-j-1).$$

Using these $a_i(n)$ $(0 \leq i \leq d-j-1)$, we put

$$\vec{a_n} = (a_0(n), \ldots, a_{n-j-1}(n))^t.$$

We shall define a $(d-j)\times(d-j)$ matrix A by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & \cdots & & & \\ & & \cdots & \cdots & & & \\ & & \cdots & \cdots & & & 1 \\ y(d) & \cdots & \cdots & \cdots & \cdots & \cdots & y(j+1) \end{pmatrix}$$

Since $\{0,1\}^{d-j}$ is a finite set, there exist non-negative integers k and p such that

$$A^p(A^k\overrightarrow{a_r}) = A^k\overrightarrow{a_r}.$$

Then by iterating powers of A to this above equality, we have

$$A^p(\overrightarrow{a_m}) = \overrightarrow{a_m} \quad \text{for} \quad m \geq r+k.$$

Therefore, by the definition of $\overrightarrow{a_m}$,

$$a(m+p) = a(m) \quad \text{for} \quad m \geq r+k-d .$$

Thus a is essentisllly periodic. Next we shall show the necessity of this theorem. Assume that a is essentially periodic. So there exist non-negative integers k and p such that, for any $n \geq k$, $a(n+p) = a(n)$.

Then we have the following two cases.

(Case I). $a(k)+a(k+1)+\ldots+a(k+p-1) = 0$.

Put $v = u_0u_1\ldots u_{p-1}(\notin \mathbb{CI})$. Then $v \in \sigma^{k+p-1}(R)' \cap R$.

(Case II). $a(k)+a(k+1)+\ldots+a(k+p-1) = 1$.

Put $v = u_0u_1\ldots u_{p-1}u_pu_{p+1}\ldots u_{2p-1}(\notin \mathbb{CI})$. Then $v \in \sigma^{k+2p-1}(R)' \cap R$. This comes from the following fact. $a(k) = a(k+p), a(k+1) = a(k+p+1), a(k+2) = a(k+p+2), \ldots,$ $a(k+p-1) = a(k+2p-1)$. So we get

$$a(k)+a(k+1)+\ldots+a(k+p-1)+a(k+p)+a(k+p+1)+\ldots+a(k+2p-1) = 0.$$

Thus we get the necessity of this theorem. Q.E.D.

Proposition 2.1.8. *Let* α *be a shift on* R. *Let* $N(\alpha)$ *be the set* $\{u \in R;$ u *is a unitary and* $u\alpha^k(R)u^* = \alpha^k(R)$, *for any integer* $k \geq 0\}$ *and* $u \in N(\alpha)$. *Then* $(Adu)\alpha$ *is a shift on* R.

Proof. Put $S = \bigcap_{k=0}^{\infty}(Adu.\alpha)^k(R)$. Take $y \in S$. Thus there is an element $x_1 \in R$ such that

$y = (Adu.\alpha)(x_1) = u\alpha(x_1)u^*$. By $u \in N(\alpha)$, $y \in \alpha(R)$. Similarly there is an element $x_2 \in R$ such that

$$y = (Adu.\alpha)^2(x_2) = u\alpha(u\alpha(x_2)u^*)u^* \in u\alpha(\alpha(R))u^* \subset \alpha^2(R).$$

Continuing this process inductively we have

$y \in \bigcap_{k=0}^{\infty}\alpha^k(R)$. Since $\bigcap_{k=0}^{\infty}\alpha^k(R) = \mathbb{C}I$, we have that $y \in \mathbb{C}I$. Thus $(Adu)\alpha$ is a shift on R. Q.E.D.

Remark 2.1.9. Let α be a Powers'binary shift on R with a unitary generator u. Put $u_n = \sigma^n(u)$. For u_0 and u_1, there exists a unitary $w \in R$ such that $wu_1w^* = u_0$, because $tr(u_1) = tr(\sigma(u_0)) = tr(u_0)$. Then we have $((Adw)\alpha)(u_0) = (Adw)(u_1) = wu_1w^* = u_0$. Thus, for any n, $((Adw)\alpha)^n(u_0) = u_0$. Therefore

$u_0 \in \bigcap_{i=0}^{\infty}((Adw)\alpha)^n(R)$. Hence $(Adw)\alpha$ is not a shift.

Recently Bures and Yin[6] obtain Theorem 2.1.7. independently.

2.2. Uncountably many non-binary shifts of index two.

Powers[19] completely classified binary shifts up to conjugacy on a hyperfinite II_1-factor R. Subsequently Price[23] ingeniously found a non-binary shift with index two on R. Inspired by the construction of Price's non-binary shift with index two on R, in this section, we shall construct uncountably many non-binary shifts on R of index two.

Let X_i be a countable discrete group, m_i a multiplier on X_i and σ_i a m_i-preserving shift of $X_i (i = 1,2,...)$. Consider an injective homomorphism $\Phi_n : X_n \longrightarrow X_{n+1}$ such that $\Phi_n \cdot \sigma_n = \sigma_{n+1}\Phi_n$ and $m_n(x,y) = m_{n+1}(\Phi_n(x),\Phi_n(y))$ for $n = 1,2,3,...$. Let $X_\infty = \varinjlim(X_n,\Phi_n)$ be the inductive limit of $\{(X_n,\Phi_n)\}$. We identify that $X_\infty = \bigcup_{n=1}^{\infty} X_n$. Then (σ_n) induces an injective homomorphism σ_∞ on X_∞ by $\sigma_\infty(x) = \sigma_n(x)$ if $x \in X_n$. Let $x,y \in X_\infty$. We may assume that $x,y \in X_n$ for some n. Put $m_\infty(x,y) = m_n(x,y)$. Then m_∞ is a multiplier on X_∞ and $\sigma_\infty : X_\infty \longrightarrow X_\infty$ becomes a shift on the group X_∞. We treat only the restricted direct product group $X_n = \amalg_{i=0}^{\infty} G_i$, $G_i \cong \mathbf{Z}_2$. Let $a : \mathbf{Z} \longrightarrow \{0,1\}$ be a signature sequence. Let m_a be the corresponding multiplier by (2.1.1). Let σ be the canonical shift. Then clearly σ preserves this multiplier m_a. Similarly let $Y = \amalg_{i=0}^{\infty} H_i$, $H_i \cong \mathbf{Z}_2$. Let F[t] be the polynomial ring over the field $F = \{0,1\}$. Fix a polynomial

$p(t) = c_0 + c_1 t + ... + c_k t^k \in F[t]$ with $c_0 = 1 = c_k$.

Let $F[t]/p(t) = \{f(t)/p(t); f(t) \in F[t]\}$. Consider an embedding

$\Psi: F[t] \longrightarrow F[t]/p(t)$ defined by $\Psi(f(t)) = p(t)f(t)/p(t) = f(t)$.

First we recall the following elementary fact. Let G be a countable discrete group such that $g^2 = 1$ for any $g \in G$. Then G is isomorphic to $\amalg_{i=0}^{\infty} G_i, G_i \cong \mathbf{Z}_2$. So we denote this group product by addition. Such a group turns out to be a vector space over F. In fact the sum is given by the addition of G and the scalar multipication is given by

$$0 \cdot x = 0 \quad \text{and} \quad 1 \cdot x = x.$$

Define a group isomorphism $\theta: X \longrightarrow F[t]$ by,

for $x = (x(0), x(1), \ldots, x(n), \ldots) \in X$,

$\theta(x) = x(0)+x(1)t+x(2)t^2+\ldots+x(n)t^n+\ldots \in F[t]$.

Define a group isomorphism $\gamma: Y \longrightarrow F[t]/p(t)$ by,

for $y = (y(o), y(1), \ldots, y(n), \ldots) \in Y$,

$\gamma(y) = (y(0)+y(1)t+\ldots+y(n)t^n+\ldots)/p(t)$.

Definition 2.2.1. For $X = \amalg_{i=0}^{\infty} G_i$, $Y = \amalg_{i=0}^{\infty} H_i$, where $G_i \cong H_i \cong \mathbf{Z}_2$, and a polynomial $p(t) \in F[t]$, by using the above maps γ, Ψ, θ. Put a group injection $\Phi_p: X \longrightarrow Y$ by $\Phi_p = \gamma^{-1}\Psi\theta$. Then,

for $x = (x(0), x(1), \ldots)$, $(\Phi_p(x))(n) = \sum_{\substack{i+j=n \\ i\geq 0, j\geq 0}} (c_i x(j))$.

The group injection $\Phi_p: X \longrightarrow Y$ is called the one <u>defined by the polynomial</u> p.

Consider the multiplication operator σ_t by t on F[t](or F[t]/p(t)).

$\sigma_t(f(t)) = tf(t)$(or $\sigma_t(f(t)/p(t)) = tf(t)/p(t)$) for

$f(t) \in F[t]$.

Then $\sigma_t = \theta\sigma\theta^{-1}$ on $F[t]$ and $\sigma_t = \gamma\sigma\gamma^{-1}$ on $F[t]/p(t)$. Thus the canonical shift is realized as the multiplication by t. Therefore $\Phi_p \cdot \sigma = \sigma \cdot \Phi_p$ on X.

The following lemma is a refinement of Price[23,Theorem 5.1].

Lemma 2.2.2. Let $a: \mathbf{Z} \longrightarrow \{0,1\}$ be a non-periodic signature sequence and $p \in F[t]$ with a nonzero constant term. Then there exits a non-periodic signature sequence $b: Z \longrightarrow \{0,1\}$ such that

(2.2.2.A)

$$m_b(\Phi_p(x),\Phi_p(y)) = m_a(x,y) \quad \text{for any } x,y \in X.$$

Poof. Let $g \in X$ such that $g(0) = 1$ and $g(i) = 0$ if $i \neq 0$, that is, $g = (1,0,0,\ldots)$. It is sufficient to prove the next (2.2.2.B) for (2.2.2.A).

(2.2.2.B)

$$\begin{cases} m_b(\Phi_p(g),\Phi_p(\sigma^j(g)) = m_a(g,\sigma^j(g)) = 1. \\ m_b(\Phi_p(\sigma^j(g)),\Phi_p(g)) = m_a(\sigma^j(g),g) \end{cases}$$

In fact, take $x = (x(i))_{i\geq 0} = \Sigma_{i\geq 0} x(i)\sigma^i(g)$ and $y = (y(i))_{i\geq 0} = \Sigma_{i\geq 0} y(i)\sigma^i(g)$ in X. Since m_a is a bicharacter,

$$m_a(x,y) = \Pi_{i\geq j\geq 0} m_a(x(i)\sigma^i(g),y(j)\sigma^j(g))$$

$$= \Pi_{i\geq j\geq 0} m_a(x(i)\sigma^{i-j}(g),y(j)g)$$

(by the σ-invariance of m_a)

$$= \Pi_{i\geq j\geq 0} m_a(x(i)\sigma^{i-j}(g), y(j)g)$$
$$\cdot \Pi_{j\geq i\geq 0} m_a(x(i)g, y(j)\sigma^{j-i}(g))$$

$$= \Pi_{i\geq j\geq 0} m_b(x(i)\Phi_p(\sigma^{i-j}(g)), y(j)\Phi_p(g))$$
$$\cdot \Pi_{j\geq i\geq 0} m_b(x(i)\Phi_p(g), y(j)\Phi_p(\sigma^{j-i}(g)))$$

by $\Phi_p\sigma = \sigma\Phi_p$ and the σ-invariance of m_b,

$$= \Pi_{i\geq j\geq 0} m_b(\Phi_p(x(i)\sigma^i(g)), \Phi_p(y(j)\sigma^j(g)))$$
$$\cdot \Pi_{j\geq i\geq 0} m_b(\Phi_p(x(i)\sigma^i(g)), \Phi_p(y(j)\sigma^j(g)))$$

$$= \Pi_{i\geq 0, j\geq 0} m_b(\Phi_p(x(i)\sigma^i(g)), \Phi_p(y(j)\sigma^j(g)))$$

$$= m_b(\Phi_p(x), \Phi_p(y)).$$

Next, we shall show that (2.2.2.B). Put $g' = (1,0,0,\ldots) \in Y$. By the definition of the multiplier m_a,

$$m_a(\sigma^j(g), g) = (-1)^{a(j)} \quad \text{for } j\geq 0.$$

Since $\Phi_p(g) = c_0 g' + c_1\sigma(g') + \ldots + c_k\sigma^k(g')$

and $\Phi_p(\sigma^j(g)) = c_0\sigma^j(g') + c_1\sigma^{j+1}(g') + \ldots + c_k\sigma^{j+k}(g')$,

We have

$$m_b(\Phi_p(\sigma^j(g)), \Phi_p(g)) = \Pi_{k\geq n\geq 0, k\geq \ell\geq 0} m_b(c_\ell\sigma^{j+\ell}(g'), c_n\sigma^n(g'))$$

$$= \Pi_{\substack{j+\ell\geq n \\ k\geq n\geq 0 \\ k\geq \ell\geq 0}} (-1)^{c_n c_\ell b(j+\ell-n)}$$

Similarly

$$m_b(\Phi_p(g),\Phi_p(\sigma^j(g))) = \prod_{\substack{k\geqq n\geqq 0\\ k\geqq \ell\geqq 0\\ n\geqq j+\ell}} m_b(c_n\sigma^n(g'),c_\ell\sigma^{j+\ell}(g'))$$

$$= \prod_{\substack{k\geqq n\geqq 0\\ k\geqq \ell\geqq 0\\ n\geqq j+\ell}} (-1)^{c_n c_\ell} b(n-j-\ell)$$

Thus we need the following equations (2.2.2.C.1),(2.2.2.C.2) in order to establish (2.2.2.B), that is,

(2.2.2.C.1)

$$a(j) = \sum_{j+\ell\geqq n} c_\ell c_n b(j+\ell-n) \quad \text{for } j\geqq 0$$

(2.2.2.C.2)

$$0 = \sum_{n\geqq j+\ell} c_\ell c_n b(n-j-\ell) \quad \text{for } j\geqq 0$$

Put $q_0 = c_0c_0+c_1c_1+\ldots+c_kc_k$, $q_1 = c_0c_1+c_1c_2+\ldots+c_{k-1}c_k,\ldots,$ $q_k = c_0c_k$. Furthermore we put $q_{-t} = q_t$ for $t = 1,2,..,k$. For (2.2.2.C.1), we demand the following.

We put

(2.2.2.D)

$$q_t^{(j)} = \begin{cases} q_t & \text{if } j+t > 0 \\ 0 & \text{if } j+t \leqq 0 \end{cases} \quad \text{for } j > 0$$

$$q_t^{(0)} = \begin{cases} q_0 & \text{if} \quad t = 0 \\ & \\ 0 & \text{if} \quad t \neq 0 \end{cases} \qquad \text{for} \quad j = 0$$

Then it is sufficient to show the existence of the signature sequence b satisfying, for $j \geq 0$,

(2.2.2.E)

$$a(j) = q_k^{(j)}b(j+k)+q_{k-1}^{(j)}b(j+k-1) + \ldots + q_{-k}^{(j)}b(j-k)$$

For $j = 0$, the above (2.2.2.E) becomes as follows.

$$a(0) = q_k^{(0)}b(k)+q_{k-1}^{(0)}b(k-1)+\ldots+q_0^{(0)}b(0)+\ldots+q_{-k}^{(0)}b(-k) = q_0b(0).$$

If we put $b(0) = 0$, then the above equation (2.2.2.E) for $j = 0$ holds. For $j = 1$,

$$a(1) = q_k^{(1)}b(k+1)+q_{k-1}^{(1)}b(k)+\ldots+q_{-1}^{(1)}b(0)+\ldots+q_{-k}^{(1)}b(1-k)$$

By (2.2.2.D), since $q_{-1}^{(1)} = \ldots = q_{-k}^{(1)} = 0$ and $q_k^{(1)} = q_k = c_0c_k = 1$, if we put $b(1) = \ldots = b(k) = 0$, then $b(k+1)$ is determined as $a(1)$ so that (2.2.2.E) for $j = 1$ is satisfied. Inductively we can take $b(s)$ for $s \geq 0$ which satisfy (2.2.2.E) for $j > 0$. In fact, put

$$b(j+k) = a(j)+q_{k-1}^{(j)}b(j+k-1)+\ldots+q_{-k}^{(j)}b(j-k).$$

Since we put $b(0) = b(1) = \ldots = 0$ for the above b, (2.2.2.C.2) holds. On the other hand, put $b(-s) = b(s)$ for $s \geq 0$. Next, we check the condition (2.2.2.E) of $b(j)$ for $j < 0$. For $j < 0$, we

put (2.2.2.F)

$$q_t^{(j)} = \begin{cases} 0 & \text{if } j+t \geq 0 \\ q_t & \text{if } j+t < 0. \end{cases}$$

Put $i = -s$ for $s > 0$, we shall show that

(2.2.2.G)

$$a(-s) = q_k^{(-s)}b(-s+k)+q_{k-1}^{(-s)}b(-s+k-1)+\ldots+q_0^{(-s)}b(-s)+\ldots.$$
$$+q_{-k+1}^{(-s)}b(-s+k-1)+q_{-k}^{(-s)}b(-s-k).$$

By the way we have already obtainded the following.

(2.2.2.H)

$$a(s) = q_k^{(s)}(b+k) + q_{k-1}^{(s)}b(s+k-1) + \ldots + q_0^{(s)}b(s) + \ldots$$
$$+ q_{-k+1}^{(s)}b(s-k+1) + q_{-k}^{(s)}b(s-k) \qquad \text{for } s > 0$$

Substituting $a(-s)$(resp. $b(-s)$) for $a(s)$(resp. $b(s)$) for $s > 0$ in (2.2.2.H), we get

(2.2.2.I)

$$a(-s) = q_k^{(s)}b(-s-k) + q_{k-1}^{(s)}b(-s-k+1) + \ldots + q_{-s}^{(s)}b(0) + \ldots.$$
$$+ q_{-k+1}^{(s)}b(-s+k-1) + q_{-k}^{(s)}b(-s+k).$$

To show (2.2.2.G), it is enough to look at the following relation.

(2.2.2.J)

$$q_{-K}^{(s)} = q_K^{(-s)},\ q_{-k+1}^{(s)} = q_{k-1}^{(-s)},\ldots,q_{k-1}^{(s)} = q_{-k+1}^{(-s)},q_k^{(s)} = q_{-K}^{(-s)}.$$

By the way the followings hold by (2.2.2.D) and (2.2.2.F).

(2.2.2.K)

$$\begin{cases} q_{-i}^{(s)} = q_{-i} & \text{for } s-i > 0 \text{ if and only if } q_i^{(-s)} = q_i \quad \text{for } -s+i < 0. \\ q_{-i}^{(s)} = 0 & \text{for } s-i \leqq 0 \text{ if and only if } q_i^{(-s)} = 0 \quad \text{for } -s+i \geqq 0. \end{cases}$$

Therefore (2.2.2.J) is satisfied. Thus (2.2.2.E) holds for $j < 0$ by $q_t^{(j)}$ of (2.2.2.F). Finally we shall show that this signature sequence b is non-periodic. For $j > 0$, $q_t^{(j)} = 0$ for $j+t \leqq 0$ where $-k \leqq t \leqq k$ and $-k \leqq t < j+t \leqq 0$. Since $b(s) = 0$ for $-k \leqq s \leqq k$, $b(j+t) = 0$. Hence

$$q_t^{(j)} b(j+t) = q_t b(j+t) \quad \text{for } j+t \leqq 0.$$

For $j < 0$, $q_t^{(j)} = 0$ for $j+t \geqq 0$ where $-k \leqq t \leqq k$. Since $b(s) = 0$ for $-k \leqq s \leqq k$ and $0 \leqq j+t < t \leqq k$, $b(j+t) = 0$. Hence $q_t^{(j)} b(j+t) = q_t b(j+t)$. For $j = 0$, $q_t^{(0)} = 0$ for $t \neq 0$ where $-k \leqq t \leqq k$. Since $b(t) = 0$ for $-k \leqq t \leqq k$, $q_t^{(0)} b(t) = q_t b(t)$ for $-k \leqq t \leqq k$.

Thus (2.2.2.E) turns out to be

(2.2.2.L)

$$a(j) = q_k b(j+k) + q_{k-1} b(j+k-1) + \ldots + q_0 b(j) + \ldots + q_{-k} b(j-k)$$

for $j \in \mathbf{Z}$.

Suppose that b is periodic. Then a is periodic by (2.2.2.L). This contradicts the assumption. Therefore there exists a non-periodic signature sequence b on $\mathbf{Z}$ which satisfies (2.2.2.A). Q.E.D.

Thus, using a polynomial $p(t) = c_0+c_1t+\ldots+c_kt^k$ with $c_0 = c_k = 1$ and a non-periodic signature sequence a on $\mathbf{Z}$ which induces a multiplier m_a on $X = \amalg_{i=0}^{\infty}G_i$, $G_i \tilde{=} \mathbf{Z}_2$, we found a group injection from X to $Y = \amalg_{i=0}^{\infty}H_i$, $H_i \tilde{=} \mathbf{Z}_2$, and a non-periodic signature sequence b which induces a multiplier m_b on Y satisfying (2.2.2.A). Continuing this procedure for a sequence of polynomials $p_\ell(t) = c_{\ell,0}+c_{\ell,1}t+\ldots+c_{\ell,k(\ell)}t^{k(\ell)}$ with $c_{\ell,0} = c_{\ell,k(\ell)} = 1$ and $X_\ell = \amalg_{i=0}^{\infty}G_i^{(\ell)}$, $G_i^{(\ell)} \tilde{=} \mathbf{Z}_2$ for $\ell = 1,2,\ldots$, we can get a family $\{m_{a_\ell};\ell = 1,2,\ldots\}$ of multipliers on X_ℓ induced by a non-periodic signature sequences a_ℓ on $\mathbf{Z}$ which satisfies (2.2.2.A). Hence we have the following proposition 2.2.3.

Proposition 2.2.3. Let $X_\ell = \amalg_{i=0}^{\infty}G_i^{(\ell)}$, $G_i^{(\ell)} \tilde{=} \mathbf{Z}_2$ and fix a signature sequence a. Consider a sequence $p = (p_1,p_2,\ldots)$ of polynomials $p_\ell(t) = c_{\ell,0}+c_{\ell,1}t+\ldots+c_{\ell,k(\ell)}t^{k(\ell)}$ with $c_{\ell,0} = c_{\ell,k(\ell)} = 1$ for $\ell = 1,2,\ldots$. Let $\Phi_{p_\ell}:X_\ell \longrightarrow X_{\ell+1}$ be the group injection defined by the polynomial p_ℓ. Then there exist multipliers m_{a_ℓ} on X_ℓ $(\ell = 1,2,\ldots)$ induced by some non-periodic signature sequences a_ℓ on $\mathbf{Z}$ which satisfy $m_{a_{\ell+1}}(\Phi_{p_\ell}(x),\Phi_{p_\ell}(y)) = m_{a_\ell}(x,y)$ for $x,y \in X_\ell$ and $a_1 = a$.

Now under the above condition, for a sequence $p = (p_1,p_2,\ldots)$ of polynomials $\{p_\ell, \ell = 1,2,3,\ldots\}$, we put $X_{[p]} = \varinjlim(X_\ell,\Phi_{p_\ell})$. Define a multiplier $m_{[a,p]}$ on $X_{[p]}$ by $m_{[a,p]}(x,y) = m_{a_\ell}(x,y)$ if $x,y \in X_\ell$. Then $R_{m_{[a,p]}}(X_{[p]})$ is a hyperfinite II_1-factor since anti-symmetric bicharactor $\omega_{m_{[a,p]}}$ is non-degenerate by

proposition 2.1.1. We have the canonical group endomorphism $\sigma_{[p]}$ which is a shift on $X_{[p]}$. Then $\sigma_{[p]}$ induces a shift $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$.

Definition 2.2.4. Under the above notation, for sequences $p = (p_1, p_2, \ldots)$ of polynomials $p_\ell(t)$ with nonzero constant terms and non-periodic signature sequences a, shifts $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$ are called <u>shifts of Price type</u>.

The normalizer of a shift σ on a hyperfinite II_1-factor R, denoted $N(\sigma)$(cf.[19]), consists of those unitary elements $u \in R$ so that $u\sigma^k(R)u^* = \sigma^k(R)$ for all $k=1,2,\ldots$. Here we shall show that the normalizer of a shift of Price type is the set of elements of the underling group up to scalar elements. This fact is proved by Price in [23]. Here we shall restate his result in terms of twisted group von Neumann algebras. We shall prepare the notations. Put $\sigma_{[a,p]} = \sigma$, $m_{[a,p]} = m$, $X_{[p]} = X$ and $R_{m_{[a,p]}}(X_{[p]}) = R$.
Let E be the conditional expectation from R onto $\sigma(R)$. Put $\theta = 2E-I$ on R. Then θ is an automorphism on R and $\theta(\lambda_m(x)) = \lambda_m(x)$ if $x \in \sigma(X)$ and $\theta(\lambda_m(x)) = -\lambda_m(x)$ if $x \notin \sigma(X)$. Put

$$\Lambda(X) = \{ \Sigma_{i=1}^{p} \mu_i \lambda_m(x_i) \; ; \; x_i \in X, \; \mu \in \mathbb{C} \text{ and } p \in \mathbb{N} \},$$
$$\Lambda(X_n) = \{ \Sigma_{i=1}^{p} \mu_i \lambda_m(x_i) \; ; \; x_i \in X_n, \; \mu_i \in \mathbb{C} \text{ and } p \in \mathbb{N} \} \text{ and}$$
$$W(X) = \{ \mu\lambda_m(x) \; ; x \in X \text{ and } \mu \in \mathbf{T} \}.$$
$$X_n^{(\ell)} = \{ x \in X_n ; \; x = (x(i)), \; x(i) = 0 \text{ for } i \geq \ell+1 \}.$$

Proposition 2.2.5. Under the condition of proposition 2.2.3., $N(\sigma) = W(X)$.

Following after [23], we shall give the proof of this proposition 2.2.5. At first we shall show that $W(X) \subseteq N(\sigma)$. Let $x \in X$. Fix $k \in \mathbb{N}$. Then since $\sigma^k(R_m(X))$ is generated by $\lambda_m(\sigma^k(X))$ and $\lambda_m(x)\lambda_m(y)\lambda_m(x)^* = m(x,y)m(\overline{y,x})\lambda_m(y)$, we have $\lambda_m(x) \in N(\sigma)$. Thus $W(X) \subseteq N(\sigma)$.

Next, we shall show that $N(\sigma) \subseteq W(X)$. This fact comes from the lemmas below.

Lemma 2.2.6. $\theta(s) = \pm s$ for $s \in N(\sigma)$. For any $\ell \in \mathbb{N}$ and $n \in \mathbb{N}$, there exist $x \in X_n^{(\ell)}$ and $s_{\ell+1} \in R$ such that $s = \lambda_m(x)\sigma^{\ell+1}(s_{\ell+1})$.

Proof. This is derived from the same method of [19]. Q.E.D.

Lemma 2.2.7. Let $s \in N(\sigma)$ and $t' \in \Lambda(X_n)$ for some $n \in \mathbb{N}$. Assume that there exist $x \in X_n^{(\ell)}$ and $s_{\ell+1} \in R$ such that $s = \lambda_m(x)\sigma^{\ell+1}(s_{\ell+1})$. Then there exist a $t \in \Lambda(X_n)$ and $t_{\ell+1} \in \Lambda(X_n)$ such that $t = \lambda_m(x)\sigma^{\ell+1}(t_{\ell+1})$ and $\|s-t\|_2 \leq \|s-t'\|_2$

Proof. Put $g = (1,0,0,0,\ldots) \in X_n$. If $\theta(s) = s$, replace t' with $t_0 = E(t')$. Then $\|s-t_0\|_2 = \|E(s-t')\|_2 \leq \|s-t'\|_2$. If $\theta(s) = -s$, replace t' with $t_0 = \lambda_m(g)E(\lambda_m(g)t')$, in which case

$\|s-t_0\|_2 = \|\lambda_m(g)E(\lambda_m(g)s)-\lambda_m(g)E(\lambda_m(g)t')\|_2$

$\leq \|E(\lambda_m(g)s-\lambda_m(g)t')\|_2 \leq \|s-t'\|_2$. In either case, if $s = \lambda_m(g)^{k_0}\sigma(s_1)$, then there exists $t_1 \in \Lambda(X_n)$ such that

$t_0 = \lambda_m(g)^{k_0}(t_1)$. Then we have $\|s_1-t_1\|_2 = \|s-t_0\|_2 \leq \|s-t'\|_2$ and $s_1 = \lambda_m(g)^{k_1}\sigma(s_2)$. Proceeding as above we may replace t_1 with an element of the form $\lambda_m(g)^{k_1}\sigma(t_2)$ where $t_2 \in \Lambda(X_n)$ such that $\|s_1-\lambda_m(g)^{k_1}\sigma(t_2)\|_2 \leq \|s_1-t_1\|_2 \leq \|s-t'\|_2$. Hence $\|s-\lambda_m(g)^{k_0}\lambda_m(\sigma(g))^{k_1}\sigma^2(t_2)\|_2 \leq \|s-t'\|_2$. Continuing this process ℓ steps yields the result. Q.E.D.

Lemma 2.2.8. Let $s \in N(\sigma)$. Let $n \in \mathbb{N}$ be a number such that, for some $t' \in \Lambda(X_n)$, $\|s-t'\|_2 < 1$. Let $N = \sup(\{q \in \mathbb{N}$; there exist $x \in X_n^{(\ell)}$, $s_{\ell+1} \in R$ and $\ell \geq q$ such that $s = \lambda_m(x)\sigma^{\ell+1}(s_{\ell+1})\})$. Then $N < \infty$.

Proof. By the assumption, there exists $k \in \mathbb{N}$ such that $\|s-\Sigma_{i=1}^{p}\gamma_i'\lambda_m(y_i')\|_2 < 1$ for $\gamma_i' \in \mathbb{C}$ and $y_i' \in X_n^{(k)}$ and $i = 1,2,..,p$. Suppose that $N = \infty$, then there are an $\ell > k$ and $x \in X_n^{(\ell)}$ such that $s = \lambda_m(x)\sigma^{\ell+1}(s_{\ell+1})$. Then by the idea of lemma 2.2.7, there exist $\gamma \in \mathbb{C}$ and $y \in X_n^{(k)}$ such that $y(i) = x(i)$ for $0 \leq i \leq k$ and $\|s-\gamma\lambda_m(y)\|_2 \leq \|s-\Sigma_{i=1}^{p}\gamma_i'\lambda_m(y_i')\|_2 < 1$.

Let $z = (z(i)) \in X_n^{(\ell)}$ for $z(i) = 0$ $(i = 0,1,..,k)$ and $z(i) = x(i)$ for $i = k+1,\ldots,\ell$. Let $s' \in R$ such that $s' = \lambda_m(z)\sigma^{\ell+1}(s_{\ell+1})$. Then $\|s-\gamma\lambda_m(y)\|_2 < 1$ yields $\|s'-\gamma I\|_2 < 1$. But $\|s'-\gamma I\|^2 = 1+|\gamma|^2-2\mathrm{Re}(\mathrm{tr}(\bar{\gamma}s'))$, and $\mathrm{tr}(s') = 0$. For if j is the first index greater than k for which $x(j) \neq 0$, $\mathrm{tr}(s') = \mathrm{tr}(\alpha^{-j}(s')) = \mathrm{tr}(\theta(\alpha^{-j}(s')) = -\mathrm{tr}(\alpha^{-j}(s')) = -\mathrm{tr}(s')$. Hence $\|s'-\gamma I\|_2^2 = 1+|\gamma|^2 > 1$, a contradiction. Thus N is finite. Q.E.D.

The proof of proposition 2.2.5. Let $s \in N(\sigma)$. Let $n \in \mathbb{N}$ be sufficiently large such that $\|s-\Sigma_{i=1}^{p}\mu_i\lambda_m(x_i)\|_2 < 1$ for some p, $\mu_i \in \mathbb{C}$ and $x_i \in X_n$ and $i = 1,2,\ldots$. By the lemma 2.2.8 , there is a maximum $\ell \in \mathbb{N}$, $s_{\ell+1} \in R$ and $x \in X_n^{(\ell)}$ such that $s = \lambda_m(x)\sigma^{\ell+1}(s_{\ell+1})$. Since s and $\lambda_m(x)$ belong to $N(\sigma)$, hence $\sigma^{\ell+1}(s_{\ell+1})$ and $s_{\ell+1}$ belong to $N(\sigma)$. Thus $\theta(s_{\ell+1}) = \pm s_{\ell+1}$ by lemma 2.2.6, if $\theta(s_{\ell+1}) = -s_{\ell+1}$,then for $g = (1,0,0,0,\ldots)$ in X, $s_{\ell+2} = \lambda_m(g)s_{\ell+1}$ belongs to $\sigma(R)$. Hence $s_{\ell+1} = \lambda_m(g)s_{\ell+2}$. But this implies a contradiction about the maximality of ℓ, therefore $\theta(s_{\ell+1}) = s_{\ell+1}$,that is, $s_{\ell+1} \in \sigma(R)$. From $s_{\ell+1} \in \sigma(R)$, $\sigma^{-1}(s_{\ell+1}) \in N(\sigma)$. Similarly $\sigma^{-1}(s_{\ell+1}) \in \sigma(R)$, thus $s_{\ell+1} \in \sigma^2(R)$. Iterating this procedure, we get $s_{\ell+1} \in \bigcap_{n\geq 1}\sigma^n(R)$. Since σ is a shift, $s_{\ell+1}$ is a scalar. So we get this proposition 2.2.5. Q.E.D.

Using this proposition 2.2.5 , we shall show the next proposition 2.2.9 which states the shifts on the hyperfinite II_1-factor induced by the shifts on the group are not conjugate if the shifts on the groups are not conjugate.

Proposition 2.2.9. Take two sequences of polynomials with non-zero constant terms , $p = (p_i)$ and $q = (q_i)$ for $i = 1,2,\ldots$, and two non-periodic signature sequences a and b. If two shifts of Price type $\sigma_{[a,p]}$ and $\sigma_{[b,q]}$ are conjugate on the hyperfinite II_1-factor, then $(\sigma_{[p]},X_{[p]})$ and $(\sigma_{[q]},X_{[q]})$ are

conjugate, where $\sigma_{[p]}$ is the induced shift on $X_{[p]}$ from $\sigma_{[a,p]}$.

Proof. The shifts $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$ induces shifts $\tilde{\sigma}_{[a,p]}: N(\sigma_{[a,p]})/\mathbf{T} \longrightarrow N(\sigma_{[a,p]})/\mathbf{T}$. By Proposition 2.2.5, $N(\sigma_{[a,p]})/T \cong X_{[p]}$ and $\tilde{\sigma}_{[a,p]} = \sigma_{[p]}$. Therefore if $\sigma_{[a,p]}$ and $\sigma_{[b,q]}$ are conjugate, then $(\sigma_{[p]}, X_{[p]})$ and $(\sigma_{[q]}, X_{[q]})$ are conjugate. Q.E.D.

In the following we shall construct uncountably many non-binary shifts. At first we shall choose countable irreducible polynomials $p_k(t)(\neq t)$ for $k = 1,2,\ldots$ which are distinct each other. Take $a = (a(1),a(2),a(3),\ldots) \in \prod_{i=1}^{\infty} \mathbf{Z}_2$. Put

$$X^a = \left\{ \begin{array}{l} g(t)/f(t) \ ; \ g(t) \in F[t] \\ \qquad f(t) \text{ satisfies that if } f(t) = p_1(t)^{k_1} \ldots p_n(t)^{k_n}, \\ \qquad k_i \neq 0, \text{ then } a(i) \neq 0. \end{array} \right\}$$

That is, X^a is the set of rational functions whose denominator may have $p_i(t)$ factor only if $a(i) \neq 0$. This X^a is, of course, isomorphic to $\coprod_{i=0}^{\infty} G_i$, where $G_i \cong \mathbf{Z}_2$. When we consider this σ_t on X^a, we denote it by σ^a.

Lemma 2.2.10. Let a and b elements in $\prod_{i=1}^{\infty} \mathbf{Z}_2$. Then $a = b$ if and only if (σ^a, X^a) and (σ^b, X^b) are conjugate.

Proof. If $a \neq b$, then there exists an n_0 such that ($a(n_0) = 1$ and $b(n_0) = 0$) or ($a(n_0) = 0$ and $b(n_0) = 1$). Hence

we may suppose that $a(n_0) = 1$ and $b(n_0) = 0$. If σ^a and σ^b are conjugate, then $p_{n_0}(\sigma^a)$ and $p_{n_0}(\sigma^b)$ are conjugate. But (Image of $p_{n_0}(\sigma^a)) = X^a$ and (Image of $p_{n_0}(\sigma^b)) = X^b$. In fact, take an element $g(t)/f(t) \in X^a$. Then $g(t)/(p_{n_0}(t)f(t)) \in X^a$ and $g(t)/f(t) = p_{n_0}(t)g(t)/p_{n_0}(t)f(t) \in \mathrm{Im}(p_{n_0}(\sigma^a))$. Hence $\mathrm{Im}(p_{n_0}(\sigma^a)) = X^a$. On the other hand, $1 \in X^b$, but $1 \notin \mathrm{Im}(p_{n_0}(\sigma^b))$. If $p_{n_0}(t)g(t)/f(t) = 1$, then $p_{n_0}(t)g(t) = f(t)$. But $p_{n_0}(t)$ does not divide $f(t)$. This is a contradiction, therefore $1 \notin \mathrm{Im}(p_{n_0}(\sigma^b))$. Thus $\mathrm{Im}(p_{n_0}(\sigma^b)) \neq X^b$. Q.E.D.

Put $X_0^a = F[t]$, $X_1^a = F[t]/p_1(t)^{a(1)}$, ... , $X_\ell^a = F[t]/(p_1(t)^{a(1)}p_2(t)^{a(2)} \ldots p_\ell(t)^{a(\ell)})^\ell$.

Then we have $\bigcup_{\ell=0}^{\infty} X_\ell^a = X^a$. Furthermore, the way of this embedding from X_ℓ^a to $X_{\ell+1}^a$ is defined by the multiplication of the polynomial

$$(p_1(t))^{a(1)}(p_2(t))^{a(2)} \ldots (p_\ell(t))^{a(\ell)}(p_{\ell+1}(t))^{(\ell+1)a(\ell+1)}.$$

In particular, the Powers' binary shift is associated to the sequence $a = (a(1),a(2), \ldots) = (0,0,0, \ldots)$ by example 2.1.2. Thus we get the following theorem 2.2.8, combined with proposition 2.2.6 and lemma 2.2.7.

Theorem 2.2.11. <u>There exist uncountably many non-conjugate non-binary shifts of index two on the hyperfinite II_1-factor.</u>

We have the following proposition similar to Price [23, Theorem 4.5] which also shows that above shifts are not binary.

Proposition 2.2.12.. Let G be a countable discrete group and σ a shift. Suppose $[G;\sigma(G)] = 2$ and for any g in G, $g^2 = 1$. Then σ is conjugate to the canonical shift on $G = \amalg_{i=0}^{\infty} \mathbf{Z}_2$ if and only if $[G:\langle \sigma^k(g); k = 0,1,2,\ldots\rangle] < \infty$ for $g(\neq 1) \in G$.

Proof. In the following , put $\{g\}^{\sim} = \langle \sigma^k(g); k = 0,1,2,..\rangle$. (sufficiency). Suppose that $[G:\{g\}^{\sim}] < +\infty$ for any $g(\neq 1) \in G$. Since $[G:\sigma(G)] = 2$, there exists an element $a \notin \sigma(G)$ such that $G = \sigma(G) \bigcup \sigma(G)a$.

Case 1. If $[G: \{a\}^{\sim}] = 1$, then we have $G = \{a\}^{\sim}$. Thus $G \cong \amalg_{i=0}^{\infty} \mathbf{Z}_2$, $\mathbf{Z}_2 = \{0,1\}$ and σ is conjugate to canonical shift on $\amalg_{i=0}^{\infty} \mathbf{Z}_2$. The conjugacy comes from the mapping Φ from $\amalg_{i=0}^{\infty} \mathbf{Z}_2$ onto G such that $\Phi(x(0),x(1),\ldots,x(k),0,0,\ldots) = a^{x(0)}\ldots\sigma^k(a)^{x(k)}$. For any x in G, x is uniquely represented by $x = a^{x(0)}\ldots\sigma^k(a)^{x(k)}$, where $x(i) \in \mathbf{Z}_2$, since $a \notin \sigma(G)$ and σ is faithful.

Case 2. If $[G:\{a\}^{\sim}] > 1$, then we have $b_1(\neq 1) \in G$ such that $b_1 \notin \{a\}^{\sim}$ and $[\{a,b_1\}^{\sim}:\{a\}^{\sim}] > 1$. The existence of b_1 comes from $\{a\}^{\sim} \neq G$. Furthermore we may assume that $b_1 \notin \sigma(G)$. Because if $b_1 \in \sigma(G)$,then, renaming $b_1 a$ by b_1, we have $b_1 \notin \{a\}^{\sim}$ and $b_1 \notin \sigma(G)$. Further we get $ab_1 \in \sigma(G)$. Because, by $b_1 \in \sigma(G)a$, then $ab_1 = a\sigma(g_1)a = aa\sigma(h_1) = \sigma(h_1) \in \sigma(G)$ from $\sigma(G) \lhd G$ and $a^2 = 1$. On the other hand, from $ab_1 \in \sigma(G)$, we denote $ab_1 = \sigma(g_1)$. If $g_1 \in \sigma(G)$, then we have $ab_1 = \sigma(\sigma(g_2))$. If $g_n \in \sigma(G)$ for any n ,then $ab_1 \in \bigcap_{k\geq 0}\sigma^k(G) = \{1\}$ since σ is a shift on G. Hence $ab_1 = 1$. Thus $a = b_1$. But $b_1 \notin \{a\}^{\sim}$. This is a

contradiction. Therefore there exists an integer m_1 such that $ab_1 = \sigma^{m_1}(b_2)$ and $b_2 \notin \sigma(G)$. Furthermore we may assume that $b_2 \notin \{a\}^{\sim}$. Because if $b_2 \in \{a\}^{\sim}$, then $ab_1 \in \{a\}^{\sim}$. So $b_1 \in a\{a\}^{\sim} \subsetneqq \{a\}^{\sim}$. This is a contradiction. As $b_1 = a\sigma^{m_1}(b_2)$, we have $\{b_2,a\}^{\sim} \supsetneqq \{b_1,a\}^{\sim}$. For the condition of b_2, we have $b_2 \notin \{a\}^{\sim}$ and $b_2 \notin \sigma(G)$. Repeating the above process, we get $ab_2 = a\sigma(g_2)a = aa\sigma(h_2) = \sigma(h_2) \in \sigma(G)$. Therefore $ab_2 = \sigma(h_2)$. If $h_2 \in \sigma(G)$, then $ab_2 = \sigma(\sigma(h_3))$. If this process continues, we have $ab_2 \in \bigcap_{k\geq 0}\sigma^k(G) = \{1\}$. Since $ab_2 = 1$, $a = b_2$. But $b_2 \notin \{a\}^{\sim}$. This is a contradiction. Therefore there exists an integer m_2 such that $ab_2 = \sigma^{m_2}(b_3)$ and $b_3 \notin \sigma(G)$. Furthermore $b_3 \notin \{a\}^{\sim}$. Because if $b_3 \in \{a\}^{\sim}$, then $ab_2 \in \{a\}^{\sim}$. Thus $b_2 \in a\{a\}^{\sim} \subsetneqq \{a\}^{\sim}$. This is a contradiction. Repeating the above process, we have

$$\ldots \supsetneqq H_3 = \{b_3,a\}^{\sim} \supsetneqq H_2 = \{b_2,a\}^{\sim} \supsetneqq H_1 = \{b_1,a\}^{\sim}.$$

Here $H_k = \{b_k,a\}^{\sim}$, where $b_k \notin \{a\}^{\sim}$ and $b_k \notin \sigma(G)$. Since $[G:H_k] = [G:\{b_k,a\}^{\sim}] \leq [G:\{a\}^{\sim}] < +\infty$, there exists an integer k such that $H_k = H_{k+1}$. Hence $b_{k+1} \in H_k$. For simplicity, we denote $b = b_k$, $m = m_k$. Since $ab_k = \sigma^{m_k}(b_{k+1})$, $b_{k+1} = \sigma^{-m}(ab)$. We denote $\{a,b\}^{\sim} = H$. Then $\sigma^{-m}(ab) \in H_k = \{a,b\}^{\sim}$. Therefore, there exist polynomials $p_0(t),q_0(t) \in F[t]$ such that

(2.2.12) $\sigma^{-m}(ab) = \langle a,p_0\rangle\langle b,q_0\rangle$ where

$\langle a,p\rangle = a^{c_0}\sigma(a^{c_1})\sigma^2(a^{c_2})\ldots\sigma^n(a^{c_n})$

for $p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n$.

Multiplying (2.2.12) by σ^m, there exist polynomials $p(t)$, $q(t)$ ε $F[t]$ such that $\langle a,p\rangle = \langle b,q\rangle$. Furthermore, without loss of generality, we may suppose that $p(t)$ and $q(t)$ are relatively prime over $F[t]$. Because if $p = p'r, q = q'r$ with degree$(r) \geq 1$, then we have $\langle a',r\rangle = \langle b',r\rangle$ where $a' = \langle a,p'\rangle, b' = \langle b,q'\rangle$. Since $\langle a',r\rangle = \langle b',r\rangle$, then $\langle a'b',r\rangle = 1$.
If $r(t) = t^p + c_{p+1}t^{p+1} + \ldots + c_d t^d$, then we have $\sigma^p(a'b')\sigma^{p+1}((a'b')^{c_{p+1}})\ldots\sigma^d((a'b')^{c_d}) = 1$. That is, $\sigma^p(a'b') = \sigma^{p+1}((a'b')^{c_{p+1}})\ldots\sigma^d((a'b')^{c_d})$. Hence

$\sigma^p(a'b') \varepsilon \bigcap_{n\geq 1}\sigma^n(G) = \{1\}$. By the injectivity of σ^p, $a'b' = 1$. Thus $a' = b'$. Therefore

$\langle a',r\rangle = \langle b',r\rangle$ if and only if $a' = b'$. That is,

$\langle a,p\rangle = \langle b,q\rangle$ if and only if $\langle a,p'\rangle = \langle b,q'\rangle$. By considering p', q', we may suppose that p and q are relatively prime over $F[t]$. So we assume that $\langle a,p\rangle = \langle b,q\rangle$ are relatively prime. By the induction about (deg p +deg q), we can show that there exists an element $c \varepsilon G$ such that $\langle c,p\rangle = b$ and $\langle c,q\rangle = a$. Then the the case $\deg(p)+\deg(q) = 0$ is trivially satisfied. So we assume that $\deg(p)+\deg(q) > 0$. We put $p(t) = \ell_0 + \ell_1 t + \ldots + \ell_n t^n$, $q(t) = m_0 + m_1 t + \ldots + m_n t^n$. Since p and q are relatively prime , if $a \notin \sigma(G), b \notin \sigma(G)$, then $\ell_0 = m_0 = 1$. Therefore, by $\langle a,p\rangle = \langle b,q\rangle$,

$ab = \sigma(a)^{\ell_1}\sigma(b)^{m_1}\ldots\sigma^n(a)^{\ell_n}\sigma^n(b)^{m_n}$. Let k be the first index $k(\geq 1)$ such that $\ell_k + m_k \neq 0$. Then $ab \varepsilon \sigma^k(G)$.
$\langle a,p(t)\rangle = \langle b,q(t)\rangle$. So $\langle a,p(t)\rangle\langle a,q(t)\rangle = \langle b,q(t)\rangle\langle a,q(t)\rangle$.

$\langle a, p(t) + q(t) \rangle = \langle ab, q(t) \rangle$. Then
$\sigma^{-k}(\langle a, p(t) + q(t) \rangle) = \sigma^{-k}(\langle ab, q(t) \rangle)$. Thus
$\langle a, (p(t) + q(t))/t^k \rangle = \langle \sigma^{-k}(ab), q(t) \rangle$. By the step of induction, there exists an element $c \in G$ such that $\langle c, q(t) \rangle = a$
$\langle c, (p(t) + q(t))/t^k \rangle = \sigma^{-k}(ab)$.

So we get $\langle c, p(t) + q(t) \rangle = ab$. That is ,
$\langle c, p(t) \rangle \langle c, q(t) \rangle = ab$. Since $\langle c, q \rangle = a$, then $\langle c, q \rangle = a$ and $\langle c, p \rangle = b$. Therefore the induction is completed.

Therefore $\{c\}^{\sim} \supseteq \{a, b\}^{\sim} \supseteq \{a, b_1\}^{\sim}$.
$[G:\{c\}^{\sim}] \leq [G:\{a, b_1\}] < [G:\{a\}^{\sim}]$. Continuing this process, there exists an element $c' \in G$ such that $[G:\{c'\}^{\sim}] = 1$. Therefore $G = \{c'\}^{\sim}$.

(necessity) Let σ be a shift on G which is conjugate to the canonical shift on $\coprod_{i=0}^{\infty} Z_2$ by a map Φ. Then there exists the element $g \in G$ such that $\Phi(e_0) = g$ where $e_0 \in \coprod_{i=0}^{\infty} Z_2$ and $e_0(0) = 1$, $e_0(i) = 0$ if $i \neq 0$. Then $G = \{g\}^{\sim}$.
If $h \in G$, then $n = g^{k_0} \sigma(g^{k_1}) \ldots \sigma^n (g^{k_n} h)$. If $k_n = 1$, then $[G:\{h\}^{\sim}] = [\{g\}^{\sim}:\{h\}^{\sim}] = 2^n < +\infty$. Because
$\#(F[t]/h(t)F[t]) = 2^n$ where $h(t) = k_0 + k_1 t + \ldots + k_n t^n$.
Q.E.D.

2.3. Outer conjugacy problem of Powers' binary shifts.

In this section we shall solve Powers'problem on outer conjugacy of binary shifts on a hyperfinite II_1 factor negatively. This work is inspired by Ocneanu [18] and Araki [1].

Let $G = \prod_{i=0}^{\infty} G_i$ be the restricted direct product of $G_i \tilde{=} \mathbf{Z}_2 = \{0,1\}$. Let a be a signature sequence. Let us define a multiplier $m_a \in Z^2(G,\mathbf{T})$ by

$$m_a(x,y) = (-1)^{\sum_{i>j} a(i-j)x(i)y(j)}$$

for $x=(x(i))$, $y=(y(j)) \in G$.

We shall define a unitary operator $\lambda_{m_a}(x)$ on $\ell^2(G)$ by

$$(\lambda_{m_a}(x)\xi)(y) = m_a(x,x^{-1}y)\xi(x^{-1}y)$$

for $x,y \in G$ and $\xi \in \ell^2(G)$.

Let $R_{m_a}(G)$ be the von Neumann algebra generated by $\{\lambda_{m_a}(x) ; x \in G\}$. In the following we shall always assume that the signature sequence a is non-periodic and identify the sequence $(a(i);i \in \mathbb{N}\cup\{0\})$ with $(a(i);i \in \mathbf{Z})$. By lemma 1.1.3 , σ induces a shift σ on $R_{m_a}(G)$ such that $\sigma(\lambda_{m_a}(x)) = \lambda_{m_a}(\sigma(x))$ for $x \in G$, where we use the same notation σ. Put $e_0 = (1,0,0,0,..) \in G$ and $e_n = \sigma^n(e_0) \in G$. Similarly put $u_0 = \lambda_{m_a}(e_0)$ and $u_n = \sigma^n(u_0)$. Then $u_n u_m = (-1)^{a(n-m)} u_m u_n$ and the hyperfinite factor of type II_1 $R=R_{m_a}(G)$ is generated by $\{u_n ; n=0,1,2,..\}$. Thus the shift $\sigma = \sigma_a$ on $R_{m_a}(G)$ is a Powers' binary shift with a signature sequence a (cf. Example 2.1.2.).

In the following we shall realize the relative commutant algebras $C_k(\sigma) = \sigma^k(R)' \cap R$ concretely.

Theorem 2.3.1. Let a be an non-periodic signature sequence. Suppose that the set $\{i \in \mathbb{N} ; a(i) \neq 0\}$ is finite.
Put $d = \max\{i \in \mathbb{N} ; a(i) \neq 0\}$
Let σ be the Powers' binary shift with a signature sequence a. Let $u_0 = \lambda_{m_a}(e_0)$ be the σ-generator. Put $u_n = \sigma^n(u)$. Then
$C_k(\sigma) = \sigma^k(R)' \cap R = \mathbb{C}1$ if $0 \leq k \leq d$ and
$C_k(\sigma) = \{u_i ; 0 \leq i \leq k-d-1\}''$ if $d+1 \leq k$.

Proof. It is clear that we have the inclusion $C_k(\sigma) \supset \mathbb{C}I$ if $0 \leq k \leq d$ and $C^k(\sigma) \supset \{u_i ; 0 \leq i \leq k-d-1\}''$ if $d+1 \leq k$. We shall show the reverse inclusion. In the following we denote $\lambda = \lambda_{m_a}$. Let $x = \Sigma_g x_g \lambda_g \in R_{m_a}(G)$. If x is in $C_k(\sigma)$, then

$$(\Sigma_g x_g \lambda_g)\lambda_{e_n} = \lambda_{e_n}(\Sigma_g x_g \lambda_g) \quad \text{for } n \geq k.$$

Hence $\Sigma_g x_g m_a(g,e_n)\lambda_{g+e_n} = \Sigma_g x_g m_a(e_n,g)\lambda_{e_n+g}$.
Thus $x_g(m_a(g,e_n)-m_a(e_n,g)) = 0$ for $n \geq k$. We may suppose that $x_g = 0$.
Then $m_a(g,e_n)=m_a(e_n,g)$ for $n \geq k$. It is enough to show that $g = 0$ if $0 \leq k \leq d$ and $g(s) = 0$ for $s \geq k - d$ if $d + 1 \leq k$.
Since $m_a(g,e_n) = (-1)^{\Sigma_{i>j}\, a(i-j)g(i)e_n(j)} = (-1)^{\Sigma_{i>n} a(i-n)g(i)}$ and $m_a(e_n,g) = (-1)^{\Sigma_{i>j}\, a(i-j)e_n(i)g(j)} = (-1)^{\Sigma_{n>j}\, a(n-j)g(j)}$, we have that $\Sigma_{i>n} a(i-n)g(i) = \Sigma_{n>j} a(n-j)g(j)$. By changing variables

from i,j to p, we have that

$$(2.3.1)\quad \Sigma_{p=1}^{d} g(p+n)a(p) = \Sigma_{p=1}^{\min(n,d)} a(p)g(n-p) \quad \text{for } n \geq k.$$

Firstly consider the case that $0 \leq k \leq d$. We shall show that $g = 0$. Suppose that $g \neq 0$. Put $m = \max\{i \in \mathbb{N}\cup\{0\} ; g(i) \neq 0\}$ and $n = m + d$. Then we have $n \geq k$. Therefore we can apply (2.3.1) in this case, so that we have $\Sigma_{p=1}^{d} g(p+m+d)a(p) = \Sigma_{p=1}^{d} g(m+d-p)a(p)$. Hence $0 = g(m)a(d)$. Since $a(d) = 1$,we have that $g(m)=0$. This is a contradiction. Thus we have $g = 0$. Next consider the case that $d+1 \leq k$. Assume that $g(s)=1$ for some $s \geq k-d$. Then we shall show the contradiction. Put $m = \max\{i \in \mathbb{N}\cup\{0\};g(i) \neq 0\}$. By the assumption we have that $m \geq k-d$. Put $n = m+d$. Then we have that $n \geq k$. Since we can apply (2.3.1), we have that $\Sigma_{p=1}^{d} g(p+m+d)a(p) = \Sigma_{p=1}^{d} g(m+d-p)a(p)$. Therefore $0 = g(m)a(d)$. Since $a(d) = 1$, $g(m) = 0$. This is a contradiction. Thus $g(s) = 0$ for $k-d \leq s$. Q.E.D.

Remark 2.3.2. In [5], Bures and Yin considered independently the relative commutant algebras for group shifts abstractly and they proved the following:

Let G be a discrete abelian group and m a multiplier of G . Let $R_m(G)$ be the von Neumann algebra as well as the above case $m = m_a$. If H is a subgroup of G, then $R_m(H)' \cap R_m(G)=R_m(D_H)$, where D_H is the subgroup $\{g \in G;\ m(g,h)=m(h,g)$ for any $h \in H\}$ of G.

Powers[19] defined the following outer conjugacy invariant $q(\sigma)$

for shifts σ : Put $q(\sigma) = \min\{k \in \mathbb{N};\ \sigma^k(R) \cap R \neq \mathbb{C}1\}$.

Remark 2.3.3. Take a signature sequence a such that the set $\{i \in \mathbb{N}; a(i) \neq 0\}$ is finite. Let degree a be the number $\max\{i \in \mathbb{N}; a(i) \neq 0\}$. Then Theorem 1 says that $q(\sigma) = (\text{degree } a)+1$.

In [19], Powers raised the following problem(cf. also[24]).

Powers' problem. If α and β are binary shifts and $q(\alpha) = q(\beta)$ then are α and β outer conjugate?

We give a negative answer to the above problem.

Corollary 2.3.4. <u>There exist binary shifts α and β such that $q(\alpha) = q(\beta)$ but α and β are not outer conjugate</u>.

Proof. Let a and b be signature sequences such that $a(2) = a(3) = 1$ and $a(i) = 0$ $(i \neq 2,3)$, $b(1) = b(3) = 1$ and $b(j) = 0$ $(j \neq 1,3)$.
Then $C_k(\sigma_a) \cong \mathbb{C}I$ for $0 \leq k \leq 3$, $C_4(\sigma_a) = \{u_0\}'' \cong \mathbb{C}^2$ and $C_5(\sigma_a) = \{u_0,u_1\}'' \cong \mathbb{C}^4$. On the other hand $C_k(\sigma_b) \cong \mathbb{C}I$ for $0 \leq k \leq 3$ and $C_4(\sigma_b) = \{u_0\}'' \cong \mathbb{C}^2$ but $C_5(\sigma_b) = \{u_0,u_1\}'' \cong M_2$. Thus $q(\sigma_a) = q(\sigma_b) = 4$ but σ_a and σ_b are not outer conjugate. Q.E.D.

Remark 2.3.5. Let a be a signature sequence such that the

set $\{i \in \mathbb{N};\ a(i) \neq 0\}$ is finite. Let order a be the number $\min\{n \in \mathbb{N};\ a(n) \neq 0\}$. Then degree a and order a are outer conjugacy invariants for Powers' binary shifts σ_a with degree a $<+\infty$. In fact $q(\sigma_a) = (\text{degree } a)+1$ and $(\text{degree } a)+(\text{order } a)+1 = \min\{k \in \mathbb{N}\ ;\ \sigma^k(R)' \cap R \text{ is not abelian}\}$. But orders and degrees are not complete outer conjugacy invariant. This is shown by the following example.

Example 2.3.6. Let a and b be signature sequences such that $a(1) = a(3) = 1$ and $a(i) = 0$ $(i \neq 1,3)$, $b(1) = b(2) = b(3) = 1$ and $b(j) = 0$ $(j \neq 1,2,3)$. Then obviously degree a = degree b and order a = order b. On the other hand, by Theorem 2.3.1, we have that

$C_7(\sigma_a) \cong M_2 \otimes \mathbb{C}^4$ and $C_7(\sigma_b) \cong M_4$. Thus $C_7(\sigma_a)$ is not isomorphic to $C_7(\sigma_b)$. Hence σ_a and σ_b are not outer conjugate.

Remark 2.3.7. In [7], M.Choda also uses the numbers $\min\{k \in \mathbb{N}\ ;\sigma^k(R)' \cap R \neq \mathbb{C}1\}$ and $\min\{k \in \mathbb{N};\ \sigma^k(R)' \cap R \text{ is not abelian}\}$ for projection shifts to show that there are at least a countable infinity of outer conjugacy classes among the projection shifts of R with the index $\lambda \in \{4\cos^2(\pi/n);\ n=3,4,\ldots\} \cup [4,\infty)$.

2.4. Multipliers on the rational function field over a finite field.

Let a be a non periodic signature sequence on $\mathbf{Z}$ and $G = \amalg_{i=0}^{\infty} G_i$ the restricted direct product of G_i, $G_i \cong \mathbf{Z}_2$. We can realize G as various subspaces of the rational function field $F(t)$ over the finite field $F = \{0,1\}$. For $x = (x(i))$ and $y = (y(i))$ in G, Let us define a multiplier

$$m_a(x,y) = (-1)^{\Sigma_{i>j} a(i-j)x(i)y(j)}.$$

Now we shall rewrite the multiplier m_a by identifying G with the additive group $(F[t],+)$ of the polynomial ring $F[t]$ over F. Let $F[[t]]$ be the ring of formal power series over F. Define $Q_a(t) \in F[[t]]$ by $Q_a(t) = \Sigma_{n=0}^{\infty} a(n)t^n$.

Lemma 2.4.1. *Let* x,y *in* $G = \amalg_{i=0}^{\infty} G_i$, $G_i \cong Z_2$. *Define* f,g *in* $F[t]$ *by* $f(t) = \Sigma_{n\geq 0} x(n)t^n$ *and* $g(t) = \Sigma_{n\geq 0} y(n)t^n$. *Then* $m_a(x,y) = (-1)^{c(f,g)}$, *where* $c(f,g)$ *is the constant term of* $f(1/t)g(t)Q_a(t)$.

Proof. As $Q_a(t) = \Sigma_{i\geq 0} a(i)t^i \in F[[t]]$, we have the following.

$$\begin{aligned} & C(f(1/t)g(t)Q_a(t)) \\ &= C((\Sigma_{i\geq 0} x(i)(1/t^i))(\Sigma_{j\geq 0} y(j)t^j)(\Sigma_{k\geq 0} a(k)t^k)) \\ &= a(1)(x(1)y(0)+x(2)y(1)+\ldots)+a(2)(x(2)y(0)+x(3)y(1)+\ldots) \\ &\quad +\ldots \\ &= \Sigma_{i>j} a(i-j)x(i)y(j). \end{aligned}$$

Q.E.D.

In the above lemma 2.4.1, we have the following elementary fact.

Lemma 2.4.2.(cf.[15]). Let $F((t))$ be the field of formal power series over F. Then there is an injective algebra homomorphism π from the rational function field $F(t)$ into $F((t))$ such that $\pi(a) = a$ for $a \in F$ and $\pi(t) = t$.

Definition 2.4.3. For $f,g \in F(t)$ and $Q \in F((t))$, let $c_Q(f,g)$ be the constant term of $\pi(f(1/t)g(t))Q(t)$. Put

$$m_Q(f,g) = (-1)^{c_Q(f,g)}.$$

Then m_Q is a multiplier on $(F(t),+)$. Define a map σ on $F(t)$ by $\sigma(f(t)) = tf(t)$. Then the map σ preserves the multiplier m_Q.

Under the same notation in 2.2, we showed the following lemma 2.2.2.

(Lemma.2.2.2.) Let $a:\mathbf{Z} \longrightarrow \{0,1\}$ be a non-periodic signature sequence and $p \in F[t]$ with a nonzero constant term. Then there exists a non-periodic signature sequence $b:\mathbf{Z} \longrightarrow \{0,1\}$ such that

$$m_b(\Phi_p(x),\Phi_p(y)) = m_a(x,y) \quad \text{for any} \quad x,y \in X.$$

Using the proof of this lemma 2.2.2 , we have the next lemma.

Lemma 2.4.4. For $h(t) \in F((t))$, $p(t) = \Sigma_{i=0}^{k} c_i t^i$ and the above signature sequences a and b in lemma 2.2.2, then we have

the constant term of $h(t)a(t)$ $(=C(h(t)a(t)))$

= the constant term of $\pi(h(t)p(1/t)p(t))b(t)$ (2.4.4)

$(= C(\pi(h(t)p(1/t)p(t))b(t)))$.

Proof. By the above lemma 2.2.2, we have

$a(j) = q_k b(j+k)+q_{k-1}b(j+k-1)+\ldots+q_0 b(j)+\ldots+q_{-k}b(j-k)$.

$q_0 = c_0c_0+c_1c_1+\ldots+c_kc_k$, $q_1 = c_0c_1+c_1c_2+\ldots+c_{k-1}c_k,\ldots,$

$q_k = c_o c_k$ and $q_{-t} = q_t (t = 0,\ldots,k)$.

In order to prove this lemma, it is sufficient to show the equality for $h(t) = t^m (m \in \mathbf{Z})$.

The left hand side of (2.4.4) $= C(t^m a(t)) = a(-m)$.

The right hand side of (2.4.4)

$= C(\pi(t^m p(1/t)p(t))b(t))$

$= q_{-k}b(-m+k)+q_{-k+1}b(-m+k-1)+\ldots+q_k b(-m-k)$.

Thus we get this lemma. Q.E.D.

Lemma 2.4.5. Let A be the subspace of F(t) defined by

$A = \{g(t)/p(t)^n \in F(t) \ ; \ g(t) \in F[t], \ n = 0,1,2,\ldots\}$,

where $p(t) \in F[t]$ with a nonzero constant term. Let a be a non-periodic signature sequence. Define $Q(t) = Q_a(t)$. Let $m = m_Q$ be the multiplier on A defined as in 2.4.3. Consider a shift σ on A defined by $\sigma(f(t)) = tf(t)$. Then the induced shift σ on $R = R_m(A)$ is exactly the non-binary shift. In particular, in

the case of $p(t) = t+1$, we have the non-binary shift considered by Price[23].

Proof. By lemma 2.4.4, we have

$$m_{a_\ell}((f(t)/p(t)^\ell),(g(t)/p(t)^\ell)) = m_{a_{\ell+1}}(f(t),g(t)) \qquad (2.4.5)$$

for $f(t), g(t) \in F[t]$.

In the situation of lemma 2.2.2,

$$m_a(f(t),g(t)) = m_b(\Phi_p(f(t)),\Phi_p(g(t)))$$
$$= m_b(p(t)f(t),p(t)g(t)).$$

Then

$$m_b(f(t),g(t)) = m_a((f(t)/p(t)),(g(t)/p(t))).$$

This comes from putting $f(t) = f(t)/p(t)$, $g(t) = g(t)/p(t)$.

So we get (2.4.5). Q.E.D.

3. Shifts with an integral index on a hyperfinite factor of type II_1.

3.1. A generalization of Powers' binary shifts.

Here we shall treat shifts on a hyperfinite II_1-factor induced by groups. Let G be a countable discrete group. Let $G_i \cong G$ for $i = 0,1,2, \ldots$ and $X = \Pi_{i=0}^{\infty} G_i$. Let $\tilde{G}_i$ be the set of elements $(x_j)_{j \geq 0}$ in X such that $x_j = 1$ for $j \neq i$. Put ρ_i be the canonical isomorphism from G to G_i in X. A function

$a: (\mathbf{Z} \setminus \{0\}) \times G \times G \longrightarrow \mathbf{T}$ is called a <u>commutation relator</u> if

(1) $a(n,gh,k) = a(n,g,k)a(n,h,k)$

(2) $a(n,g,hk) = a(n,g,h)a(n,g,k)$

(3) $a(n,g,h) = \overline{a(-n,h,g)}$

for any $n \in \mathbf{Z} \setminus \{0\}$, $g,h,k \in G$.

Let Comm(G) be the set of all commutation relators. Let $\sigma: X \longrightarrow X$ be the canonical shift on X. Let Bich(X,T) be the set of all functions $m: X \times X \longrightarrow T$ such that

(a) m is a bicharacter

(b) $m(\sigma(x),\sigma(y)) = m(x,y)$

(c) $m(\rho_i(g),\rho_j(h)) = 1$ if $i \leq j$

Then we have the following lemma.

Lemma 3.1.1. <u>There is a one to one correspondence between the elements in</u> Comm(G) <u>and the elements in</u> Bich(X,T) <u>such that</u>

(3.1.1)

$$m(x,y) = \Pi_{(i,j)} a(i-j;x(i),y(j)),$$

where $(i,j) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ and $i > j$,

$$a(n;g,h) = m(\rho_i(g),\rho_j(h))/m(\rho_j(h),\rho_i(g)) \quad \text{for } n = i-j .$$

Proof. Take an element $a(n;g,h)$ from $Comm(G)$ and put $m(x,y)$ as (3.1.1). Then this function $m(x,y)$ on $X \times X$ defines an element in $Bich(X,T)$. We shall show this statement. At first we shall show that (a) m is a bicharacter.

$$\begin{aligned} m(x,yz) &= \Pi_{i>j} a(i-j;\ x(i),(yz)(j)) \\ &= \Pi_{i>j} a(i-j;x(i),y(j)z(j)) \\ &= \Pi_{i>j} a(i-j;x(i),y(j))a(i-j;x(i)z(j)) \\ &= [\Pi_{i>j} a(i-j;x(i),y(j))][\Pi_{i>j} a(i-j;x(i),z(j))] \\ &= m(x,y)m(x,z) \end{aligned}$$

Similarly we get

$$m(xy,z) = m(x,z)m(y,z) \quad \text{for any fixed } z \text{ in } X.$$

Secondly we shall show that

(b) $\quad m(\sigma(x),\sigma(y)) = m(x,y)$

$$\begin{aligned} m(\sigma(x),\sigma(y)) &= \Pi_{i>j} a(i-j;(\sigma(x))(i),(\sigma(y))(j)) \\ &= \Pi_{i>j} a((i-1))-(j-1);\ x(i-1),y(j-1)) \\ &= \Pi_{i>j} a(i-j;x(i),y(j)) = m(x,y) \end{aligned}$$

Thirdly we shall show that

(c) $\quad m(\rho_k(g),\rho_\ell(h)) = 1 \qquad$ if $k \leq \ell$.

In fact $\quad m(\rho_k(g),\rho_\ell(h)) = \Pi_{i>j} a(i-j;\rho_k(g)(i),\rho_\ell(h)(j)) = 1.$

Conversely take an element m from $Bich(X,T)$. Then we shall show that (1),(2),(3) hold for $a(n;g,h)$ defined by

$a(n;g,h) = m(\rho_i(g),\rho_j(h))/m(\rho_j(h),\rho_i(g))$ for $n = i-j$.

Because if $n = i-j = i'-j'$, then $a(i-j;g,h) = a(i'-j';g,h)$ by the shift invariance of m. Using the property (a) of m,

$$a(i-j;gh,k) = m(\rho_i(gh),\rho_j(k)/m(\rho_j(k),\rho_i(gh))$$

$$= m(\rho_i(g)\rho_i(h),\rho_j(k))/m(\rho_j(k),\rho_i(g)\rho_i(h))$$

$$= m(\rho_i(g),\rho_j(k))m(\rho_i(h),\rho_j(k))/m(\rho_j(k),\rho_i(g))m(\rho_j(k),\rho_i(h))$$

$$= a(i-j;g,k)a(i-j;h,k)$$

At last we shall show that (3) holds.

$$a(i-j;g,h)a(j-i;h,g)$$
$$=\{(m(\rho_i(g),\rho_j(h))/m(\rho_j(h),\rho_i(g)))\}$$
$$\{(m(\rho_j(h),\rho_i(g))/m(\rho_i(g),\rho_j(h)))\}$$
$$= 1.$$

Therefore $a(i-j;g,h) = \overline{a(-(i-j);h,g)}$.

Starting from $a \in \mathrm{Comm}(G)$, we get $m \in \mathrm{Bich}(X,T)$ and using this m, we get $\tilde{a} \in \mathrm{Comm}(G)$. We shall show that this $\tilde{a}$ equals to a.

$$\tilde{a}(i-j;g,h) = m(\rho_i(g),\rho_j(h))/m(\rho_j(h),\rho_i(g))$$
$$= m(\rho_i(g),\rho_j(h)) = a(i-j;g,h) \quad \text{if } i > j,$$

$$= 1/m(\rho_j(h),\rho_i(g)) = 1/a(j-i;h,g) = a(i-j;g,h)$$
$$\text{if } i < j.$$

Thus we get $\tilde{a} = a$. Starting from $m \in Bich(X,T)$, we get $a \in Comm(G)$ and using this a, we get $\tilde{m} \in Bich(X,T)$. We shall show that this $\tilde{m}$ equals to m.

$$\begin{aligned}\tilde{m}(x,y) &= \prod_{i>j} a(i-j;x(i),y(j)) \\ &= \prod_{i>j} m(\rho_i(x(i)),\rho_j(y(j)))/m(\rho_j(y(j)),\rho_i(x(i))) \\ &= \prod_{i>j} m(\rho_i(x(i)),\rho_j(y(j))), \\ &\quad , \text{ since } m(\rho_j(y(j)),\rho_i(x(i)) = 1, \\ &= m(x,y).\end{aligned}$$

Thus we get $\tilde{m} = m$. Q.E.D.

Definition 3.1.2. Let u be a mapping from $X_0 = \bigcup_{i=0}^{\infty} \tilde{G}_i$ to the unitary group $U(B(H))$ of $B(H)$ on a separable Hilbert space H. Then u is called a generator representation with respect to a mapping a in $Comm(G)$ if u satisfies the following .

$$u(\rho_i(g))u(\rho_j(h)) = a(i-j;g,h)u(\rho_j(h))u(\rho_i(g)),$$

where ρ_i is a canonical injection from G into $\tilde{G}_i$ in X and $u|_{\tilde{G}_i}$ which restricts the representation u on X_0 to G_i into $U(B(H))$ is a unitary representation.

Next, we shall give a relation between projective representations and generator representations.

Lemma 3.1.3. Let G be a countable discrete group and $X = \prod_{i=0}^{\infty} G_i$, where $G_i \cong G$. Fix a commutation relator $a \in Comm(G)$ and the corresponding multiplier $m \in Bich(X,T)$ as in lemma 3.1.1. Then there exists a one to one correspodence

between the set of all projective representations u from X into U(B(H)) and the set of all generator representations from $X_0 = \bigcup_{i=0}^{\infty} \tilde{G}_i$ into U(B(H)).

Proof. Take the generator representation u with respect to a in Comm(G) from X_0 into U(B(H)).

For $x = (x(0),x(1),\ldots,x(n),1,1,\ldots)$ we put

$\tilde{u}(x) = u(x(0)) \ldots u(x(n))$. Then putting

$y = (y(o),y(1), \ldots ,y(n),1,1,\ldots)$,

$$\begin{aligned}\tilde{u}(x)\tilde{u}(y) &= u(x(0) \ldots u(x(n))u(y(0))u(y(1)) \ldots u(y(n)) \\ &= \Pi_{i>j} a(i-j;x(i),y(j))u((xy)(0)) \ldots u((xy)(n)) \\ &= \Pi_{i>j} a(i-j;x(i),y(j))\tilde{u}(xy) \\ &= m(x,y)\tilde{u}(xy) \qquad \text{for any } x,y \text{ in } X.\end{aligned}$$

Thus stating from the generator representation, we gwt the projective representation of X on the same Hilbert space. Conversely we shall start from the projective representation v of X. Then for g in G, put $\tilde{v}(\rho_i(g)) = v(\rho_i(g))$, where $\rho_i(g)$ in $\tilde{G}_i$ in X. Then we shall show that $\tilde{v}$ is the generator representation. At first we have that

$$\tilde{v}(\rho_i(g))\tilde{v}(\rho_j(h)) = m(\rho_i(g),\rho_j(h))\tilde{v}(\rho_i(h)\rho_j(h)).$$

Furthermore, for $i \leq j$, we have that

$$\tilde{v}(\rho_i(g))\tilde{v}(\rho_j(h)) = \tilde{v}(\rho_i(g)\rho_j(h)),$$

in particular, for $i = j$, $\tilde{v}$ is a unitary representation.

For $i < j$, we have that

$$\tilde{v}(\rho_i(g))\tilde{v}(\rho_j(h))$$
$$= \tilde{v}(\rho_j(h)\rho_i(g))$$
$$= \overline{m(\rho_j(h),\rho_i(g))}\tilde{v}(\rho_j(h))\tilde{v}(\rho_i(g))$$
$$= \Pi_{s>t}\overline{a(s-t;\rho_j(h)(s),\rho_i(g)(t))}\tilde{v}(\rho_j(h))\tilde{v}(\rho_i(g))$$
$$= \overline{a(j-i;h,g)}\tilde{v}(\rho_j(h))\tilde{v}(\rho_i(g))$$
$$= a(i-j;g,h)\tilde{v}(\rho_j(h))\tilde{v}(\rho_i(g)).$$

For $i > j$, we have that

$$\tilde{v}(\rho_i(g))\tilde{v}(\rho_j(h))$$
$$= m(\rho_i(g),\rho_j(h))\tilde{v}(\rho_i(g)\rho_j(h))$$
$$= \Pi_{s>t}a(s-t;\rho_i(g)(s),\rho_j(h)(t))\tilde{v}(\rho_i(g)\rho_j(h))$$
$$= a(i-j;g,h)\tilde{v}(\rho_j(h),\rho_i(g))$$
$$= a(i-j;g,h)\overline{m(\rho_j(h),\rho_i(g))}\tilde{v}(\rho_j(h))\tilde{v}(\rho_i(g))$$
$$= a(i-j;g,h)\tilde{v}(\rho_j(h))\tilde{v}(\rho_i(g))$$

Thus $\tilde{v}$ is a generator representation.

Starting from the generator representation u with respect to a in Comm(G) from X_0 into U(B(H)) and using this u, we shall construct the projective representation $\tilde{u}$ with the multiplier m corresponding to a from X into U(B(H)) and using this $\tilde{u}$, we construct the generator representation $\tilde{\tilde{u}}$ with respect to a in Comm(G) from X_0 into U(B(H)). Then we shall show that $\tilde{\tilde{u}} = u$. By the way of the correspondence we get

$\tilde{\tilde{u}}(\rho_i(g)) = \tilde{u}(\rho_i(g)) = u(\rho_i(g))$. Thus we get $\tilde{\tilde{u}} = u$. Conversely starting from the projective representation v with

the $m \in Bich(X,T)$ and using this v, we shall construct the generator representation $\tilde{v}$ with respect to a in $Comm(G)$ and using this $\tilde{v}$, we shall construct the projective representation $\tilde{\tilde{v}}$ with the $m \in Bich(X,T)$. Then we shall show that $\tilde{\tilde{v}} = v$.

By the way of the correspondence,

for $x = (x(0),x(1), \ldots ,x(n),1,1,\ldots)$,

$$\tilde{\tilde{v}}(x) = \tilde{v}(\rho_0(x(0)))\ldots\tilde{v}(\rho_n(x(n)))$$

$$= v(\rho_0(x(0))) \ldots v(\rho_n(x(n))) = v(x).$$

Thus we get $\tilde{\tilde{v}} = v$. Q.E.D.

Before discribing our theorem we shall prepare definitions and its examples.

Definition 3.1.4. Let M be a von Neumann algebra and α be a shift of M. Let X be a countable discrete group and σ be a shift of X. Let m be a σ-invariant multiplier on X. Then $(R_m(X),\sigma_m)$ is called a <u>realization</u> of (M,α) if there exists an isomorphism θ from M onto $R_m(X)$ such that $\alpha\theta = \theta\sigma_m$, where σ_m is the shift on $R_m(X)$ induced by σ.

Definition 3.1.5 (Suzuki). Let G be a countable discrete group and put $X = \amalg_{i=0}^{\infty} G_i$ where $G_i \cong G$. Let Φ be a homomorphism from G into X. Let Σ_Φ be the subgroup generated by $\{\sigma^n(\Phi(a)); n = 0,1,2,\ldots, a \in G\}$. A map Φ is called E-<u>homomorphism</u> if $\Sigma_\Phi = X$. A map Φ is called <u>trivial</u> if $\Phi(G) = \tilde{G}_0$. A group G is called an ES-<u>group</u> if any

E-homomorphism is trivial.

Next, we shall look at examples of ES-groups below.

Remark 3.1.6. The following groups are ES-groups.
(1) $\mathbf{Z}_p$, where p is a prime number (2) $\mathbf{Z}_n$, where n is a square free integer (3) $\mathbf{Z}$ (4) $\mathbf{Z} \times \mathbf{Z}_n$, where n is a square free integer (5) S_3 (the symmetric group of degree three) (6) $K \setminus \mathbf{Z}_p = (K \times K \times \ldots \times K) \rtimes \mathbf{Z}_p$ where K is a finite simple group and $p \nmid \#K$.

Remark 3.1.7. For $G = \mathbf{Z}_4 = \{0,1,2,3\}$, this is not an ES-group. This is shown as follows.
Take $x = (1,2,0,0,0,\ldots)$. Then $\sigma(x) = (0,1,2,0,0,0,\ldots)$, $2\sigma(x) = (0,2,4,0,0,0,\ldots)$. And we get $x+2\sigma(x) = (1,4,4,0,0,0,\ldots) = (1,0,0,0,\ldots)$. Therefore $x = (1,2,0,0,0,\ldots)$ is a σ-generator and x is not in $\widetilde{G_0}$.

In the below, we shall prove the following theorem 3.1.8. which generalize the Powers' result.

Theorem 3.1.8. Let α and β be shifts of a II_1-factor M such that $\alpha(M)' \cap M = \mathbb{C}I$ and $\beta(M)' \cap M = \mathbb{C}I$. Let G and H be ES-groups. Let $(R_m(X),\sigma_m)$ (resp. $(R_n(Y),\sigma_n)$) be a realization of (M,α) (resp. (M,β)), where $X = \amalg_{i=0}^{\infty} G_i$, $G_i \cong G$ for any i and $Y = \amalg_{i=0}^{\infty} H_i$, $H_i \cong H$ for any i. Then (M,α) is conjugate to (M,β) if and only if H is isomorphic to G via Ψ and $b(n;g,h) = a(n;\Psi(g),\Psi(h))$ for any $g,h \in H$ where a(resp. b)

corresponds to m(resp. n).

At first we shall show the next lemma.

Lemma 3.1.9. Let M be a II_1 factor. Let G and H be countable discrete groups and put $X = \amalg_{i=0}^{\infty} G_i$, $G_i \cong G$, $Y = \amalg_{i=0}^{\infty} H_i$, $H_i \cong H$. Let α and β be shifts of M such that $\alpha(M)' \cap M = \mathbb{C}I$ and $\beta(M)' \cap M = \mathbb{C}I$. Assume that there exists a realization $(R_m(X), \sigma_m)$ (resp. $(R_n(Y), \sigma_n)$ for (M,α) (resp. (M,β)), where $m \in Bich(X,T)$ (resp. $n \in Bich(Y,T)$). Then we have a group isomorphism from H onto G.

Proof. Since $(R_m(X), \sigma_m)$ is a realization of (M,α), we may put $R_m(X) = M$. By the assumption of this lemma, putting $N = \sigma_m(R_m(X))$, then (1) M is generated by $\mathcal{N}_M(N) = \{u \in U(M); uNu^* = N\}$ and (2) $\alpha(M)' \cap M = N' \cap M = \mathbb{C}I$. Hence by Nakamura-Takeda[16;lemma 3], we have that $M = G \ltimes N$. By Nakamura-Takeda[17,Theorem], for any $w \in \mathcal{N}_M(N) = \{u \in U(M); uNu^* = N\}$, there exists $g \in G$ such that $w = \lambda_g a_g$, where $a_g \in N$. Therefore $\mathcal{N}_M(N)/N$ is isomorphic to G. Similarly regarding M as $R_n(Y)$ and N as $\sigma_n(R_n(Y))$, $\mathcal{N}_M(N)/N$ is isomorphic to H. Thus it is shown that G is isomorphic to H. Q.E.D.

Lemma 3.1.10. Let α be a shift of a II_1-factor M such that $\alpha(M)' \cap M = \mathbb{C}I$. Let G be a countable discrete group and put $X = \amalg_{i=0}^{\infty} G_i$, where $G_i \cong G$ for any i. Assume that

$(R_m(X), \sigma_m)$ is a realization of (M,α), where m in $\mathrm{Bich}(X,T)$. Suppose w is in the normalizer of α (i.e. $w \in N(\alpha)$). Then there exist an $x \in X$ and $\mu \in \mathbf{T}$ such that $w = \mu u(x)$, where $u(x) = \theta(\lambda_m(x))$ via an isomorphism θ from $R_m(X)$ onto M.

Proof. It is sufficient to prove this lemma for $M = R_m(X)$ and $\alpha = \sigma_m$. By the assumption of this lemma, we can conlude that $M = G \ltimes N$, where $N = \alpha(M)$. From $w \in N(\alpha)$, by [17], $w = \lambda_m(\rho_0(g_0))a_{g_0}$, where $a_{g_0} \in N$ and $g_0 \in G$.
Then $\lambda_m(\rho_0(g_0)) \in N(\alpha)$,
since $\lambda_m(\rho_0(g_0))\sigma_m^k(R_m(X))\lambda_m(\rho_0(g_0))^* \subset \sigma_m^k(R_m(X))$. Since $\lambda_m(\rho_0(g_0))^{-1}w \in N(\alpha)$, we have that $a_{g_0} = \sigma_m(w_{g_0}) \in N(\alpha)$ and $w_{g_0} \in N(\alpha)$. Because $\sigma_m(w_{g_0})\sigma_m^k(R_m(X))\sigma_m(w_{g_0})^* \subset \sigma_m^k(R_m(X))$, $k = 1,2, \ldots$, multiplying σ_m^{-1},

$$w_{g_0}\sigma_m^{k-1}(R_m(X))w_{g_0}^* \subset \sigma_m^{k-1}(R_m(X)).$$

Since $w_{g_0} \in N(\alpha)$, by [17], we have that $w_{g_0} = \lambda_m(\rho_1(g_1))a_{g_1}$ and $a_{g_1} \in N(\alpha)$. Continuing this process we get the following expression of w.

$$w = \lambda_m(\rho_0(g_0))\lambda_m(\rho_1(g_1)) \ldots \lambda_m(\rho_n(g_n))\sigma_m^{n+1}(w_{g_n}).$$

We put the following.

$$\ell = \sup\{n;\ w = \lambda_m(\rho_0(g_0))\lambda_m(\rho_1(g_1)) \ldots \lambda_m(\rho_n(g_n))\sigma_m^{n+1}(w_{g_n}) \text{ with } g_n \neq 1\ \}.$$

We shall show that ℓ is finite. So we shall assume that ℓ is infinite.
Choose k so large and take an integer $n > k$ such that $w = \lambda_m(\rho_0(g_0))\lambda_m(\rho_1(g_1)) \ldots \lambda_m(\rho_n(g_n))\sigma_m^{n+1}(w_{g_n})$ since ℓ is infinite. Take $y = (y(0), y(1), \ldots, y(k), 1, 1, \ldots)$.

$\lambda_m(y)*w$

$= \lambda_m((y(0),y(1),\ldots,y(k),\ldots))*\lambda_m(\rho_0(g_0)) \ \ldots \ \lambda_m(\rho_n(g_n))\sigma_m^{n+1}(w_{g_n})$

$= \nu\lambda_m(\rho_0((y(0)^{-1}(g_0)))\lambda_m(\rho_1((y(1)^{-1}(g_1)) \ \ldots\ldots\ldots$

$\ldots\ldots\lambda_m(\rho_k(y(k)^{-1}(g_k))\lambda_m(\rho_{k+1}(g_{k+1}))\ldots\ldots\lambda_m(\rho_n(g_n))\sigma_m^{n+1}(w_{g_n}),$

where $\nu \in \mathbf{T}$.

Thus $\lambda_m(y)*w$ is approximated by the linear combinations of $\lambda_m(h)$, $h \neq 1$. Hence $\omega_{\delta_1}(\lambda_m(y)*w) = 0$. Since $\{\lambda_m(y); y \in X\}$ is a total se of $R_m(X)$, this is a contradiction. Hence ℓ is finite. Thus

$$w = \lambda_m(\rho_0(g_0))\lambda_m(\rho_1(g_1)) \ \ldots \ \lambda_m(\rho_\ell(g_\ell))\sigma_m^{\ell+1}(w_{g_\ell})$$
$$= \lambda_m(\rho_0(g_0))\lambda_m(\rho_1(g_1)) \ \ldots \ \lambda_m(\rho_\ell(g_\ell))\sigma_m^{t}(w_t)$$
$$\text{for all } t \geq \ell+1.$$

Since ℓ is finite, for $t \geq \ell+1$, $\lambda_m(\rho_t(g_t)) = I$ and only the power of σ_m is increasing. Therefore $\lambda_m(\rho_\ell(g_\ell))* \ \ldots \ \lambda_m(\rho_0(g_0))*w = \sigma_m^t(w_t) \in \bigcap_{s\geq 1}\sigma_m^s(M) = \mathbb{C}I$. Thus

$\lambda_m(\rho_\ell(g_\ell))* \ \ldots \ \lambda_m(\rho_0(g_0))*w = \mu$, where $\mu \in T$. So we have that $w = \mu\lambda_m(\rho_0(g_0)) \ \ldots \ \lambda_m(\rho_\ell(g_\ell))$.

Thus $w = \mu\lambda_m(x)$, where $x = (g_0, \ \ldots \ , g_\ell, 1, 1, \ \ldots \ )$. Q.E.D.

Proposition 3.1.11. Let α (resp. β) be a shift of a II_1-factor M such that $\alpha(M)' \cap M = \mathbb{C}I$ (resp. $\beta(M)' \cap M = \mathbb{C}I$). Assume that α and β are conjugate, i.e., there exists an automorphism γ of M such that $\gamma\alpha\gamma^{-1} = \beta$. Let G be an ES-group and put $X = \amalg_{i=0}^{\infty} G_i$ and $X_0 = \bigcup_{i=0}^{\infty} \tilde{G}_i$, where $G_i \cong G$ for any i, and $\tilde{G}_i$ is the i-component set in X. Assume that $(R_m(X),\sigma_m)$ (resp. $(R_n(X),\sigma_n)$) is a realization of (M,α) (resp.

$(M,\beta))$,where m and n in Bich(X,T). Then there exist an automorphism $\Phi \in$ AutG a character χ of G into T such that

$$\gamma^{-1}(v(\rho_0(g)) = \chi(g)u(\Phi(g))$$

where $u(x) = \theta(\lambda_m(x))$(resp. $v(x) = \theta'(\lambda_n(x))$ via an isomorphism θ(resp. θ') from $R_m(X)$ (resp. $R_n(X)$) onto M.

Proof. For any $g \in G$, we have $\mu \in \mathbf{T}$ and $x \in X$ such that $\gamma^{-1}(v(\rho_0(g))) = \mu u(x)$ by lemma 3.1.10. At first we shall show that μ and x is uniquely determined by this given g. If $\mu u(x) = \mu'(u(y)) = \gamma^{-1}(v(\rho_0(g))$, then $\theta^{-1}(\mu u(x)) = \theta^{-1}(\mu\ u(y))$. So $\mu\lambda_m(x) = \mu'\lambda_m(y)$. Regarding $R_m(X)$ in $L^2(R_m(X),\omega_\delta)$ via ξ , $\{\xi(\lambda_m(x));\ x \in X\}$ is a CONS for $L^2(R_m(X),\omega_{\delta_1})$. Then $\mu\xi(\lambda_m(x)) = \mu\ \xi(\lambda_m(y))$. Hence $x = y$ and $\mu = \mu'$. Therefore we put this unique x (resp. μ) by $\Phi(g)$(resp.$\chi(g)$). Next we shall show that Φ is a E-map and χ is a character. We shall show that Φ is a homomorphism from G into X. Take $\gamma^{-1}(v(\rho_0(g))) = \chi(g)u(\Phi(g))$ and $\gamma^{-1}(v(\rho_0(h)) = \chi(h)u(\Phi(h))$, where g and h in G. We have that

$$\gamma^{-1}(v(\rho_0(gh)) = \chi(gh)u(\Phi(gh)) \quad \text{and}$$

$$\gamma^{-1}(v(\rho_0(g)))\gamma^{-1}(v(\rho_0(h)))$$

$$= \chi(g)u(\Phi(g))\chi(h)u(\Phi(h))$$

$$= \chi(g)\chi(h)u(\Phi(g))u(\Phi(h))$$

$$= \chi(g)\chi(h)m(\Phi(g),\Phi(h))u(\Phi(g)\Phi(h)).$$

Since v is a unitary representation of $\tilde{G}_0$,

(3.1.11)

$$\chi(gh)u(\Phi(gh)) = \chi(g)\chi(h)m(\Phi(g),\Phi(h))u(\Phi(g)\Phi(h)).$$

Multiplying θ^{-1} to both side and taking the map ξ to both side of (3.1.11),

$$\xi(\lambda_m(\Phi(gh))) = \xi(\lambda_m(\Phi(g)\Phi(h))).$$

Thus $\Phi(gh) = \Phi(g)\Phi(h)$. Therefore Φ is a homomorphism from G into X. Multiplying α^k to $\gamma^{-1}(v(\rho_0(g))) = \chi(g)u(\Phi(g))$, then, by using the realization,

$$\gamma^{-1}(v(\sigma^k(\rho_0(g))) = \chi(g)u(\sigma^k(\Phi(g))).$$

Since $\{\sigma^k(\rho_0(g)); g \in G, k = 0,1,2, \ldots\}$ generates X, $\{v(\sigma^k(\rho_0(g)); g \in G, k = 0,1,2, \ldots\}$ generates M. Hence if $\{\sigma^k(\Phi(g)); g \in G, k = 0,1,2, \ldots\}$ does not generate X, then

$\{u(\sigma^k(\Phi(g)); g \in G, k = 0,1,2, \ldots\}$ does not generate M. This contradicts the fact that $\{v(\sigma^k(\rho_0(g)));g \in G, k \in \mathbb{N}\cup\{0\}\}$ generates M. Thus

$<\sigma^n(\Phi(G)); n = 0,1,2,\ldots > = X$. That is, Φ is an E-map. Since G is an ES-group, Φ is trivial. Thus $\Phi(G) = \tilde{G}_0$. As $u|_{\tilde{G}_o}$ is a unitary representation,

$$u(\Phi(gh)) = u(\Phi(g)\Phi(h)) = u(\Phi(g))u(\Phi(h)).$$

By (3.1.11),

$$\chi(gh)u(\Phi(gh)) = \chi(g)\chi(h)u(\Phi(g))u(\Phi(h)).$$

So $\chi(gh) = \chi(g)\chi(h)$. Hence χ is a character. Thus we get this proposition. Q.E.D.

In the below, we shall prove Theorem 3.1.8. By lemma 3.1.9., it is sufficient to prove the theorem for G = H. At first we shall show the necessity of this theorem 3.1.8. By proposition 3.1.11,

$$\gamma^{-1}(v(\rho_i(g))v(\rho_j(h))) = \gamma^{-1}(b(i-j;g,h)v(\rho_j(h)v(\rho_i(g)))$$
$$= b(i-j;g,h)\chi(h)u(\rho_j(\Phi(h)))\chi(g)u(\rho_i(\Phi(g))),$$

$$u(\rho_j(\Phi(g)))u(\rho_i(\Phi(h))) = a(j-i;\Phi(g),\Phi(h))u(\rho_j(\Phi(h)))u(\rho_i(\Phi(g)))$$

and

$$\gamma^{-1}(v(\rho_i(g))v(\rho_j(h))$$
$$= \chi(g)u(\rho_i(\Phi(g)))\chi(h)u(\rho_j(\Phi(h)))$$
$$= a(i-j;\Phi(g),\Phi(h))\chi(h)u(\rho_i(\Phi(h)))\chi(g)u(\rho_j(\Phi(g)))$$
$$= a(i-j;\Phi(g),\Phi(h))\overline{b(i-j;g,h)}\gamma^{-1}(v(\rho_i(g))v(\rho_j(h)))$$

Therefore $a(i-j;\Phi(g),\Phi(h))\overline{b(i-j;g,h)} = 1.$

Thus $b(i-j;g,h) = a(i-j;\Phi(g),\Phi(h))$.

Conversely we shall show the sufficiency of this theorem 3.1.8. At first we put

$U(\xi(\lambda_n(x))) = \xi(\lambda_m(\Psi(x)))$ for $x \in X$, where Ψ satisfies the condition of Theorem 3.1.8.

Then U is an isometry from $L^2(R_n(X),\omega_{\delta_1})$ onto $L^2(R_m(X),\omega_{\delta_1})$. This fact follows from the computation below.

$$(\xi(\lambda_m(\Psi(x)))|\xi(\lambda_m(\Psi(y))))$$
$$= (\lambda_m(\Psi(x))\delta_1|\lambda_m(\Psi(y))\delta_1)$$

$= (m(\Psi(x),1)\delta_{\Psi(x)} | m(\Psi(y),1)\delta_{\Psi(y)})$

$= m(\Psi(x),1)m(\Psi(y),1)(\delta_{\Psi(x)} | \delta_{\Psi(y)})$

$= (\delta_{\Psi(x)} | \delta_{\Psi(y)}) = \delta_{x,y}$, since Ψ is an automorphism of G.

On the other hand

$$(\xi(\lambda_n(x)) | \xi(\lambda_n(y))) = (\lambda_n(x)\delta_1 | \lambda_n(y)\delta_1)$$
$$= (n(x,1)\delta_x | n(y,1)\delta_y)$$
$$= n(x,1)n(y,1)(\delta_x | \delta_y) = \delta_{x,y}.$$

Next using this surjective isometry U, we shall construct an isomorphism from $R_n(X)$ onto $R_m(X)$. By the condition

$b(n;g,h) = a(n;\Psi(g),\Psi(h))$, we have

$$m(\Psi(x),\Psi(y)) = \prod_{i>j} a(i-j;(\Psi(x))(i);(\Psi(y))(j))$$
$$= \prod_{i>j} b(i-j;x(i),y(j)) = n(x,y).$$

Thus we get

$n(x,y) = m(\Psi(x),\Psi(y))$ for any x and y in X.

$(U^*\lambda_m(\Psi(x))U)\xi(\lambda_n(y))$

$= U^*\lambda_m(\Psi(x))\xi(\lambda_m(\Psi(y)))$

$= U^*m(\Psi(x),\Psi(y))\xi(\lambda_m(\Psi(xy)))$

$= m(\Psi(x),\Psi(y))\xi(\lambda_n(xy))$

$= n(x,y)\xi(\lambda_n(y))$

Putting $U\lambda_n(x)U^* = \tilde{\Psi}(\lambda_n(x))$, $\tilde{\Psi}$ is an isomorphism from $R_n(X)$ on to $R_m(X)$. Then we shall show that $\tilde{\Psi}$ gives the conjugacy for σ_m and σ_n. That is, putting

$$\tilde{\Psi}(\lambda_n(x)) = \lambda_m(\Psi(x)),$$
$$\tilde{\Psi}(\lambda_n(\sigma_n(x)) = \lambda_m(\Psi(\sigma_m(x))) = \lambda_m(\sigma_m(\Psi(x))).$$

Then $\theta^{-1}\alpha = \sigma_m\theta^{-1} = \tilde{\Psi}\sigma_n\tilde{\Psi}^{-1}\theta^{-1} = \tilde{\Psi}\theta'^{-1}\beta\theta'\tilde{\Psi}^{-1}\theta^{-1}$.
So $\beta(\theta'\tilde{\Psi}^{-1}\theta^{-1}) = (\theta'\tilde{\Psi}^{-1}\theta^{-1})\alpha$. This proves the conjugacy for α and β . Thus we have the proof of theorem 3.1.8 .
Q.E.D.

As a corollary of this theorem 3.1.8 , we have a Powers' result [19].

Corollary 3.1.12 (Powers). Two binary shifts α and β of R are conjugate if and only if their anticommutator sets $S(\alpha)$ and $S(\beta)$ coincide.

Proof. We put $X = \amalg_{i=0}^{\infty} G_i$, $G_i \cong \mathbf{Z}_2$. For $S(\alpha)$ and $S(\beta)$, take functions $a(k;g,h)$ and $b(k;g,h)$ from $(\mathbf{Z}\setminus\{0\}) \times \mathbf{Z}_2 \times \mathbf{Z}_2$ into $\{\gamma \in \mathbb{C}; \gamma^2 = 1\} = \{1,-1\}$.

$S(\alpha) = \{k \in \mathbb{N};\ a(k;1,1) = -1\}$ and
$S(\beta) = \{k \in \mathbb{N}; b(k;1,1) = -1\}$, where $\mathbf{Z}_2 = \{0,1\}$.
By the theorem 3.1.8, we have that
$R = (R_m(X),\sigma_m) \cong (R_n(X),\sigma_n)$, where m (resp. n) correspods to a (resp. b) and σ_m (resp. σ_n) corresponds to α(resp. β) if and only if there exists a $\Phi \in \mathrm{Aut}\mathbf{Z}_2$ such that $b(n;g,h) = a(n;\Phi(g),\Phi(h))$ for any g and h in $\mathbf{Z}_2$. Since $\mathrm{Aut}\,\mathbf{Z}_2 = \{1\}$, it is equivalent to

$b(n;g,h) = a(n;g,h)$ for any g and h in $\mathbf{Z}_2$

if and only $b(n;1,1) = a(n;1,1)$

if and only $S(\alpha) = S(\beta)$.

Thus we get the result of Powers as a corollary of Theorem 3.1.8. Q.E.D.

In the below we shall look at examples.

Example 3.1.13. Take $G = \mathbf{Z}_3$ and $X = \coprod_{i=0}^{\infty} G_i$, $G_i \cong G$. We note that $\mathbf{Z}_3 = \{0,1,2\}$ and Aut $\mathbf{Z}_3 = \{\Phi_1, \Phi_2\}$, where $\Phi_1(1) = 1$, $\Phi_2(1) = 2$. Furthermore, $a(n;g,h) \in \{\gamma \in T;\ \gamma^3 = 1\}$. Then by the theorem 3.1.8 , $(R_m(X), \sigma_m) \cong (R_n(X), \sigma_n)$ if and only if $b(n;g,h) = a(n;\Phi(g),\Phi(h))$, where $\Phi \in$ Aut $\mathbf{Z}_3$, if and only if $b(n;1,1) = a(n;1,1)$ or $b(n;1,1) = a(n;2,2) = a(n,1,1)^4 = a(n;1,1)$ if and only if $S(\alpha;1) = S(\beta;1)$, $S(\alpha;\exp(2\pi i/3)) = S(\beta;\exp(2\pi i/3))$, $S(\alpha;\exp(4\pi i/3)) = S(\beta;\exp(4\pi i/3))$, where $S(\alpha;\gamma) = \{\ k \in \mathbb{N}; u\alpha^k(u) = \gamma\alpha^k(u)u\ \}$.

Remak 3.1.14. For $G = \mathbf{Z}_k$, $a(n;g,h)^k = a(n;k\cdot g,h) = a(n;0,h) = 1$. Therefore $a(n;g,h) \in \{\gamma \in \mathbb{C}; \gamma^k = 1\} \subset T$.

Example 3.1.15. Put $G = \mathbf{Z}_5 = \{0,1,2,3,4\}$. Even if $a(n;1,1) \neq b(n;1,1)$, $(R_m(X), \sigma_m)$ can be conjugate to $(R_n(X), \sigma_n)$ whenever $a(n;1,1) = b(n;1,1)^{\ell}$ for $n \in Z$,where $\ell = 4$. This is implied by the fact that $1^2 \equiv 4^2 \equiv 1 \pmod 5$, $2^2 \equiv 3^2 \equiv 4 \pmod 5$.

3.2. Price type shifts with an integral index.

In this section, we shall consider the general case of 2.2.

Let $n \geq 2$ be an integer and $G = \amalg_{i=0}^{\infty} G_i$, $G \cong \mathbf{Z}_n$. Let σ be a canonical shift on G. Let $a = (a(j))_{j \in \mathbf{Z}}$ be an infinite sequence satisfying $a(j) \in \mathbf{Z}_n$, $a(0) = 0$ and $a(-j) = -a(j)$(cf.[24], [5],[6]).

Suppose that

(3.2.0) for all primes p dividing n, the above sequence $a = (a(j))$ fails to be periodic mod p.(cf. [5],[24]).

Put $\gamma = e^{2\pi i/n}$. Using the above sequence $a = (a(i))$, let us define a multiplier m_a by

$m_a(x,y) = \gamma^{\sum_{i>j} a(i-j)x(i)y(j)}$ for $x = (x(i)), y = (y(j))$ in G(cf.(2.1.1)). Then $m = m_a$ preserves σ so that σ induces a σ_m on $R_m(G)$. On the other hand Bures and Yin [5] showed that all the following statements (1),(2),(3) are equivalent.

(1) the sequence $a = (a(i))$ satisfies (3.2.0).

(2) $R_m(G)$ is a factor. (3) $\sigma_m(R_m(G))' \cap R_m(G) = \mathbb{C}I$.

Definition 3.2.1.([5]).

A shift α of the hyperfinite II_1-factor R is called an n-shift if α is conjugate to a shift induced from the canonical shift σ on $G = \amalg_{i=0}^{\infty} G_i$, $G_i \tilde{=} \mathbf{Z}_n$ and nondegenerate σ-preserving multiplier m on $G \times G$.

Definition 3.2.2.

By the same way of 2.2, for sequences $p = (p_\ell)_{\ell \geq 1}$ of polynomials $p_\ell(t)$ where $p_\ell(t) = \Sigma_{i=0}^{k(\ell)} c_{\ell,i} t^i$ and $c_{\ell,0} = c_{\ell,k(\ell)} = 1$, and the sequence a satisfying (3.2.0), we can make shifts $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$. We call them shifts of Price type.

In the below, put $\sigma_{[a,p]} = \sigma$, $X_{[p]} = X$.

Proposition 3.2.3.

For a shift σ of Price type, $\sigma(R)' \cap R = \mathbb{C}I$.

Proof. We use theorem 1.2 and proposition 3.1[5]. Take $x(\neq \lambda I)$ in $\sigma(R)' \cap R$. Let $\{\delta_g; g \in X\}$ be the canonical orthonormal basis of $\ell^2(X)$. Then we have $x\delta_e = \Sigma_{g \in G} c_g \delta_g$, $\Sigma_{g \in G} |c_g|^2 < +\infty$. For any $h \in \sigma(X)$, $(\lambda_m(h)x)\delta_e = \Sigma_{g \in G} c_g m(h,g) \delta_{gh}$ and $(x\lambda_m(h))\delta_e = x(\rho_m(h^{-1})\delta_e) = \Sigma_{g \in G} c_g m(g,h) \delta_{gh}$, where $\rho_m(g)(\delta_h) = m(h,g^{-1})\delta_{hg}$ for $g,h \in X$. As $\lambda_m(h)x = x\lambda_m(h)$, we have $c_g m(h,g) = c_g m(g,h)$. Thus if $c_g \neq 0$, $m(h,g) = m(g,h)$ for any $h \in \sigma(X)$. Since x is not scalar, there exists $g(\neq 1)$ in X such that $\lambda_m(g) \in \sigma(R)' \cap R$. On the other hand, as $X = X_{[p]} = \varinjlim X_p$, there exists a number ℓ such that $g \in X_\ell$. On the other hand $\lambda_m(g)|_{\ell^2(X_\ell)} = \lambda_{m_\ell}(g)$. Obviously $\lambda_{m_\ell}(g)$ is not scalar and $\lambda_{m_\ell}(g) \in \sigma(R_{m_\ell}(X_\ell))' \cap R_{m_\ell}(X_\ell)$. But $\sigma(R_{m_\ell}(X_\ell))' \cap R_{m_\ell}(X_\ell) = \mathbb{C}I$ by theorem 1.2 [5]. This is a contradiction. Therefore $\sigma(R)' \cap R = \mathbb{C}I$. Q.E.D.

Next we use the following facts proved in [5].

Proposition 3.2.4. ([5 ,proposition 1.4]).

Let G be a countable discrete abelian group and m a nondegenerate multiplier of G and σ a shift of G which preserves m. Let σ_m be the shift of the hyperfinite II_1 factor $R_m(G)$ induced by σ. Then $N(\sigma_m)/\mathbf{T} \cong G$ if and only if $\sigma_m(R)' \cap R = \mathbb{C}$.

Using this proposition 3.2.4. , proposition 3.2.3. and the same method of lemma 2.2.10., we get the following.

Theorem 3.2.5.

There exist uncountably many non-conjugate Price type shifts of index n, which is not conjugate to n-shifts,of index n on the hyperfinite II_1-factor.

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