Solutions of New General Relativity

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CONTENTS

Abstract

New general relativity is a theory of gravitation based on the Weitzenböck spacetime endowed with absolute parallelism which contains torsion and identically vanishing curvature. The gravitational equations which are derived from gravitational action quadratic in the torsion with arbitrary weights c_1, c_2 and c_3 can naturally incorporate the Dirac field as a source term.

In this paper, we show some exact solutions of the gravitational field equations in new general relativity. Stationary and axially symmetric solutions in the case with $c_1 = c_2 = 0$, homogeneous and isotropic solutions, and Kasner type solutions are presented. Our stationary and axially symmetric solutions have a non-vanishing axial-vector part of the torsion which is coupled with the intrinsic spin of matter. In the homogeneous and isotropic solutions, we show that the absolute parallelism is inconsistent with the closed, homogeneous and isotropic universe. The Kasner type solutions in n+1 dimensional Weitzenböck spacetime are examined in detail and several differences from those of general relativity are discussed.

§ 1 Introduction

In the history of physics, it has been confirmed that fundamental forces of nature are expressed in the framework of gauge theory.[1] In 1956, Utiyama pointed out that general relativity[2)~4)] is a gauge theory of a group of local Lorentz transformations.[5] This idea was extended by Kibble.[6] He introduced a group of translations of world coordinate in addition to the group of local Lorentz transformations.

In 1967, Hayashi and Nakano proposed a translation gauge theory of gravitation with a group of 'global' rather than 'local' Lorentz transformations.[7] Fundamental entity of their translation gauge theory is not the metric tensor but vierbein fields. The results of general relativity which have been confirmed by observations can be also reproduced by the translation gauge theory. Furthermore, the framework of this theory includes in a consistent manner a notion of intrinsic spin angular-momentum of source matter. Miyamoto and Nakano estimated the energy of the spin-spin gravitational interaction of the Dirac spinor fields.[8]

On the other hand, geometrical extension of the gravitational physics has been discussed ever since the discovery of general relativity. In 1928, Einstein introduced a notion of absolute parallelism to unify the gravitation and the electro-magnetism.[9] The spacetime underlying this theory is called the Weitzenböck spacetime, which is characterized by torsion. His idea, however, failed because the theory could not give correct gravitational field equations coupled with the electromagnetic field.[10] In 1967, MØller revived the notion of absolute parallelism to construct a new theory of gravity.[11] Pellegrini and Plebanski found a Lagrangian formulation for the absolute parallelism.[12]

In 1977, these two streams were unified by Hayashi.[13] He found that geometry underlying the translation gauge theory is the Weitzenböck spacetime. He called this theory 'new general relativity'. Finally, the basis of new general relativity was completed by Hayashi and Shirafuji.[14]

The basic entity of new general relativity which is called parallel vector fields is vierbein fields endowed with absolute parallelism. Strength of gravitation is caused only by the torsion fields.[*] The gravitational Lagrangian of new general relativity is constructed from three terms of invariants quadratic in the torsion and a cosmological term which is usually neglected. It is the most attractive point of new general relativity in contrast to general relativity that the Dirac spinor field can be regarded as a source of gravitation without difficulty. This seems very important for a microscopic theory of gravitation and for unification with the theories of matters. In investigations of nature of the gravitational

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*) A theory of gravitation based on the Riemann-Cartan spacetime which is characterized by the torsion and curvature is called Poincaré gauge theory.[15]

system governed by new general relativity, exact solutions of the gravitational field equations will greatly contribute to these problems. Unfortunately, exact solutions of new general relativity have not yet been studied with sufficient generality because of complexity of the gravitational field equations. A static and isotropic solution was found by Hayashi and Shirafuji.[14] They also showed Birkhoff´s theorem of new general relativity. Other types of solutions, for example, axially symmetric solutions and cosmological solutions, etc., had not been discussed until 1981.

In this paper, a series of theoretical efforts of the present author for finding exact solutions in several cases are summarized and examined in details. Here stationary and axially symmetric solutions,[16] homogeneous and isotropic solutions,[17] and homogeneous and anisotropic solutions[18] are discussed.[*]

It should be emphasized that our stationary and axially symmetric gravitational field couples with intrinsic spin of the system. This is the most characteristic point of new general relativity. On the other hand, we are interested in cosmological models of new general relativity. It is very important to examine whether homogeneity and isotropy are compatible with the absolute parallelism. We shall answer this question and show that new general relativity leads to

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*) Prototypes of these solutions in general relativity have been studied by many people.[19]

'Friedmann type' metric in this case. In the early universe, we cannot postulate apriori the isotropy of the universe. The simplest models of the homogeneous and anisotropic universe which we shall call 'Kasner type' universe are obtained.

In order to prepare for the present work, we briefly review new general relativity in § 2 . In § 3 , the stationary and axially symmetric solutions are discussed. We examine the homogeneous and isotropic universe in § 4 , and the Kasner type universe in § 5 . The last section is devoted to summary and discussion.

§ 2 New general relativity

2-1) Gravitational field equations

In new general relativity, fundamental entity is parallel vector fields, $\mathbf{b}^k = (b^k{}_\mu)$, with those inverse, $\mathbf{b}_k = (b_k{}^\mu)$.*) The metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = \eta_{km}\, b^k{}_\mu b^m{}_\nu \quad , \tag{2.1a}$$

with its inverse,

$$g^{\mu\nu} = \eta^{km}\, b_k{}^\mu b_m{}^\nu \quad . \tag{2.1b}$$

Here η_{km} and η^{km} are the metric of the Minkowski spacetime:

$$\eta_{km} = \eta^{km} = \mathrm{diag}(-1,1,1,1) \ . \tag{2.2}$$

We use $g^{\mu\nu}$ and $g_{\mu\nu}$ for raising and lowering the Greek indices, and η^{km} and η_{km} for raising and lowering the Latin indices, respectively.

Spacetime of new general relativity is the Weitzenböck spacetime endowed with absolute parallelism for the internal Lorentz frame as

$$D_\nu \mathbf{b}^k = 0 \quad , \tag{2.3a}$$

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*) In this paper, we use Greek letters for world indices and Latin letters for internal Lorentz indices labeling the parallel vector fields. The middle part of the Greek alphabet, λ, μ, ν, $\cdots$, refers to 0, 1, 2 and 3, while the initial part, α, β, γ, $\cdots$, denotes 1, 2 and 3. In a similar way, the middle part of the Latin alphabet, i, j, k, $\cdots$, means 0, 1, 2 and 3, while the initial part, a, b, c, $\cdots$, denotes 1, 2 and 3.

or equivalently,

$$D_\nu b^k{}_\mu \equiv \partial_\nu b^k{}_\mu - \Gamma^\lambda{}_{\mu\nu} b^k{}_\lambda = 0 \quad . \tag{2.3b}$$

From this equation, the affine connection $\Gamma^\lambda{}_{\mu\nu}$ can be solved as

$$\Gamma^\lambda{}_{\mu\nu} = b_k{}^\lambda \partial_\nu b^k{}_\mu \quad . \tag{2.4}$$

Curvature tensor field and torsion tensor field are given by

$$R^\rho{}_{\sigma\mu\nu}(\Gamma) \equiv \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu} = 0 \ , \tag{2.5}$$

$$T^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \ . \tag{2.6}$$

Here we use (2.4) in (2.5), which leads to the identically vanishing curvature.

From a gauge theoretical point of view, translation gauge fields $c^k{}_\mu$ are defined by $c^k{}_\mu \equiv b^k{}_\mu - \delta^k{}_\mu$, and the torsion field is regarded as a field strength.

The line element ds in this spacetime is represented as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad . \tag{2.7}$$

Orbits of a freely falling point particle are given by the geodesic equations:

$$\delta \int ds = 0 \quad , \tag{2.8a}$$

or equivalently,

$$\frac{d^2 x^\lambda}{d\tau^2} + \{^{\lambda}_{\mu\nu}\} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad , \tag{2.8b}$$

with $\{^{\lambda}_{\mu\nu}\}$ denoting the Christoffel symbol and $d/d\tau$ representing differentiation with respect to a proper time τ along the trajectory of the particle.

In order to construct gravitational Lagrangian L_G , we require that L_G is quadratic in the torsion field besides a cosmological term, and that it is invariant under general coordinate transformations, under "global," proper and orthochronous Lorentz transformations, and under parity operation.

For the general form quadratic in the torsion, it is useful to decompose the torsion field to its irreducible parts,

$$t_{\lambda\mu\nu} \equiv \frac{1}{2}(T_{\lambda\mu\nu}+T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\lambda\nu}v_{\mu}+g_{\mu\nu}v_{\lambda}-2g_{\lambda\mu}v_{\nu}) \quad , \qquad (2.9a)$$

$$v_{\mu} \equiv T^{\lambda}{}_{\lambda\mu} \quad , \qquad (2.9b)$$

$$a_{\mu} \equiv \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma} \quad , \qquad (2.9c)$$

with $\varepsilon_{\mu\nu\rho\sigma}$ being the totally antisymmetric tensor normalized as $\varepsilon_{0123} = -\sqrt{-g}$.

The gravitational Lagrangian L_G is given by

$$L_G = a_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + a_2(v^{\mu}v_{\mu}) + a_3(a^{\mu}a_{\mu}) - \Lambda \quad . \qquad (2.10)$$

Here a_1, a_2 and a_3 are parameters which should be determined by observations and Λ is a cosmological constant. To clarify difference from general relativity, we rewrite the expression (2.10) as the following way,

$$L_G = \frac{1}{2\kappa}[\, R(\{\}) + 2c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + 2c_2(v^{\mu}v_{\mu}) + 2c_3(a^{\mu}a_{\mu}) - 2\kappa\Lambda \,] + (\text{total derivative}) \quad , \qquad (2.11)$$

where

$$c_1 = \kappa a_1 + \frac{1}{3}\,, \quad c_2 = \kappa a_2 - \frac{1}{3}\,, \quad c_3 = \kappa a_3 + \frac{3}{4} \,. \qquad (2.12)$$

The symbol κ represents the Einstein constant; $\kappa = 8\pi G$, and $R(\{\})$ denotes the Riemann-Christoffel scalar curvature constructed from the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\}$. Finally, the gravitational action I_G is given by

$$I_G = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[\ R(\{\}) + 2c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + 2c_2(v^{\mu}v_{\mu}) + 2c_3(a^{\mu}a_{\mu}) - 2\kappa\Lambda\ ] \quad . \tag{2.13}$$

Next, we turn our attention to matter fields. In the Weitzenböck spacetime accompanied with the absolute parallelism, covariant derivative for the Dirac spinor field ψ is represented as

$$D_{\lambda}\psi = \partial_{\lambda}\psi \quad , \tag{2.14}$$

because of the requirement (2.3). Lagrangian L_D for the Dirac spinor field ψ is given by*)

$$L_D = \frac{i}{2}\, b_k{}^{\mu}\ \{\bar{\psi}\gamma^k D_{\mu}\psi - (D_{\mu}\bar{\psi})\gamma^k\psi\} - m\bar{\psi}\psi \quad , \tag{2.15a}$$

or equivalently,

$$L_D = \frac{i}{2}\, b_k{}^{\mu}\ \{\bar{\psi}\gamma^k\nabla_{\mu}\psi - (\nabla_{\mu}\bar{\psi})\gamma^k\psi\} - \frac{3}{4}\, a_k\bar{\psi}\gamma^5\gamma^k\psi - m\bar{\psi}\psi \quad , \tag{2.15b}$$

where ∇_{μ} is the formal covariant differentiation of the spinor field,

$$\nabla_{\mu}\psi \equiv (\partial_{\mu} + \frac{i}{2}\,\Delta_{ij\mu}s^{ij})\ \psi \quad , \tag{2.16}$$

with respect to the Ricci rotation coefficients $\Delta_{ij\mu}$,

$$\Delta_{ijk} = b_k{}^{\mu}\Delta_{ij\mu} \equiv -\frac{1}{2}\,(T_{ijk} - T_{jik} - T_{kij}) \quad , \tag{2.17}$$

and the generators of the Lorentz group s^{ij} ,

$$s^{ij} \equiv \frac{i}{4}\,[\gamma^i, \gamma^j] \quad . \tag{2.18}$$

The Dirac equations are written as

$$\{\ ib_k{}^{\mu}\gamma^k(D_{\mu} + \frac{1}{2}\,v_{\mu}) - m\ \}\ \psi = 0 \quad , \tag{2.19a}$$

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*) Our convention of the gamma matrices is as follows:

$$\{\gamma^i, \gamma^j\} = -\,2\eta^{ij} \ , \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ .$$

or equivalently,

$$\{ ib_k{}^{\mu}\gamma^k\nabla_\mu - \frac{3}{4} a_k\gamma^5\gamma^k - m \} \psi = 0 \quad . \qquad (2.19b)$$

From this equation, new general relativity predicts spin pressesion of the spinor field:[14)]

$$\frac{D}{d\tau} s^\mu = (\Delta^{\lambda\mu\nu}-\Delta^{\lambda\nu\mu})u_\lambda s_\nu \quad , \qquad (2.20a)$$

or equivalently,

$$\frac{\nabla}{d\tau} s^\mu = -\frac{3}{2} \varepsilon^{\mu\nu\rho\sigma}u_\nu a_\rho s_\sigma \quad , \qquad (2.20b)$$

with s^μ representing the spin vector and u_ν denoting four-velocity of semi-classical wave packet. Here $D/d\tau \equiv u^\mu D_\mu$ and $\nabla/d\tau \equiv u^\mu\nabla_\mu$ mean covariant differentiation along the classical trajectory $x^\mu(\tau)$ which is given by the geodesic equations (2.8b).

Next, we study the Yang-Mills field $A_\mu = \{A^a{}_\mu\}$ whose field strength $F_{\mu\nu} = \{F^a{}_{\mu\nu}\}$ should be defined as[*)]

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \alpha[A_\mu,A_\nu] \quad , \qquad (2.21)$$

where α is a coupling constant. The Lagrangian L_{YM} for the Yang-Mills field is quadratic in the field strength $F_{\mu\nu}$,

$$L_{YM} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} \quad . \qquad (2.22)$$

Equations of motion are given by

$$\nabla_\nu F^{\mu\nu} - \alpha[A_\nu,F^{\mu\nu}] = j^\mu \quad , \qquad (2.23)$$

where j^μ represents a source current. This is just the

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*) If we define the field strength,

$$F_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu - \alpha[A_\mu,A_\nu] \quad ,$$

it is not gauge invariant. Noticing that $\partial_\mu A_\nu-\partial_\nu A_\mu$ is also a tensor, we therefore adopt the definition (2.21).

Yang-Mills equations in general relativity.

A total action is a sum of the gravitational action I_G and matter action I_M which is an integration of matter Lagrangian L_M. Variating the total action I with respect to the parallel vector fields $b^k{}_\mu$, we obtain the gravitational field equations:

$$I^{\mu\nu} \equiv G^{\mu\nu}(\{\}) + 2D_\lambda F^{\mu\nu\lambda} + 2v_\lambda F^{\mu\nu\lambda} + 2H^{\mu\nu} - g^{\mu\nu}L' = \kappa T^{\mu\nu} , \tag{2.24}$$

where

$$F^{\mu\nu\lambda} \equiv c_1(t^{\mu\nu\lambda}-t^{\mu\lambda\nu}) + c_2(g^{\mu\nu}v^\lambda-g^{\mu\lambda}v^\nu) - \frac{c_3}{3}\varepsilon^{\mu\nu\lambda\rho}a_\rho$$
$$= - F^{\mu\lambda\nu} , \tag{2.25a}$$

$$D_\lambda F^{\mu\nu\lambda} \equiv \partial_\lambda F^{\mu\nu\lambda} + \Gamma^\mu{}_{\rho\lambda}F^{\rho\nu\lambda} + \Gamma^\nu{}_{\rho\lambda}F^{\mu\rho\lambda} + \Gamma^\lambda{}_{\rho\lambda}F^{\mu\nu\rho} , \tag{2.25b}$$

$$H^{\mu\nu} \equiv T^{\rho\sigma\mu}F_{\rho\sigma}{}^\nu - \frac{1}{2}T^{\nu\rho\sigma}F^\mu{}_{\rho\sigma} = H^{\nu\mu} , \tag{2.25c}$$

$$L' \equiv c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + c_2(v^\mu v_\mu) + c_3(a^\mu a_\mu) - \kappa\Lambda , \tag{2.25d}$$

$$T^{\mu\nu} \equiv (1/\sqrt{-g})\eta^{kj}b_j{}^\mu\, \delta(\sqrt{-g}L_M)/\delta b^k{}_\nu . \tag{2.25e}$$

Here the tensor $G^{\mu\nu}(\{\})$ is the Einstein tensor of general relativity which is made of the Christoffel symbols. The tensor $T^{\mu\nu}$ denotes an energy-momentum tensor of the matter fields, which is respectively defined for the Dirac spinor field and for the Yang-Mills field as

$$T^{\mu\nu}_D = -\frac{i}{2}b_k{}^\nu(\bar\psi\gamma^k D^\mu\psi-(D^\mu\bar\psi)\gamma^k\psi) + g^{\mu\nu}L_D , \tag{2.26}$$

$$T^{\mu\nu}_{YM} = F^{\mu\rho}F^{\nu\sigma}g_{\rho\sigma} + g^{\mu\nu}L_{YM} . \tag{2.27}$$

It should be noted that the gravitational equations (2.24) are in general asymmetric for the indices μ and ν, which allows the spinor field as a source of gravitation.

2-2) Weak field approximation

In order to consider physical meanings of solutions of the gravitational field equations (2.24) in case they being found, we can derive useful information from the weak field approximation which we here discuss. In the weak field approximation, the parallel vector fields $b^k{}_\mu$ can be written as

$$b^k{}_\mu = \delta^k{}_\mu + a^k{}_\mu \quad , \qquad (|a^k{}_\mu| << 1) \quad . \tag{2.28}$$

In the lowest order of $a^k{}_\mu$, the Greek indices cannot be distinguished from the Latin indices: We adopt the Greek indices,

$$b_{\mu\nu} = \eta_{\mu\nu} + \eta_{\mu\rho} a^\rho{}_\nu \quad . \tag{2.29}$$

Here we use the Minkowski metric $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ for raising and lowering the indices. We shall decompose the field $a_{\mu\nu}$ into its symmetric and antisymmetric parts,

$$a_{\mu\nu} = \frac{1}{2} h_{\mu\nu} + A_{\mu\nu} \quad , \tag{2.30}$$

with $h_{\mu\nu} = h_{\nu\mu}$ and $A_{\mu\nu} = - A_{\nu\mu}$. The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad . \tag{2.31}$$

By substituting (2.30) into (2.24) without the cosmological constant Λ and keeping only the lowest terms, gravitational field equations for $h_{\mu\nu}$ and $A_{\mu\nu}$ are given as follows:*)

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*) In this paper, we express symmetric and antisymmetric parts of a tensor $T^{\mu\nu}$ as $T^{(\mu\nu)} \equiv \frac{1}{2} (T^{\mu\nu}+T^{\nu\mu})$ and $T^{[\mu\nu]} \equiv \frac{1}{2} (T^{\mu\nu}-T^{\nu\mu})$, respectively.

$$
\begin{aligned}
&- \tfrac{1}{2}(1-3c_1)\Box\bar{h}^{\mu\nu} + \tfrac{1}{2}(1-2c_1+c_2)(\partial^\mu\partial_\rho\bar{h}^{\rho\nu}+\partial^\nu\partial_\rho\bar{h}^{\rho\mu}) \\
&\quad - \tfrac{1}{2}(1-c_1+2c_2)\eta^{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \tfrac{1}{2}(c_1+c_2)(\eta^{\mu\nu}\Box\bar{h}-\partial^\mu\partial^\nu\bar{h}) \\
&\quad + (c_1+c_2)(\partial^\mu\partial_\rho A^{\rho\nu}+\partial^\nu\partial_\rho A^{\rho\mu}) = \kappa T^{(\mu\nu)} \quad , \qquad (2.32a)
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{1}{2}(c_1+c_2)(\partial^\mu\partial_\rho\bar{h}^{\rho\nu}-\partial^\nu\partial_\rho\bar{h}^{\rho\mu}) + (c_1-\tfrac{4}{9}c_3)\Box A^{\mu\nu} \\
&\quad +(c_2+\tfrac{4}{9}c_3)(\partial^\mu\partial_\rho A^{\rho\nu}-\partial^\nu\partial_\rho A^{\rho\mu}) = \kappa T^{[\mu\nu]} \quad , \qquad (2.32b)
\end{aligned}
$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and $\bar{h} = \bar{h}^\rho{}_\rho = - h^\rho{}_\rho = - h$.
It will be found that these equations are very important to understand physical meanings of integration constants in obtained exact solutions.

2-3) A static and isotropic solution

A static and isotropic solution in vacuum in new general relativity without a cosmological constant Λ has been obtained by Hayashi and Shirafuji.[14)] This gravitational field is induced by a static, isotropic and spinless source localized at the origin.

From the discussion in Appendix A, if we require isotropy of space, it is possible to find a set of coordinates, $x^0 = t$ and x^α , of which the parallel vector fields $b^k{}_\mu$ are form invariant under space rotation,

$$x'^\alpha = R_{\alpha\beta} x^\beta \ , \quad \mathbb{b}'^a = R_{ac} \mathbb{b}^c \ , \tag{2.33}$$

where $R = (R_{\alpha\beta}) = (R_{ab})$ is a constant 3×3 orthogonal matrix,

$$R\, R^t = R^t\, R = I \quad , \quad \det R = 1 \quad . \tag{2.34}$$

The most general form of the isotropic parallel vector fields $b^k{}_\mu$ can be given by

$$b^k{}_\mu = {\scriptstyle k\downarrow} \begin{pmatrix} C & G \dfrac{x^\alpha}{r} \\ H \dfrac{x^a}{r} & D\delta_{a\alpha} + E \dfrac{x^a x^\alpha}{r^2} + F\varepsilon_{a\alpha\beta} \dfrac{x^\beta}{r} \end{pmatrix} , \tag{2.35}$$

with $r \equiv (x^\alpha x^\alpha)^{1/2}$ and $\varepsilon_{a\alpha\beta}$ being a totally antisymmetric tensor of the 3 dimensional Euclidean space normalized as $\varepsilon_{123} = 1$. The functions, C, D, E, F, G and H, depend on t and r. By a suitable redefinition of t and r, the parallel vector fields (2.35) are reduced without loss of generality to the form,

$$b^k{}_\mu = {}_k^{\downarrow}\begin{pmatrix} C & 0 \\ H\frac{x^a}{r} & D\delta_{a\alpha} + F\varepsilon_{a\alpha\beta}\frac{x^\beta}{r} \end{pmatrix} . \qquad (2.36)$$

By assuming the parallel vector fields static, the functions, C, D, F and H, depend only on r. If we impose time reversal invariance for the parallel vector fields, the function H should be vanishing. If we further require form invariance of the parallel vector fields for space inversion, F should be also vanishing. This means that macroscopic spin polarization $\mathbb{S}$ of the source of gravitation is negligibly small. It is reminded that the spin $\mathbb{S}$ is changed to $-\mathbb{S}$ by the space inversion. Finally, by rewriting C and D as $\sqrt{A}$ and $\sqrt{B}$, the parallel vector fields are reduced to the form,

$$b^k{}_\mu = {}_k^{\downarrow}\begin{pmatrix} \sqrt{A} & 0 \\ 0 & \sqrt{B}\ \delta_{a\alpha} \end{pmatrix} . \qquad (2.37)$$

By substituting (2.37) into (2.24) in vacuum with $\Lambda = 0$, the gravitational field equations are given by

$$\varepsilon(A'/A)' + (1-2\varepsilon)(B'/B)' + \frac{2}{r}\{\varepsilon(A'/A)+(1-2\varepsilon)(B'/B)\}$$
$$+ \frac{\varepsilon}{4}(A'/A)^2 + \frac{\varepsilon}{2}(A'/A)(B'/B) + \frac{1}{4}(1-4\varepsilon)(B'/B)^2 = 0 , \qquad (2.38a)$$

$$(1-2\varepsilon)(A'/A)' + (B'/B)' + \frac{1}{r}\{(1-2\varepsilon)(A'/A)+(B'/B)\}$$
$$+ \frac{1}{2}(1-3\varepsilon)(A'/A)^2 + \varepsilon(A'/A)(B'/B) = 0 , \qquad (2.38b)$$

$$(1-2\varepsilon)(A'/A)' +(B'/B)' - \frac{1}{r}\{(1-2\varepsilon)(A'/A)+(B'/B)\}$$
$$+ \frac{1}{2}(1-4\varepsilon)(A'/A)^2 - (1-3\varepsilon)(A'/A)(B'/B) - \frac{1}{2}(B'/B)^2 = 0 , \qquad (2.38c)$$

where

$$\varepsilon = (c_1+c_2)/(1+c_1+4c_2) \quad . \tag{2.39}$$

By demanding the boundary condition,

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \quad , \tag{2.40}$$

the solution of Eqs. (2.38) are obtained by

$$A(r) = (1-\alpha/pr)^{p}(1+\alpha/qr)^{-q} \quad ,$$

$$B(r) = (1-\alpha/pr)^{2-p}(1+\alpha/qr)^{2+q} \quad , \tag{2.41}$$

where

$$p \equiv \frac{2}{1-5\varepsilon}\,\{\sqrt{(1-\varepsilon)(1-4\varepsilon)} - 2\varepsilon\} \quad ,$$

$$q \equiv \frac{2}{1-5\varepsilon}\,\{\sqrt{(1-\varepsilon)(1-4\varepsilon)} + 2\varepsilon\} \quad . \tag{2.42}$$

Here α is an integration constant with the dimension of length and ε is assumed as $\varepsilon < 1/4$.

2-4) Comparison with observations

We consider meaning of the constant α and restrict the parameters c_1 and c_2 by the Newtonian limit and solar-system experiments. The constant α is restricted to the mass of the isotropic and non-relativistic source at the origin with $T^{00} \gg |T^{\alpha\beta}| \sim 0$. By the use of the expression (2.37) in Eqs. (2.32), the equations in the weak field approximation are given by

$$- (1+c_1+4c_2)[\ \varepsilon A'' + (1-2\varepsilon)B'' + \frac{2}{r}\{\varepsilon A'+(1-2\varepsilon)B'\}\ ] = \kappa T^{00} \quad , \tag{2.43a}$$

$$(1-2\varepsilon)A' + B' = 0 \quad , \tag{2.43b}$$

in which only T^{00} is taken into account as localized static source.

The solution is given by

$$A(r) = 1 - \frac{2}{(1-\varepsilon)(1-4\varepsilon)(1+c_1+4c_2)}\ \frac{GM}{r} \quad , \tag{2.44a}$$

$$B(r) = 1 + \frac{2(1-2\varepsilon)}{(1-\varepsilon)(1-4\varepsilon)(1+c_1+4c_2)}\ \frac{GM}{r} \quad , \tag{2.44b}$$

where G is the Newton constant and M represents the mass of the source centered at the origin,

$$M \equiv 4\pi \int T^{00} r^2\, dr \quad . \tag{2.45}$$

On the other hand, Newton's law of motion demands

$$g_{00} = - A = - 1 + 2\, \frac{GM}{r} \quad , \tag{2.46}$$

because of the geodesic equation (2.8b). By comparing (2.44a) with (2.46), it should be require that

$$(1-\varepsilon)(1-4\varepsilon)(1+c_1+4c_2) = 1 \quad , \tag{2.47a}$$

or equivalently,

$$4c_1 + c_2 + 9c_1c_2 = 0 \quad . \tag{2.47b}$$

In terms of the parameter ε defined by (2.39), c_1 and c_2 satisfying (2.47) can be written as

$$c_1 = -\frac{\varepsilon}{3(1-\varepsilon)} \quad , \quad c_2 = \frac{4\varepsilon}{3(1-4\varepsilon)} \quad . \tag{2.48}$$

From this restriction, the parameter α is related to the mass of the central source as

$$\alpha = GM \quad . \tag{2.49}$$

The final form of the static and isotropic metric of new general relativity is expressed by

$$\begin{aligned} ds^2 = &- (1-GM/pr)^p(1+GM/qr)^{-q}dt^2 \\ &+ (1-GM/pr)^{2-p}(1+GM/qr)^{2+q}dx^\alpha dx^\alpha \quad . \end{aligned} \tag{2.50}$$

It should be noted that in the case with $\varepsilon = 0$, (2.50) reduce to the Schwarzschild metric,[20]

$$ds^2 = -\left(\frac{1-GM/2r}{1+GM/2r}\right)^2 dt^2 + (1+GM/2r)^4 dx^\alpha dx^\alpha \quad , \tag{2.51}$$

because of the definition (2.42).

By the use of this metric (2.50), several comparisons with observations such as, for example, solar deflection and periherion advance, etc. , have been discussed. All the known data[21] have been consistently explained by a choice of the parameters ,

$$c_1 = 0.001 \pm 0.001 \quad , \quad c_2 = -0.005 \pm 0.005 \quad , \tag{2.52a}$$

namely,

$$\varepsilon = -0.004 \pm 0.004 \quad . \tag{2.52b}$$

It is very small. This is the reason why a special case $c_1 = c_2 = 0$ (or $\varepsilon = 0$) is discussed.

2-5) The case with $c_1 = c_2 = 0$

It is important to discuss a special case with $c_1 = c_2 = 0$ because the framework is somewhat different from the case with $c_1 \neq 0 \neq c_2$. The notion of the absolute parallelism is extended in this case. In the weak field approximation, definite particle picture can easily obtained.

In the case with $c_1 = c_2 = 0$, the gravitational action (2.23) is reduced to

$$I_G = \frac{1}{2\kappa} \int d^4x \ \sqrt{-g} \ [\ R(\{\}) + 2c_3(a^\mu a_\mu) - 2\kappa\Lambda \ ] \quad . \qquad (2.53)$$

It should be noted that this action is invariant under a restricted local Lorentz transformation which preserves the form of the axial-vector field a_μ . By dividing the gravitational field equations into symmetric and anti-symmetric parts for the indices μ and ν , they become in this case,

$$P^{\mu\nu} \equiv G^{\mu\nu}(\{\}) + K^{\mu\nu} + \kappa\Lambda g^{\mu\nu} = \kappa T^{(\mu\nu)} \quad , \qquad (2.54)$$

$$Q^{\mu\nu} \equiv b_i{}^\mu b_j{}^\nu \partial_\rho(\sqrt{-g} \ J^{ij\rho}) = \lambda\sqrt{-g} \ T^{[\mu\nu]} \quad , \qquad (2.55)$$

where

$$\lambda = 9\kappa/4c_3 \quad , \qquad (2.56)$$

$$K^{\mu\nu} \equiv \frac{\kappa}{\lambda} [\ \frac{1}{2} \{\varepsilon^{\mu\rho\sigma\lambda}(T^\nu{}_{\rho\sigma} - T_{\rho\sigma}{}^\nu) + \varepsilon^{\nu\rho\sigma\lambda}(T^\mu{}_{\rho\sigma} - T_{\rho\sigma}{}^\mu)\} \ a_\lambda - \frac{3}{2} \ a^\mu a^\nu \ - \frac{3}{4} \ g^{\mu\nu} a^\rho a_\rho \ ] \quad , \qquad (2.57)$$

$$J^{\mu\nu\rho} = b_i{}^\mu b_j{}^\nu J^{ij\rho} \equiv - \frac{3}{2} \ \varepsilon^{\mu\nu\rho\sigma} a_\sigma \quad . \qquad (2.58)$$

According to the restricted local Lorentz invariance of the gravitational action (2.53), there might be a local Lorentz transformation, $b'^k{}_\mu(x) = \Lambda^k{}_m(x) b^m{}_\mu(x)$, which leaves not only the axial-vector field a_μ but also the gravitational field equations (2.54) and (2.55) unchanged.[14),22)] This transformation also does not change the equations of motion for the gauge fields and the Dirac field.

For example, if $a_\mu = 0$, a local Lorentz transformation which preserves the relation $a'_\mu = 0$ does not change the gravitaional field equations because they coincide with the Einstein equations. This local Lorentz transformation cannot be observed by experiments. Therefore, we can regard these two parallel vector fields $b'^k{}_\mu(x)$ and $b^k{}_\mu(x)$ as equivalent objects.

In the case with $c_1 = c_2 = 0$, Hayashi and Shirafuji showed Birkhoff's theorem in new general relativity.[14)] They started with the most general isotropic parallel vector fields (2.36). By substituting (2.36) into the gravitational field equations (2.54) and (2.55) in vacuum, $T^{\mu\nu} = 0$, without the cosmological constant Λ , the equations for the functions, C, D, F and H, are obtained. They found that the axial-vector field a_μ should be identically vanishing if the boundary condition,

$$\lim_{r\to\infty} b^k{}_\mu = \delta^k{}_\mu \quad , \tag{2.59}$$

is imposed. Then the gravitational field equations are reduced to the Einstein equations which satisfy Birkhoff's theorem.[23)] In the case with $c_1 = c_2 = 0$, the isotropic gravitational field in empty space with the boundary condition (2.59) is the 'static' Schwarzschild solution.

Next, we examine weak field approximation. In the case with $c_1 = c_2 = 0$, the gravitational field equations in the weak field approximation (2.32) are reduced to

$$\Box\bar{h}_{\mu\nu} - (\partial_\mu\partial^\rho\bar{h}_{\rho\nu}+\partial_\nu\partial^\rho\bar{h}_{\rho\mu}) + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} = -2\kappa T_{(\mu\nu)}, \quad (2.60)$$

$$\Box A_{\mu\nu} - (\partial_\mu\partial^\rho A_{\rho\nu}-\partial_\nu\partial^\rho A_{\rho\mu}) = -\lambda T_{[\mu\nu]} . \quad (2.61)$$

Here the equations for the fields $\bar{h}_{\mu\nu}$ and $A_{\mu\nu}$ are completely separated.

Under the following transformations,

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu J_\nu - \partial_\nu J_\mu \quad , \quad (2.62)$$

$$A'_{\mu\nu} = A_{\mu\nu} + \partial_\mu H_\nu - \partial_\nu H_\mu \quad , \quad (2.63)$$

with J_μ and H_μ being small functions, the gravitational field equations (2.60) and (2.61) preserve their forms. These transformatioms can be regarded as gauge transformations. Using these degrees of freedom, we can take gauge conditions,

$$\partial^\rho\bar{h}_{\rho\mu} = 0 \quad , \quad (2.64)$$

$$\partial^\rho A_{\rho\mu} = 0 \quad . \quad (2.65)$$

Then the field equations (2.60) and (2.61) become

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{(\mu\nu)} \quad , \quad (2.66)$$

$$\Box A_{\mu\nu} = -\lambda T_{[\mu\nu]} \quad . \quad (2.67)$$

It should be noted that the equation (2.67) represents coupling of the antisymmetric field $A_{\mu\nu}$ with intrinsic spin of matter because of the Tetrode formula,

$$T_{[\mu\nu]} = \frac{1}{2}\partial^\rho S_{\mu\nu\rho} \quad , \quad (2.68)$$

with $S_{\mu\nu\rho}$ being a spin tensor.

Solutions of $(0,\alpha)$ components of Eqs. (2.66) and (2.67) which contain effects of the orbital and spin angular-momenta of the matter fields are given by

$$h_{0\alpha}(t,x) = \frac{\kappa}{4\pi} \varepsilon_{\alpha\beta\gamma} \frac{x^{\beta} J^{\gamma}(t-r)}{r^{3}} , \qquad (2.69a)$$

with

$$J_{\alpha}(t) = \varepsilon_{\alpha\beta\gamma} \int d^{3}x \ x^{\beta} T^{(\gamma 0)}(t,x) , \qquad (2.69b)$$

and

$$A_{0\alpha}(t,x) = \frac{\lambda}{8\pi} \varepsilon_{\alpha\beta\gamma} \frac{x^{\beta} S^{\gamma}(t-r)}{r^{3}} , \qquad (2.70a)$$

with

$$S_{\alpha}(t) = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} \int d^{3}x \ S^{\beta\gamma 0}(t,x) . \qquad (2.70b)$$

Here the quantities J_{α} and S_{α} are the components of the volume integrations of the total and spin angular-momenta, respectively. We will utilize these expressions of (2.69) and (2.70) in § 3 .

Spin and parity of the fields $\bar{h}_{\mu\nu}$ and $A_{\mu\nu}$ are 2^{+} and 0^{-}, respectively. The field $\bar{h}_{\mu\nu}$ is well-known as graviton field of general relativity. On the other hand, the field $A_{\mu\nu}$ is a characteristic field of new general relativity. From the positivity of energy of the field A_{μ} , the parameter λ should be positive.

§ 3 Stationary and axially symmetric solutions

In this section, we examine stationary and axially symmetric solutions of the gravitational field equations (2.54) and (2.55) in empty space with $\Lambda = 0$. In general relativity, many people have given axially symmetric solutions of the Einstein equations.[19),24)~32)] One of the remarkable points of new general relativity is that a notion of intrinsic spin of matter can be consistently included in the framework of the theory. If there is a source of gravitation whose intrinsic spin cannot be neglected, the axial-vector field a_μ has a finite value. It is very interesting to get a solution with non-vanishing axial-vector field a_μ . We here obtain stationary and axially symmetric gravitational fields coupled with the orbital and intrinsic spin angular-momenta of the system.

First of all, we shall determine a form of stationary and axially symmetric parallel vector fields. As for the general coordinate system, we use the cylindrical coordinate system with $\eta^0 = t$, $\eta^1 = \eta^1$, $\eta^2 = \eta^2$ and $\eta^3 = \phi$, where ϕ is a polar angle around the direction of $\mathbb{b}^{(2)}$.*) Then the parallel vector field $\mathbb{b}^{(2)}$ is parallel to the symmetric axis. The axial symmetry of $b^k{}_\mu$ is defined as follows: The parallel vector fields $b^k{}_\mu$ are form invariant under the transformation,

$$\phi' = \phi + \delta\phi \quad , \tag{3.1a}$$

---

*) In this paper, numbers of the Latin indices are enclosed in parenthesis in order to avoid confusion.

$$\mathbb{b}'^{(1)} = \mathbb{b}^{(1)}\cos\,\delta\phi - \mathbb{b}^{(3)}\sin\,\delta\phi \quad ,$$
$$\mathbb{b}'^{(3)} = \mathbb{b}^{(1)}\sin\,\delta\phi + \mathbb{b}^{(3)}\cos\,\delta\phi \quad . \qquad (3.1b)$$

They are given by

$$b^k{}_\mu = \overset{\downarrow}{k}\begin{pmatrix} A & E & I & M \\ B\cos\phi - D\sin\phi & F\cos\phi - H\sin\phi & J\cos\phi - L\sin\phi & N\cos\phi - Q\sin\phi \\ C & G & K & P \\ B\sin\phi + D\cos\phi & F\sin\phi + H\cos\phi & J\sin\phi + L\cos\phi & N\sin\phi + Q\cos\phi \end{pmatrix}, \qquad (3.2)$$

where A, B,···, P and Q are functions of η^1, η^2 and t .

We require that the parallel vector fields $b^k{}_\mu$ are stationary, thus having no dependence on time t . We further assume that $b^k{}_\mu$ are form invariant under the PT operation;

$$t' = -t \ , \quad \phi' = -\phi \ , \qquad (3.3a)$$
$$\mathbb{b}'^{(0)} = -\mathbb{b}^{(0)} \ , \quad \mathbb{b}'^{(3)} = -\mathbb{b}^{(3)} \ . \qquad (3.3b)$$

The parallel vector fields become

$$b^k{}_\mu = \overset{\downarrow}{k}\begin{pmatrix} A & 0 & 0 & M \\ -D\sin\phi & F\cos\phi & J\cos\phi & -Q\sin\phi \\ 0 & G & K & 0 \\ D\cos\phi & F\sin\phi & J\sin\phi & Q\cos\phi \end{pmatrix} \quad , \qquad (3.4)$$

where A, D,···, M and Q depend on η^1 and η^2 . Using a freedom of coordinate transformation for η^1 and η^2 , we can take the metric $g_{\mu\nu}$ constructed from (3.4) as

$$g_{11} = g_{22} \ , \quad g_{12} = 0 \ . \qquad (3.5)$$

Then the parallel vector fields $b^k{}_\mu$ can be written as

$$b^k{}_\mu = \overset{\downarrow}{k}\begin{pmatrix} A & 0 & 0 & M \\ -D\sin\phi & C\cos u\cos\phi & C\sin u\cos\phi & -Q\sin\phi \\ 0 & -C\sin u & C\cos u & 0 \\ D\cos\phi & C\cos u\sin\phi & C\sin u\sin\phi & Q\cos\phi \end{pmatrix} \quad , \qquad (3.6)$$

where C and u are functions of η^1 and η^2 .

Defining new functions, f, ω, τ, ρ and χ, which depend on η^1 and η^2 , we can rewrite the expression (3.6) to the form,

$$
b^k{}_\mu = \overset{\downarrow}{k}\begin{pmatrix}
f^{1/2}\cosh\frac{\chi}{2} & 0 & 0 & \rho f^{-1/2}\sinh\frac{\chi}{2} - f^{1/2}\omega\cosh\frac{\chi}{2} \\
-f^{1/2}\sinh\frac{\chi}{2}\sin\phi & f^{-1/2}e^{\tau}\cos u\cos\phi & f^{-1/2}e^{\tau}\sin u\cos\phi & -(\rho f^{-1/2}\cosh\frac{\chi}{2} - f^{1/2}\omega\sinh\frac{\chi}{2})\sin\phi \\
0 & -f^{-1/2}e^{\tau}\sin u & f^{-1/2}e^{\tau}\cos u & 0 \\
f^{1/2}\sinh\frac{\chi}{2}\cos\phi & f^{-1/2}e^{\tau}\cos u\sin\phi & f^{-1/2}e^{\tau}\sin u\sin\phi & (\rho f^{-1/2}\cosh\frac{\chi}{2} - f^{1/2}\omega\sinh\frac{\chi}{2})\cos\phi
\end{pmatrix} . \tag{3.7}
$$

From the expression (3.7), the metric becomes the well-known Papapetrou[25]- Ernst[26] form,

$$
ds^2 = - f(dt-\omega d\phi)^2 + f^{-1}[e^{2\tau}((d\eta^1)^2+(d\eta^2)^2)+\rho^2 d\phi^2]. \tag{3.8}
$$

It should be noted that the functions f and ρ cannot be vanishing because of

$$
\sqrt{-g} = \rho f^{-1}e^{2\tau} > 0 \quad . \tag{3.9}
$$

For later use, we show the asymptotic form of ((0),α) and (a,0) components of the parallel vector fields. In the asymptotic region far from the central source, our cylindrical coordinate η^μ is related to the rectangular coordinate x^μ as

$$
x^0 \sim \eta^0, \quad x^1 \sim \eta^1\cos\phi, \quad x^2 \sim \eta^2, \quad x^3 \sim \eta^1\sin\phi. \tag{3.10}
$$

In this cylindrical coordinate system, the solutions (2.69) and (2.70) in the weak field approximation are reduced to

$$b_{(0)1} \sim 0 ,$$

$$b_{(0)2} \sim 0 ,$$

$$b_{(0)3} \sim \frac{1}{8\pi} (\kappa J+\lambda S) \frac{(\eta^1)^2}{r^3} ,$$

$$b_{(1)0} \sim - \frac{1}{8\pi} (\kappa J-\lambda S) \frac{\eta^1}{r^3} \sin\phi ,$$

$$b_{(2)0} \sim 0 ,$$

$$b_{(3)0} \sim \frac{1}{8\pi} (\kappa J-\lambda S) \frac{\eta^1}{r^3} \cos\phi , \qquad (3.11)$$

with $r^2 \equiv (\eta^1)^2 + (\eta^2)^2$. Here we assume the volume integrations of the total and spin angular-momenta are in the direction of $\mathbb{b}^{(2)}$:

$$S^2 \equiv S , \quad J^2 \equiv J \equiv L + S ,$$

$$S^\alpha = 0 , \quad J^\alpha = 0 , \qquad (\alpha \neq 2) \qquad (3.12)$$

with L representing a volume integration of the orbital angular-momentum.

On the other hand, the $((0),\alpha)$ and $(a,0)$ components of the parallel vector fields (3.7) are given in the lowest order of the weak field approximation as

$$b_{(0)1} = 0 ,$$

$$b_{(0)2} = 0 ,$$

$$b_{(0)3} \sim - \frac{1}{2} \eta^1 \chi + \omega ,$$

$$b_{(1)0} \sim - \frac{1}{2} \chi \sin\phi ,$$

$$b_{(2)0} = 0 \ ,$$

$$b_{(3)0} \sim \frac{1}{2} \chi \cos\phi \ . \tag{3.13}$$

In this approximation, we should take $f \to 1$ and $\rho \to \eta^1$ because of the functions ω and χ being small quantities. Comparing (3.13) with (3.11), we get the asymptotic forms of ω and χ in the weak field approximation as

$$\omega^{asy} \sim \frac{\kappa}{4\pi} (L+S) \frac{(\eta^1)^2}{r^3} \ , \tag{3.14a}$$

$$\chi^{asy} \sim \frac{\kappa}{4\pi} [L+(1-\lambda/\kappa)S] \frac{\eta^1}{r^3} \ . \tag{3.14b}$$

Now, let us look for exact solutions. Substituting (3.7) into the gravitational field equations (2.54) and (2.55) with $\Lambda = 0$, we obtain the field equations for the symetric part,

$$P^{00} = e^{-2\tau} [f^{-2}I + \frac{\omega}{\rho} K - (1-\frac{f^2\omega^2}{\rho^2})L - \frac{1}{\rho} N] = 0 \ , \tag{3.15a}$$

$$P^{03} = P^{30} = e^{-2\tau} [\frac{1}{2\rho} K + \frac{f^2\omega}{\rho^2} L] = 0 \ , \tag{3.15b}$$

$$P^{33} = e^{-2\tau} \frac{f^2}{\rho^2} L = 0 \ , \tag{3.15c}$$

$$P^{11} + P^{22} = e^{-4\tau} \frac{f^2}{\rho} N = 0 \ , \tag{3.15d}$$

$$P^{11} - P^{22} = 2e^{-4\tau} \frac{f^2}{\rho} F = 0 \ , \tag{3.15e}$$

$$P^{12} = P^{21} = e^{-4\tau} \frac{f^2}{\rho} H = 0 \ , \tag{3.15f}$$

$$\text{Other components} \equiv 0 \ , \tag{3.15g}$$

and for the antisymmetric part,

$$Q^{03} = - Q^{30} = - \frac{1}{2\rho} U = 0 \ , \tag{3.16a}$$

$$Q^{12} = - Q^{21} = \frac{1}{2} e^{-\gamma} \frac{f}{\rho} V \quad , \tag{3.16b}$$

$$\text{Other components} \equiv 0 \quad , \tag{3.16c}$$

where

$$I \equiv f\nabla^2 f + \frac{f}{\rho} \nabla\rho\nabla f - (\nabla f)^2 + \frac{f^4}{\rho^2} (\nabla\omega)^2$$
$$- \frac{\kappa}{\lambda} \frac{f^3}{\rho^2} (A_1 \frac{\partial\omega}{\partial\eta^1} + A_2 \frac{\partial\omega}{\partial\eta^2}) = 0 \quad , \tag{3.17a}$$

$$K \equiv \nabla\cdot(\frac{f^2}{\rho} \nabla\omega) - \frac{\kappa}{\lambda} \{ \frac{f}{\rho^2} e^{\gamma}\cosh\frac{\chi}{2} (A_1\cos u + A_2\sin u)$$
$$+ \frac{1}{\rho} (A_1 \frac{\partial f}{\partial\eta^1} + A_2 \frac{\partial f}{\partial\eta^2}) - \frac{f}{\rho^2} (A_1 \frac{\partial\rho}{\partial\eta^1} + A_2 \frac{\partial\rho}{\partial\eta^2}) \} = 0 \, , \tag{3.17b}$$

$$L \equiv \nabla^2\gamma + \frac{1}{4} f^{-2}(\nabla f)^2 + \frac{1}{4} \frac{f^2}{\rho^2} (\nabla\omega)^2 + \frac{\kappa}{\lambda} \{ \frac{1}{4\rho^2} ({A_1}^2 + {A_2}^2)$$
$$- \frac{1}{2\rho} (A_1 \frac{\partial\chi}{\partial\eta^1} + A_2 \frac{\partial\chi}{\partial\eta^2}) \} = 0 \quad , \tag{3.17c}$$

$$N \equiv \nabla^2\rho - \frac{\kappa}{\lambda} \frac{1}{\rho} e^{\gamma}\sinh\frac{\chi}{2} (A_1\cos u + A_2\sin u) = 0 \quad , \tag{3.17d}$$

$$F \equiv - \frac{1}{2} (\frac{\partial^2\rho}{(\partial\eta^1)^2} - \frac{\partial^2\rho}{(\partial\eta^2)^2}) + (\frac{\partial\gamma}{\partial\eta^1} \frac{\partial\rho}{\partial\eta^1} - \frac{\partial\gamma}{\partial\eta^2} \frac{\partial\rho}{\partial\eta^2})$$
$$- \frac{1}{4} \rho f^{-2} \{ (\frac{\partial f}{\partial\eta^1})^2 - (\frac{\partial f}{\partial\eta^2})^2 \} + \frac{1}{4} \frac{f^2}{\rho} \{ (\frac{\partial\omega}{\partial\eta^1})^2 - (\frac{\partial\omega}{\partial\eta^2})^2 \}$$
$$+ \frac{\kappa}{\lambda} \{ - \frac{1}{4} \frac{f}{\rho} (A_1 \frac{\partial\omega}{\partial\eta^1} - A_2 \frac{\partial\omega}{\partial\eta^2})$$
$$+ \frac{1}{4} (A_1 \frac{\partial\chi}{\partial\eta^1} - A_2 \frac{\partial\chi}{\partial\eta^2}) \} = 0 \quad , \tag{3.17e}$$

$$H \equiv - \frac{\partial^2\rho}{\partial\eta^1\partial\eta^2} + (\frac{\partial\rho}{\partial\eta^1} \frac{\partial\gamma}{\partial\eta^2} + \frac{\partial\rho}{\partial\eta^2} \frac{\partial\gamma}{\partial\eta^1}) - \frac{1}{2} \rho f^{-2} \frac{\partial f}{\partial\eta^1} \frac{\partial f}{\partial\eta^2}$$
$$+ \frac{1}{2} \frac{f^2}{\rho} \frac{\partial\omega}{\partial\eta^1} \frac{\partial\omega}{\partial\eta^2} + \frac{\kappa}{\lambda} \{ - \frac{1}{4} \frac{f}{\rho} (A_1 \frac{\partial\omega}{\partial\eta^2} + A_2 \frac{\partial\omega}{\partial\eta^1})$$
$$+ \frac{1}{4} (A_1 \frac{\partial\chi}{\partial\eta^2} + A_2 \frac{\partial\chi}{\partial\eta^1}) \} = 0 \, , \tag{3.17f}$$

$$U \equiv \frac{\partial A_1}{\partial\eta^1} + \frac{\partial A_2}{\partial\eta^2} - \frac{1}{\rho} e^{\gamma}\cosh\frac{\chi}{2} (A_1\cos u + A_2\sin u) = 0 \, , \tag{3.17g}$$

$$V \equiv \sinh\frac{\chi}{2}\ (\ A_1 \sin u - A_2 \cos u\ ) = 0\ . \qquad (3.17h)$$

Here ∇ is the two dimensional gradient operator, $(\frac{\partial}{\partial\eta^1}, \frac{\partial}{\partial\eta^2})$, and A_1 and A_2 are related to the axial-vector field a_μ as

$$a_1 = -\frac{1}{3\rho} A_2 = -\frac{1}{3\rho}\ (\ 2e^{\tau}\sinh\frac{\chi}{2} \sin u - f\frac{\partial\omega}{\partial\eta^2} + \rho\frac{\partial\chi}{\partial\eta^2}\ )\ , \qquad (3.18a)$$

$$a_2 = \frac{1}{3\rho} A_1 \equiv \frac{1}{3\rho}\ (\ 2e^{\tau}\sinh\frac{\chi}{2} \cos u - f\frac{\partial\omega}{\partial\eta^1} + \rho\frac{\partial\chi}{\partial\eta^1}\ ), \qquad (3.18b)$$

$$a_0 = a_3 = 0\ . \qquad (3.18c)$$

Combining (3.17g) with (3.17b), we obtain

$$\tilde{K} \equiv K - \frac{\kappa}{\lambda}\frac{f}{\rho} U$$
$$= \nabla\cdot(\ \frac{f^2}{\rho}\nabla\omega) - \frac{\kappa}{\lambda}\ \{\ \frac{\partial}{\partial\eta^1}(\ \frac{f}{\rho} A_1) + \frac{\partial}{\partial\eta^2}(\ \frac{f}{\rho} A_2)\ \} = 0\ . \qquad (3.19)$$

The equations (3.17e) and (3.17f) are not independent of the others by the relation,

$$\frac{\partial F}{\partial\eta^1} + \frac{\partial H}{\partial\eta^2} = -\frac{1}{2}\frac{\partial}{\partial\eta^1}[N] + \frac{\partial\tau}{\partial\eta^1}[N] + \frac{\partial\rho}{\partial\eta^1}[L] + \frac{1}{2}\frac{\partial\omega}{\partial\eta^1}[\tilde{K}]$$
$$- \frac{1}{2}\rho f^{-3}\frac{\partial f}{\partial\eta^1}[I] + \frac{\kappa}{\lambda}\ \{\ \frac{1}{2}\frac{\partial\chi}{\partial\eta^1}[U] - \frac{1}{2\rho}\frac{\partial}{\partial\eta^2}[\rho V]$$
$$+ \frac{\partial u}{\partial\eta^1}[V]\ \} = 0\ , \qquad (3.20a)$$

$$-\frac{\partial F}{\partial\eta^2} + \frac{\partial H}{\partial\eta^1} = -\frac{1}{2}\frac{\partial}{\partial\eta^2}[N] + \frac{\partial\tau}{\partial\eta^2}[N] + \frac{\partial\rho}{\partial\eta^2}[L] + \frac{1}{2}\frac{\partial\rho}{\partial\eta^2}[\tilde{k}]$$
$$- \frac{1}{2}\rho f^{-3}\frac{\partial f}{\partial\eta^2}[I] + \frac{\kappa}{\lambda}\ \{\ \frac{1}{2}\frac{\partial\chi}{\partial\eta^2}[U] + \frac{1}{2\rho}\frac{\partial}{\partial\eta^1}[\rho V]$$
$$+ \frac{\partial u}{\partial\eta^2}[V]\ \} = 0\ . \qquad (3.20b)$$

We now take notice of Eq. (3.17h). It requires

$$A_1 \sin u - A_2 \cos u = 0\ , \qquad (3.21)$$

or

$$\chi \equiv 0 \ . \tag{3.22}$$

We examine these two cases separately.

(i) The case $A_1 \sin u - A_2 \cos u = 0$

Using the asymptotic form (3.14) in (3.18) with taking the limit $f \to 1$ and $\gamma \to 0$, the functions A_1 and A_2 are expressed in a region far from the central body as

$$A_1^{asy} \sim - \frac{\lambda S}{4\pi} \left(2 \frac{\eta^1}{r^3} - 3 \frac{(\eta^1)^3}{r^5} \right) ,$$

$$A_2^{asy} \sim 3 \frac{\lambda S}{4\pi} \frac{(\eta^1)^2 \eta^2}{r^5} \ . \tag{3.23}$$

Here the function u seems to vanish in this far region. Hence, from (3.21), it should be satisfied that $|A_1| >> |A_2|$, which is inconsistent with (3.23) except for $S = 0$.

In the exceptional case with $S = 0$, or with

$$A_1 \equiv 2e^{\gamma} \sinh\frac{\chi}{2} \cos u - f \frac{\partial \omega}{\partial \eta^1} + \rho \frac{\partial \chi}{\partial \eta^1} = 0 ,$$

$$A_2 \equiv 2e^{\gamma} \sinh\frac{\chi}{2} \sin u - f \frac{\partial \omega}{\partial \eta^2} + \rho \frac{\partial \chi}{\partial \eta^2} = 0 , \tag{3.24}$$

the gravitational field equations (3.17) are reduced to those of general relativity:

$$f\nabla^2 f + \frac{f}{\rho} \nabla\rho\cdot\nabla f - (\nabla f)^2 + \frac{f^4}{\rho^2} (\nabla\omega)^2 = 0 , \tag{3.25a}$$

$$\nabla\cdot\left(\frac{f^2}{\rho} \nabla\omega \right) = 0 , \tag{3.25b}$$

$$\nabla^2 \gamma + \frac{1}{4} f^{-2} (\nabla f)^2 + \frac{1}{4} \frac{f^2}{\rho^2} (\nabla\omega)^2 = 0 , \tag{3.25c}$$

$$\nabla^2 \rho = 0 , \tag{3.25d}$$

$$- \frac{1}{2} \left(\frac{\partial^2 \rho}{(\partial\eta^1)^2} - \frac{\partial^2 \rho}{(\partial\eta^2)^2} \right) + \left(\frac{\partial\gamma}{\partial\eta^1} \frac{\partial\rho}{\partial\eta^1} - \frac{\partial\gamma}{\partial\eta^2} \frac{\partial\rho}{\partial\eta^2} \right)$$

$$- \frac{1}{4} \rho f^{-2} \{ (\frac{\partial f}{\partial \eta^1})^2 - (\frac{\partial f}{\partial \eta^2})^2 \}$$

$$+ \frac{1}{4} \frac{f^2}{\rho} \{ (\frac{\partial \omega}{\partial \eta^1})^2 - (\frac{\partial \omega}{\partial \eta^2})^2 \} = 0 \quad , \tag{3.25e}$$

$$- \frac{\partial^2 \rho}{\partial \eta^1 \partial \eta^2} + (\frac{\partial \rho}{\partial \eta^1} \frac{\partial \gamma}{\partial \eta^2} + \frac{\partial \rho}{\partial \eta^2} \frac{\partial \gamma}{\partial \eta^1}) - \frac{1}{2} \rho f^{-2} \frac{\partial f}{\partial \eta^1} \frac{\partial f}{\partial \eta^2}$$

$$+ \frac{1}{4} \frac{f^2}{\rho} \frac{\partial \omega}{\partial \eta^1} \frac{\partial \omega}{\partial \eta^2} = 0 \quad . \tag{3.25f}$$

By taking a solution $\rho = \eta^1$ of (3.25d), these equations are reduced to the Ernst equation which has been well investigated by many people.[28)~30)]

We expect Eqs. (3.24) to have a solution for χ and u. The function χ will be determined by the following equation,

$$(\nabla \chi)^2 + 2 \frac{f}{\rho} \nabla \omega \nabla \chi + \frac{f^2}{\rho^2} (\nabla \omega)^2 - \frac{4}{\rho^2} e^{2\gamma} (\sinh \frac{\chi}{2})^2 = 0 \ , \tag{3.26}$$

where the functions, f, ω and γ, are determined by Eqs. (3.25). The existence of the solution for χ in (3.26) is not at present well-known. In the weak field approximation, however, it is given by

$$\omega^{asy} \sim \frac{\kappa L}{4\pi} \frac{(\eta^1)^2}{r^3} \quad , \qquad \chi^{asy} \sim \frac{\kappa L}{4\pi} \frac{\eta^1}{r^3} \quad . \tag{3.27}$$

We will have to examine the existence of the solution of (3.26) exactly.

(ii) The case with $\chi = 0$

We examine the case (3.22) in which the axial-vector field a_μ has the form,

$$a_1 = \frac{f}{3\rho} \frac{\partial \omega}{\partial \eta^2} \ , \quad a_2 = - \frac{f}{3\rho} \frac{\partial \omega}{\partial \eta^1} \ , \quad a_0 = a_3 = 0 \ . \tag{3.28}$$

The gravitational field equations (3.17) are reduced to

$$f\nabla^2 f + \frac{f}{\rho}\nabla\rho\nabla f - (\nabla f)^2 + (1+\kappa/\lambda)\frac{f^4}{\rho^2}(\nabla\omega)^2 = 0 \quad , \tag{3.29a}$$

$$(1+\kappa/\lambda)\nabla\cdot\left(\frac{f^2}{\rho}\nabla\omega\right) = 0 \quad , \tag{3.29b}$$

$$\nabla^2\gamma + \frac{1}{4}f^{-2}(\nabla f)^2 + \frac{1}{4}(1+\kappa/\lambda)\frac{f^2}{\rho^2}(\nabla\omega)^2 = 0 \quad , \tag{3.29c}$$

$$\nabla^2\rho = 0 \quad , \tag{3.29d}$$

$$-\frac{1}{2}\left(\frac{\partial^2\rho}{(\partial\eta^1)^2} - \frac{\partial^2\rho}{(\partial\eta^2)^2}\right) + \left(\frac{\partial\gamma}{\partial\eta^1}\frac{\partial\rho}{\partial\eta^1} - \frac{\partial\gamma}{\partial\eta^2}\frac{\partial\rho}{\partial\eta^2}\right)$$
$$-\frac{1}{4}\rho f^{-2}\left(\left(\frac{\partial f}{\partial\eta^1}\right)^2 - \left(\frac{\partial f}{\partial\eta^2}\right)^2\right)$$
$$+\frac{1}{4}(1+\kappa/\lambda)\frac{f^2}{\rho}\left(\left(\frac{\partial\omega}{\partial\eta^1}\right)^2 - \left(\frac{\partial\omega}{\partial\eta^2}\right)^2\right) = 0 \quad , \tag{3.29e}$$

$$-\frac{\partial^2\rho}{\partial\eta^1\partial\eta^2} + \left(\frac{\partial\rho}{\partial\eta^1}\frac{\partial\gamma}{\partial\eta^2} + \frac{\partial\rho}{\partial\eta^2}\frac{\partial\gamma}{\partial\eta^1}\right) - \frac{1}{2}\rho f^{-2}\frac{\partial f}{\partial\eta^1}\frac{\partial f}{\partial\eta^2}$$
$$+\frac{1}{4}(1+\kappa/\lambda)\frac{f^2}{\rho}\frac{\partial\omega}{\partial\eta^1}\frac{\partial\omega}{\partial\eta^2} = 0 \quad , \tag{3.29f}$$

$$\nabla\cdot(f\nabla\omega) - \frac{f}{\rho}e^{\gamma}\left(\frac{\partial\omega}{\partial\eta^1}\cos u + \frac{\partial\omega}{\partial\eta^2}\sin u\right) = 0 \quad . \tag{3.29g}$$

The last equation is obtained from the antisymmetric part of the gravitational field equations.

If we replace ω by

$$\tilde{\omega} \equiv (1+\kappa/\lambda)^{1/2}\omega \quad , \tag{3.30}$$

Eqs. (3.29a) ~ (3.29f) are reduced to the Einstein equations (3.25) which have been already solved. Thus our problem is to find a solution for the function u which satisfies Eq. (3.29g). We emphasize that Eq. (3.29g) does not include a derivative of the function u , so we can easily get a solution. In the case with $\tilde{\omega} \neq 0$, it is given by

$$\sin u = \frac{1}{X^2+Y^2} [YZ \pm X(X^2+Y^2-Z^2)^{1/2}] ,$$

$$\cos u = \frac{1}{X^2+Y^2} [XZ \mp Y(X^2+Y^2-Z^2)^{1/2}] , \qquad (3.31a)$$

where

$$X \equiv e^{\gamma} \frac{\partial \omega}{\partial \eta^1} , \quad Y \equiv e^{\gamma} \frac{\partial \omega}{\partial \eta^2} , \quad Z \equiv \rho f^{-1} \nabla\cdot(f\nabla\omega) . \qquad (3.31b)$$

Here the double signs in (3.31a) are in the same order.

We consider physical meaning of this gravitational field. Our solutions couple with the orbital and intrinsic spin angular-momenta of the system under a special condition. In the weak field approximation, the functions ω and χ are given by (3.14). The condition $\chi = 0$ demands the relation,

$$L = (\lambda/\kappa-1) S . \qquad (3.32)$$

In this sense, our solutions might as well be called as 'special' solutions.

We now examine the function u which does not appear in the metric (3.8). From (3.31a), the reality condition for the parallel vector fields is given by

$$X^2 + Y^2 \geq Z^2 , \qquad (3.33)$$

which may not be always satisfied.*) In the case of (3.33),

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*) It might be possible to reformulate new general relativity so that the complex parallel vector fields could be accepted. At present, it is not known to us whether such a reformulation is indeed possible.

|sin u| and |cos u| are equal to or smaller than unity. On the other hand, in the weak field approximation, the asymptotic functions X^{asy}, Y^{asy} and Z^{asy} are given by

$$X^{asy} \sim \frac{\kappa}{4\pi} (L + S)\left(2 \frac{\eta^1}{r^3} - 3 \frac{(\eta^1)^3}{r^5} \right) , \tag{3.34a}$$

$$Y^{asy} \sim - \frac{3\kappa}{4\pi} (L + S) \frac{(\eta^1)^2\eta^2}{r^5} , \tag{3.34b}$$

$$Z^{asy} \sim \frac{\kappa}{4\pi} (L + S)\left(2 \frac{\eta^1}{r^3} - 3 \frac{(\eta^1)^2}{r^5} \right) \sim X^{asy} , \tag{3.34c}$$

which lead to

$$\sin u = \frac{1}{(X^{asy})^2+(Y^{asy})^2} [X^{asy}Y^{asy} \pm X^{asy}|Y^{asy}|] ,$$

$$\cos u = \frac{1}{(X^{asy})^2+(Y^{asy})^2} [(X^{asy})^2 \mp Y^{asy}|Y^{asy}|] . \tag{3.35}$$

If we require that u vanish asymptotically, we must choose the upper signs in the region $Y < 0$ ($\eta^2 > 0$) and the lower one in the region $Y > 0$ ($\eta^2 < 0$) . This choice might break the condition of continuity of the function u .

As an example, we examine the reality and continuity of u using the Kerr solution:[25),28)] It is given by

$$f = \frac{r^2-2GMr+a^2\cos^2\theta}{r^2+a^2\cos^2\theta} , \tag{3.36a}$$

$$\tilde{\omega} = \frac{2GMar\sin^2\theta}{r^2-2GMr+a^2\cos^2\theta} = (1+\kappa/\lambda)^{1/2} \omega , \tag{3.36b}$$

$$e^{2\gamma} = \frac{r^2-2GMr+a^2\cos^2\theta}{r^2-2GMr+(GM)^2\sin^2\theta+a^2\cos^2\theta} , \tag{3.36c}$$

$$\rho = (r^2-2GMr+a^2)^{1/2}\sin\theta , \tag{3.36d}$$

in terms of the polar coordinate (t, r, θ, ϕ) relating with the cylindrical coordinate $(t, \eta^1, \eta^2, \phi)$ as

$$\eta^1 = (r^2-2GMr+a^2)^{1/2}\sin\theta\ , \tag{3.37a}$$

$$\eta^2 = (r-GM)\cos\theta\ . \tag{3.37b}$$

Here G is the Newton constant, M denotes mass of the system and a is a parameter representing the total angular-momentum J of the system as $J = (1+\kappa/\lambda)^{-1/2}ma$.

After numerical calculation, we find that the function u cannot be continuous by requiring $\lim_{r\to\infty} u = 0$, and cannot be real in a region near the event horizon. We show the region where the parallel vector fields cannot be real, in Figure. However, a remark should be in order. A certain class of local Lorentz transformation might have some relevance on this point. Such a transformation leaves the axial-vector field a_μ unchanged. If this transformation reflects an unphysical freedom of the theory, we might utilize it to save from the difficulty of the parallel vector fields. At present, physical meaning of such transformation is left as an open question.

In the case with $\omega = 0$, the axial-vector field a_μ vanishes identically and the gravitational field equations are reduced to those of general relativity with $\omega = 0$.[24), 29)] From the results of the weak field approximation of (3.14), $L = S = 0$ are required. This gravitational field is created by a source without orbital and spin angular-momenta. The function u cannot be determined from the gravitational field equations. We have a freedom of unphysical local Lorentz transformation which changes u in this case.

It should be noted that in the region in which a matter field exists, Eqs. (3.29) might not agree with the Einstein equations by the replacement (3.30) because of Eq. (3.29b). For example, it is difficult to find Kerr-Newman type solutions in which the electro-magnetic field is contained as a source of gravitation.[26),31),32)]

§ 4 Homogeneous and isotropic universe

In this section, we examine models of the homogeneous and isotropic universe. Recently, grand unified theories of weak, electro-magnetic and strong interactions predict that the cosmological constant Λ might has a finite value as energy of vacuum in the early universe.[33] Therefore, we should consider the gravitational field equations with Λ .

At first, we construct homogeneous and isotropic parallel vector fields using the method of Appendix A . In this case, the metric is given by the Robertson-Walker metric[34] which takes the following form in a comoving frame,

$$ds^2 = - dt^2 + R(t)^2 (\delta_{\alpha\beta} + \frac{k x^\alpha x^\beta}{1-kr^2}) dx^\alpha dx^\beta \quad , \qquad (4.1)$$

where R(t) is the radius of the universe depending only on time and k is a constant with dimension $(\text{length})^{-2}$. Since the parallel vector fields are isotropic, we can use the form (2.35). Then we assume the form invariance under space inversion, $x'^\alpha = - x^\alpha$ and $\mathbb{b}'_{(a)} = - \mathbb{b}_{(a)}$, which enables us to drop a term proportional to $\varepsilon_{a\alpha\beta} \frac{x^\beta}{r}$ in (2.35). The parallel vector fields which are related to the metric (4.1) are represented as

$$b^i{}_\mu = \overset{\downarrow}{i} \begin{pmatrix} \cosh\chi & R \dfrac{\sinh\chi}{(1-kr^2)^{1/2}} \dfrac{x^\alpha}{r} \\ \sinh\chi & R (\delta_{a\alpha} + (\dfrac{\cosh\chi}{(1-kr^2)^{1/2}} - 1) \dfrac{x^a x^\alpha}{r^2}) \end{pmatrix} , \qquad (4.2)$$

where χ is an unknown function of r and t to be determined by requiring that $b^i{}_\mu$ is form invariant under an isometry and a 'global' Lorentz transformation.

The isometry of the Roberson-Walker metric (4.1) consists of spatial rotations and quasitranslations:

$$t' = t \ , \quad x'^\alpha = x^\alpha + \varepsilon^\alpha{}_\beta x^\beta + c^\alpha (1-kr^2)^{1/2} \ , \tag{4.3}$$

with $\varepsilon^\alpha{}_\beta = -\varepsilon^\beta{}_\alpha$ and c^α being infinitesimal constants. The infinitesimal Lorentz transformation which compensates an infinitesimal coordinate transformation to preserve the form of the parallel vector fields, is given by (A.6) in Appendix A as

$$\omega^i{}_m = b^{i\rho} b_m{}^\lambda \nabla_\lambda \xi_\rho + K^i{}_{m\lambda} \xi^\lambda \ . \tag{4.4}$$

From (4.3), the Killing vector ξ^μ is given by

$$\xi^0 = 0 \ , \quad \xi^\alpha = \varepsilon^\alpha{}_\beta x^\beta + c^\alpha (1-kr^2)^{1/2} \ . \tag{4.5}$$

Substituting (4.2) and (4.5) into (4.4), we obtain

$$\omega^{(0)}{}_{(0)} = 0 \ , \tag{4.6a}$$

$$\omega^{(0)}{}_a = \omega^a{}_{(0)} = \chi'(1-kr^2)^{1/2} c^a - \left(\chi'(1-kr^2)^{1/2} - \frac{\sinh\chi}{r} \right) \times \left(c^a - \frac{c^\alpha x^\alpha x^a}{r^2} \right), \tag{4.6b}$$

$$\omega^a{}_b = \varepsilon^a{}_b + \left((1-kr^2)^{1/2} - \cosh\chi \right) \frac{c^a x^b - c^b x^a}{r^2} \ , \tag{4.6c}$$

with χ' representing $\partial\chi/\partial r$. In order for $\omega^i{}_m$ to be x-independent, the function χ should be chosen as

$$\cosh\chi = (1-kr^2)^{1/2} \ , \quad \sinh\chi = (-k)^{1/2} r \ . \tag{4.7}$$

The choice with $\sinh\chi = -(-k)^{1/2} r$ is also possible. However, this case is obtained from the case with (4.7) by time reversal. So we shall restrict ourselves to the choice

(4.7), since the gravitational field equations are invariant under time reversal.

Thus the parallel vector fields of the homogeneous and isotropic spacetime are given by

$$b^{i}{}_{\mu} = \underset{i}{\downarrow}\begin{pmatrix} (1-kr^{2})^{1/2} & \dfrac{R(-k)^{1/2}}{(1-kr^{2})^{1/2}}\, x^{\alpha} \\ (-k)^{1/2} x^{a} & R\,\delta_{a\alpha} \end{pmatrix} . \tag{4.8}$$

They satisfy the requirement of x-independence of $\omega^{i}{}_{m}$ as

$$\omega^{(0)}{}_{(0)} = 0 ,$$

$$\omega^{(0)}{}_{a} = \omega^{a}{}_{(0)} = (-k)^{1/2} c^{a} ,$$

$$\omega^{a}{}_{b} = \varepsilon^{a}{}_{b} . \tag{4.9}$$

If the universe is closed (namely, if $k > 0$) , the parallel vector fields (4.8) become complex valued, and also the transformation parameter, $\omega^{(0)}{}_{a}$ and $\omega^{a}{}_{(0)}$, are pure imaginary. In the present formulation of new general relativity, the parallel vector fields are assumed to be real. Therefore, we cannot take (4.8) as the parallel vector fields if the universe is closed. In other words, new general relativity with $\varepsilon \neq 0$ seems to be incompatible with the closed, homogeneous and isotropic universe. On the other hand, if the universe is open (namely, if $k \leqq 0$) , the parallel vector fields are uniquely determined by the requirement of homogeneity and isotropy.

In the special case with $\varepsilon = 0$, however, the situation is changed and the underlying spacetime which is called the extended Weitzenböck spacetime allows a restricted local

Lorentz transformation which leaves the axial-vector field a_μ and the gravitational field equations unchanged. On the other hand, in the expression (4.2), a_μ is identically vanishing. As is mentioned in Appendix A, a freedom of local Lorentz transformation which leaves a_μ vanishing is really allowed. Then the function χ cannot be determined by the gravitational field equations and is left arbitrary.

Consequently, although in the case with $k > 0$, the parameter ε should be chosen as $\varepsilon = 0$, in the case with $k \leq 0$, there is no restriction for ε .

Next, we turn our attention to the function $R(t)$ of (4.1) which describes the evolution of the universe. We shall now use the gravitational field equations to derive the equations for $R(t)$. At first, we assume that the parameter ε is nonvanishing. Then the universe should be open.

As for the source of the gravitational field, we assume as usual that cosmic matter can be approximated by perfect fluid with the energy-momentum tensor,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \quad , \tag{4.10}$$

with u^μ being the four-velocity of the fluid. In the comoving frame, it is given by

$$T^{00} = \rho \ , \quad T^{0\alpha} = T^{\alpha 0} = 0 \ , \quad T^{\alpha\beta} = p\ g^{\alpha\beta} \ , \tag{4.11}$$

where p and ρ are functions of time t denoting the energy density and the pressure of the perfect fluid in the comoving frame.

Using (4.8) and (4.11) in (2.24), we finally get

$$I^{00} = \frac{3}{1-4\varepsilon}\,\frac{\dot{R}^2 + k}{R^2} - \kappa\Lambda = \kappa\rho \quad , \tag{4.12a}$$

$$I^{0\alpha} = I^{\alpha 0} = 0 \quad , \tag{4.12b}$$

$$I^{\alpha\beta} = [-\frac{1}{1-4\varepsilon}\,\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} + \kappa\Lambda\,]\,g^{\alpha\beta} = \kappa p\, g^{\alpha\beta} \quad , \tag{4.12c}$$

or equivalently,

$$\dot{R}^2 + k = \frac{\kappa}{3}(1-4\varepsilon)(\rho + \Lambda)R^2 \quad , \tag{4.13}$$

$$2R\ddot{R} + \dot{R}^2 + k = -\,\kappa(1-4\varepsilon)(p - \Lambda)R^2 \quad . \tag{4.14}$$

These equations give the conservation law of energy,

$$\frac{d}{dR}[(\rho + \Lambda)R^3] = -\,3(p - \Lambda)R^2 \quad . \tag{4.15}$$

We can take (4.13) and (4.15) as the independent equations. The equation of energy conservation (4.15) is also satisfied in general relativity and, therefore, (4.13) characterize the gravitational field equations of new general relativity. These equations (4.13) and (4.15) are reduced to those of general relativity with a cosmological constant by replacing the Einstein constant κ as $\tilde{\kappa} \equiv (1-4\varepsilon)\kappa$. Since the parameter ε is expected to be very small, new general relativity gives homogeneous and isotropic models of the open universe nearly the same as those of general relativity.

In the case with $\Lambda = 0$ and $p = 0$, or $\Lambda = 0$ and $p = \rho/3$, the solutions are the well-known Friedmann models.[35] In the case with $\Lambda > 0$ and $p = \rho = 0$, it is called the de Sitter universe.[36] Here we can get the homogeneous and isotropic solutions of new general relativity with $\varepsilon \neq 0$.

When the parameter ε is vanishing, the underlying spacetime is the extended Weitzenböck spacetime. The function χ of (4.2) is left undetermined. The gravitational equations coincide with those of general relativity. New general relativity with $\varepsilon = 0$ thus gives same homogeneous and isotropic models of the universe as those of general relativity.

§ 5 Kasner type universe

In this section, we examine models of the universe which is homogeneous and anisotropic. We now observe that the present universe is isotropic with high accuracy. In the early universe, however, we do not know the isotropy of the universe. In general relativity, one of the models of the anisotropic universe is the well-known Kasner solution. Here we study this type of the universe in new general relativity. For forthcoming applications of the Kaluza-Klein theory,[37),38)] we consider solutions of new general relativity in n+1 dimensional spacetime. Short review of n+1 dimensional new general relativity is given in Appendix B .

We restrict ourselves to diagonal and space-independent parallel vector fields,

$$b^k{}_\mu = \overset{\downarrow}{k}\begin{pmatrix} 1 & 0 \\ 0 & \exp\lambda_\alpha(t)\ \delta_{a\alpha} \end{pmatrix}, \tag{5.1}$$

where the indices a and α run from 1 to n. The functions λ_α depend only on time. We shall call the form (5.1) "Kasner type" parallel vector fields. This spacetime is spatially homogeneous.

We substitute the expression (5.1) into the gravitational field equations (B.9) without matter fields to get the equations for λ_α as

$$\frac{1}{2}\left[-(1-3c_1)\sum_{\alpha=1}^{n}(\dot\lambda_\alpha)^2 + (1-\frac{3}{n}c_1+2c_2)(\sum_{\alpha=1}^{n}\dot\lambda_\alpha)^2 \right] - \kappa\Lambda = 0 , \tag{5.2a}$$

$$(1-3c_1)\left[\ddot\lambda_\alpha + \dot\lambda_\alpha\sum_{\beta=1}^{n}\dot\lambda_\beta - \frac{1}{2}\sum_{\beta=1}^{n}(\dot\lambda_\beta)^2 \right]$$

$$- (1-\frac{3}{n}c_1+2c_2)[\sum_{\beta=1}^{n}\ddot{\lambda}_\beta + \frac{1}{2}(\sum_{\beta=1}^{n}\dot{\lambda}_\beta)^2] + \kappa\Lambda = 0 \quad , \qquad (5.2b)$$

where the dots represent derivatives with respect to t . At present, we have no way to restrict the values of c_1 and c_2 in general dimensions. In view of the fact that $|c_1|$ and $|c_2|$ are both known to be very small in the case of $n = 3$, we hereafter postulate

$$1 - 3c_1 > 0 \ , \quad 1 - \frac{3}{n} c_1 + 2c_2 > 0 \ . \qquad (5.3)$$

We take summations over α in (5.2b),

$$(1-3c_1)[\sum_{\alpha=1}^{n}\ddot{\lambda}_\alpha + (\sum_{\alpha=1}^{n}\dot{\lambda}_\alpha)^2 - \frac{n}{2}\sum_{\alpha=1}^{n}(\dot{\lambda}_\alpha)^2]$$

$$- (1-\frac{3}{n}c_1+2c_2)n[\sum_{\alpha=1}^{n}\ddot{\lambda}_\alpha + \frac{1}{2}(\sum_{\alpha=1}^{n}\dot{\lambda}_\alpha)^2] + n\kappa\Lambda = 0 \quad . \qquad (5.4)$$

Using (5.2a) in (5.4), we get the equation,

$$\sum_{\alpha=1}^{n}\ddot{\lambda}_\alpha + (\sum_{\alpha=1}^{n}\dot{\lambda}_\alpha)^2 = 2n\kappa\Lambda/[(n-1)+2nc_2] \quad . \qquad (5.5)$$

The spacetime is classified by the sign of the cosmological constant Λ . Solutions for $\sum_{\alpha=1}^{n}\dot{\lambda}_\alpha$ are obtained as

for $\Lambda = 0$,

$$\text{(i)} \quad \sum_{\alpha=1}^{n}\dot{\lambda}_\alpha = 0 \ , \qquad (5.6a)$$

$$\text{(ii)} \quad \sum_{\alpha=1}^{n}\dot{\lambda}_\alpha = \frac{1}{t+\delta} \ , \qquad (5.6b)$$

for $\Lambda > 0$,

$$\text{(i)} \quad \sum_{\alpha=1}^{n}\dot{\lambda}_\alpha = \pm\, 2p \ , \qquad (5.7a)$$

$$\text{(ii)} \quad \sum_{\alpha=1}^{n}\dot{\lambda}_\alpha = 2p\,\coth(2pt+\delta) \ , \qquad (5.7b)$$

$$\text{(iii)} \quad \sum_{\alpha=1}^{n}\dot{\lambda}_\alpha = 2p\,\tanh(2pt+\delta) \ , \qquad (5.7c)$$

for $\Lambda < 0$,

(i) $$\sum_{\alpha=1}^{n} \dot{\lambda}_\alpha = \pm 2pi \quad , \tag{5.8a}$$

(ii) $$\sum_{\alpha=1}^{n} \dot{\lambda}_\alpha = 2p \cot(2pt+\delta) \quad , \tag{5.8b}$$

where

$$p \equiv [\, n\kappa|\Lambda|/2\{(n-1)+2nc_2\} \,]^{1/2} \quad , \tag{5.9}$$

with δ's being integration constants. Using constant transformations of time t , we can take $\delta = 0$ without loss of generality. Here we consider that the universe begins at $t = 0$, except for (5.7a) and (5.8a). The solutions (5.7c) and (5.8a) lead to complex valued parallel vector fields. We discard these two cases because we consider real spacetime.

Inserting (5.6)~(5.8) into (5.2a), we get the expressions of $\sum_{\alpha=1}^{n} (\dot{\lambda}_\alpha)^2$,

for $\Lambda = 0$,

(i) $$\sum_{\alpha=1}^{n} (\dot{\lambda}_\alpha)^2 = 0 \quad , \tag{5.10a}$$

(ii) $$\sum_{\alpha=1}^{n} (\dot{\lambda}_\alpha)^2 = \frac{1-(3/n)c_1+2c_2}{1-3c_1}\, t^{-2} \quad , \tag{5.10b}$$

for $\Lambda > 0$,

(i) $$\sum_{\alpha=1}^{n} (\dot{\lambda}_\alpha)^2 = \frac{4p^2}{n} \quad , \tag{5.11a}$$

(ii) $$\sum_{\alpha=1}^{n} (\dot{\lambda}_\alpha)^2 = \frac{4p^2}{1-3c_1} [\, (1-\frac{3}{n}c_1+2c_2)(\coth 2pt)^2 - \frac{1}{n}\{(n-1)+2nc_2\} \,] \quad , \tag{5.11b}$$

for $\Lambda < 0$,

(ii) $$\sum_{\alpha=1}^{n} (\dot{\lambda}_\alpha)^2 = \frac{4p^2}{1-3c_1} [\, (1-\frac{3}{n}c_1+2c_2)(\cot 2pt)^2$$

$$+ \frac{1}{n} \{(n-1)+2nc_2\}] \quad . \tag{5.12}$$

The expression (5.10a) leads to

$$\lambda_\alpha = \text{constant} \ , \tag{5.13}$$

representing the n+1 dimensional Minkowski spacetime.

Substituting (5.6)~(5.8) and (5.10)~(5.12) into (5.2b), we get the equations for each λ_α as

for $\Lambda = 0$,

$$\text{(ii)} \quad \ddot{\lambda}_\alpha + \frac{1}{t} \dot{\lambda}_\alpha = 0 \quad , \tag{5.14}$$

for $\Lambda > 0$,

$$\text{(i)} \quad \ddot{\lambda}_\alpha \pm 2p\dot{\lambda}_\alpha - 4p^2/n = 0 \quad , \tag{5.15a}$$

$$\text{(ii)} \quad \ddot{\lambda}_\alpha + 2p(\coth 2pt)\dot{\lambda}_\alpha - 4p^2/n = 0 \quad , \tag{5.15b}$$

for $\Lambda < 0$,

$$\text{(i)} \quad \ddot{\lambda}_\alpha + 2p(\cot 2pt)\dot{\lambda}_\alpha + 4p^2/n = 0 \quad , \tag{5.16}$$

solutions of (5.14)~(5.16) are given by

for $\Lambda = 0$,

$$\text{(ii)} \quad \lambda_\alpha = \log t^{q_\alpha} + \tau_\alpha \quad , \tag{5.17}$$

for $\Lambda > 0$,

$$\text{(i)} \quad \lambda_\alpha = \frac{2pt}{n} + q_\alpha \exp(-2pt) + \tau_\alpha \quad , \tag{5.18a}$$

$$\text{(ii)} \quad \lambda_\alpha = \log (\sinh pt)^{q_\alpha} (\cosh pt)^{\frac{2}{n}-q_\alpha} + \tau_\alpha \quad , \tag{5.18b}$$

for $\Lambda < 0$,

$$\text{(ii)} \quad \lambda_\alpha = \log (\sin pt)^{q_\alpha} (\cos pt)^{\frac{2}{n}-q_\alpha} + \tau_\alpha \quad , \tag{5.19}$$

where q_α and τ_α are integration constants. Using constant scale transformations of the coordinate x^α, we can take

$\tau_\alpha = 0$ without loss of generality. The solutions are considered to be realizable while $-\infty < t < \infty$ for (5.18a), $t > 0$ for (5.17) and (5.18b), and $0 < t < \pi/2p$ for (5.19). In (5.18a), we choose expanding universe, taking the upper sign of (5.7a).

Substituting the derivatives of (5.17)~(5.19) into (5.6)~(5.8) and (5.10)~(5.12), we find that the constraints for q_α should be irrespectively of the signs of Λ ,

$$\sum_{\alpha=1}^{n} q_\alpha = 1 , \qquad \sum_{\alpha=1}^{n} (q_\alpha)^2 = \frac{1-(3/n)c_1+2c_2}{1-3c_1} , \tag{5.20}$$

except for the solution (5.18a). In the case of (5.18a), q_α should be vanishing.

The parallel vector fields are given by

for $\Lambda = 0$,

(i) $$b^k{}_\mu = \delta^k{}_\mu , \tag{5.21a}$$

(ii) $$b^k{}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & t^{q_\alpha}\,\delta_{a\alpha} \end{pmatrix} , \tag{5.21b}$$

for $\Lambda > 0$,

(i) $$b^k{}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \exp(2pt/n)\,\delta_{a\alpha} \end{pmatrix} , \tag{5.22a}$$

(ii) $$b^k{}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & (\sinh pt)^{q_\alpha}(\cosh pt)^{\frac{2}{n}-q_\alpha}\,\delta_{a\alpha} \end{pmatrix} , \tag{5.22b}$$

for $\Lambda < 0$,

(ii) $$b^k{}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & (\sin pt)^{q_\alpha}(\cos pt)^{\frac{2}{n}-q_\alpha}\,\delta_{a\alpha} \end{pmatrix} , \tag{5.23}$$

with the condition (5.20) and the definition (5.9). Of course, our solutions include those of general relativity with a cosmological constant as a limiting case of $c_1 = c_2 = 0$. The solution (5.21b) corresponds to the Kasner solution[39] in general relativity. The solution (5.22a) should be called the de Sitter universe[36] in new general relativity. It should be noted that this solution was obtained in § 4 by assuming the homogeneity and isotropy of the universe. In general relativity, the solutions with $\Lambda \neq 0$ like (5.22b) and (5.23) were discussed by Saunders.[40]

The solutions obtained above are classified into two cases as whether space is isotropic or not. The cases (i) are isotropic. In the case with $\Lambda < 0$, there is no real isotropic solution having the expression (5.1).

Hereafter, we mainly discuss the solutions in 3+1 dimensional spacetime. As already mentioned in § 2 , the condition (5.3) seems to be reasonable in 3+1 dimension. We should take n = 3 in the expressions (5.21)~(5.23). In terms of the parameter ε , the definition (5.9) and the constraints (5.20) are respectively expressed as

$$p \equiv [\, \frac{3}{4}\,(1-4\varepsilon)\kappa|\Lambda|\,]^{1/2} \quad , \tag{5.24}$$

$$\sum_{\alpha=1}^{3} q_\alpha = 1 \quad , \qquad \sum_{\alpha=1}^{3} (q_\alpha)^2 = \frac{1-2\varepsilon}{1-4\varepsilon} \ . \tag{5.25}$$

In the constraints (5.25), all the three parameters q_α cannot have a same value, exept for the case with $\varepsilon = 1$ which is inconsistent with (2.52). When two of them have a same value, they are given by

$$q_1 = q_2 = \frac{1}{3}\,[1-\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}] \ , \ q_3 = \frac{1}{3}\,[1+2\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}] \ , \tag{5.26a}$$

$$q_1 = \frac{1}{3}\left[1-2\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] , \quad q_2 = q_3 = \frac{1}{3}\left[1+\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] . \quad (5.26b)$$

When all of them are different, they are restricted in the region,

$$\frac{1}{3}\left[1-2\left(\frac{1-\varepsilon}{1-2\varepsilon}\right)^{1/2}\right] < q_1 < \frac{1}{3}\left[1-\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] ,$$

$$\frac{1}{3}\left[1-\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] < q_2 < \frac{1}{3}\left[1+\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] ,$$

$$\frac{1}{3}\left[1+\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] < q_3 < \frac{1}{3}\left[1+2\left(\frac{1-\varepsilon}{1-4\varepsilon}\right)^{1/2}\right] . \quad (5.27)$$

Here we assume without loss of generality, $q_1 \leqq q_2 < q_3$ or $q_1 < q_2 \leqq q_3$.

For (5.21b) with $n = 3$ and $\Lambda = 0$, in the case with $\varepsilon = 0$, distances parallel to two of three spatial axes must expand and those parallel to another axis must contract except for (5.26a).*) However, in the case with $\varepsilon > 0$, distances parallel to one of three axes expand and those parallel to the other axes contract for the condition (5.26a). In the case with $\varepsilon < 0$, distances parallel to all the axes expand for (5.26a). In general relativity, there is no solution of these two types. Developments of the other universes with $\Lambda \neq 0$ can be easily examined.

Finally, a comment should be noted. We are mainly considering the anisotropic space in the early universe. However, mechanisms such as quantum mechanical back reaction which suppress this anisotropy in the development of the early universe have been discussed by many people.[41] This problem should be reconsidered in the framework of new general relativity.

Footnote to p. 50

*) In the case with $\varepsilon = 0$, the condition, $q_1 = q_2 = 0$ and $q_3 = 1$, leads to the following line element,

$$ds^2 = - dt^2 + dx^2 + dy^2 + t^2 dz^2 .$$

By the coordinate transformation,

$$\tau = t \cosh z \ , \quad \xi = x \ , \quad \eta = y \ , \quad \zeta = t \sinh z \ ,$$

the line element is reduced to Minkowskian. Then the parallel vector fields become

$$b^k{}_\mu = \overset{\downarrow}{k}\begin{pmatrix} \tau/(\tau^2-\zeta^2)^{1/2} & 0 & 0 & -\zeta/(\tau^2-\zeta^2)^{1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\zeta/(\tau^2-\zeta^2)^{1/2} & 0 & 0 & \tau/(\tau^2-\zeta^2)^{1/2} \end{pmatrix} . \qquad (\#)$$

Furthermore, new general relativity allows a restricted local Lorentz transformation which leaves the axial-vector field a_μ vanishing.[14),22)] Due to this local Lorentz covariance, the parallel vector fields (#) can be regarded as to be equivalent to $b^k{}_\mu = \delta^k{}_\mu$.

§ 6 Summary and discussion

The geometry underlying new general relativity is the Weitzenböck spacetime imposed with absolute parallelism. This spacetime is characterized only by the torsion field. The gravitational field equations which are derived from the gravitational Lagrangian quadratic in the torsion can include the Dirac spinor field as a source of gravitation. In this sense, new general relativity seems more suitable for microscopic system than general relativity.

In this paper, we have obtained some exact solutions of the gravitational field equations in new general relativity for spinning black hole and the universe. These solutions have several differences from related solutions of the Einstein equations. In the following, we will express these differences for each solution separately.

In the past, no one has exactly discussed possible effects of finite polarization of spin to the black hole solution. This problem is, in some sense, beyond the scope of general relativity because the symmetry of the gravitational field equations is inconsistent with that of energy-momentum tensor for the Dirac field. In new general relativity, a part of the torsion field couples with the antisymmetric part of the energy-momentum tensor. Actually, in the case with $\varepsilon = 0$, we could find stationary and axially symmetric solutions which are coupled with the intrinsic spin of source. Although the metric of the solutions can be obtained from relevant solutions of general relativity, the existence of the axial-vector field a_μ

reflecting the effects of the intrinsic spin is essential. If such a type of black hole is permitted, fermion feels the torsion and is expected to make precession of the spin.

The solutions thus obtained have unusual feature that reality and continuity of the parallel vector fields are not satisfied even outside of the event horizon. However, this difficulty might be superficial. There is a freedom of local Lorentz transformation which preserves the form of the axial-vector field a_{μ} . This local Lorentz transformation cannot be observed by means of experiments with the Dirac fields and gauge fields because the equations of motion for these fields are covariant under this transformation. Therefore, this local Lorentz transformation might be regarded as an unphysical one and could be used to settle the difficulty for the parallel vector fields. It is still an open problem how to interprete this local Lorentz transformation. We believe that our stationary and axially symmetric solutions will present a good testing ground to this problem.

Next, we turn our attention to cosmological solutions. Models of the universe have been discussed in other theories of gravitation with torsion in order to avoid singularity at the initial time of the universe by many people.[42)] On this problem, we showed that in new general relativity the homogeneous and isotropic universe composed of matter without macroscopic spin polarization is quite similar to those of general relativity. Then the closed, homogeneous and isotropic universe is incompatible with the absolute parallelism. However, the situation is changed in the case with $\varepsilon = 0$. There exists a freedom of unphysical local Lorentz transformation which leaves not only the axial-vector field a_μ but also the gravitational field equations unchanged. By means of this transformation, we obtained models of the universe equivalent to those of general relativity. It is very interesting to consider effects of macroscopic spin polarization of the cosmic matter which might have influence on the singularity problem at the beginning of the universe.

We have postulated the isotropy of the universe so far. However, we should also construct anisotropic models of the universe because it has been recently clarified that anisotropy of the universe in higher dimensional spacetime presents interesting possibility toward a unified theory of gravitation and matter. A mechanism of dimensional reduction in the Kaluza-Klein theory has been proposed based on the anisotropic expansion of the Kasner metric in general relativity.[38)] In connection with this mechanism, the Dirac hypothesis[43)] can be realized. We obtained the Kasner type solutions in the n+1 dimensional Weitzenböck spacetime. When the parameters c_1 and c_2 are not vanishing, expansion of the universe could be qualitatively different from that of general relativity.

Here we neglected matter fields and stayed in classical level but it has been suggested that quantum mechanical creation of particles in anisotropically expanding spacetime may restore the isotropy.[41)] This problem should be reconsidered by means of our solutions in new general relativity.

Finally, new general relativity is a limiting case of Poincaré gauge theory which is characterized by the torsion and the curvature.[15)] It is very important to look for exact solutions of the gravitational field equations in the framework of Poincaré gauge theory.

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Appendix A : Absolute parallelism and symmetry of spacetime

We discuss symmetry of spacetime for the parallel vector fields with absolute parallelism. Consider a transformation of parallel vector fields induced by a coordinate transformation, $x^{\mu} \to x'^{\mu}$, and a global Lorentz transformation,

$$b'^{k}{}_{\mu}(x') = A^{k}{}_{m} \frac{\partial x^{\nu}}{\partial x'^{\mu}} b^{m}{}_{\nu}(x) \quad , \tag{A.1}$$

where $A^{k}{}_{m}$ are constants satisfying

$$\eta_{ij} A^{i}{}_{k} A^{j}{}_{m} = \eta_{km} \quad . \tag{A.2}$$

When the transformed parallel vector fields $b'^{k}{}_{\mu}(x')$ are the same functions of their argument x'^{μ} as the original parallel vector fields $b^{k}{}_{\mu}(x)$ of their argument x^{μ} ,

$$b'^{k}{}_{\mu}(y) = b^{k}{}_{\mu}(y) \qquad \text{for all } y \ , \tag{A.3}$$

we shall call (A.1) a symmetry transformation in the Weitzenböck spacetime.

For an infinitesimal symmetry transformation of the Weitzenböck spacetime,

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \qquad (\ |\xi^{\mu}| \ll 1 \ ), \tag{A.4a}$$

$$A^{k}{}_{m} = \delta^{k}{}_{m} + \omega^{k}{}_{m} \qquad (\ \omega_{km} + \omega_{mk} = 0 \ , \quad |\omega_{km}| \ll 1 \ ), \tag{A.4b}$$

the functions ξ^{μ} must be a Killing vector satisfying the Killing equations,

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0 \quad . \tag{A.5}$$

Here ∇_{μ} denotes a covariant derivative with respect to the Christoffel symbol. The condition (A.3) gives

$$\omega_{km} = b_{k}{}^{\rho} b_{m}{}^{\sigma} \nabla_{\sigma}\xi_{\rho} + K_{km\lambda}\xi^{\lambda} \ , \tag{A.6}$$

where $K_{km\mu}$ is the contorsion tensor related to the torsion tensor $T_{\lambda\mu\nu}$ as

$$K_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu}+T_{\mu\lambda\nu}-T_{\nu\lambda\mu}) = -K_{\mu\lambda\nu} . \qquad (A.7)$$

We can easily see ω_{km} in (A.6) antisymmetric because of the properties of the indices in (A.5) and (A.7). Since only global Lorentz transformations are allowed, ω_{km} in (A.6) should be x-independent. This condition imposes a restriction on the parallel vector fields. The Killing equations (A.5) and this condition can be rewritten as

$$D_\mu\xi_\nu + D_\nu\xi_\mu + (T_{\mu\nu\lambda}+T_{\nu\mu\lambda})\xi^\lambda = 0 , \qquad (A.8a)$$

$$D_\lambda[D_\nu\xi_\mu + T_{\mu\nu\rho}\xi^\rho] = 0 , \qquad (A.8b)$$

with D_μ denoting a covariant derivative in the Weitzenböck spacetime.

In the special case with $c_1 = c_2 = 0$ ($\varepsilon = 0$) , the situation is changed. There might be a restricted local Lorentz transformation which leaves the axial-vector part a_μ of the torsion tensor and the gravitational field equations unchanged. As for the matter fields, for example the Dirac fields and gauge fields, the equations of motion are covariant under this local Lorentz transformation. We cannot observe the effects of this local Lorentz transformation, so we should regard these two parallel vector fields which are connected to each other by this local Lorentz tranfsformation, as physically equivalent objects.

New general relativity with $c_1 = c_2 = 0$ allows a local Lorentz transformation which preserves the form of the axial-vector field a_μ and that of the gravitational field equations. Especially, in the case with $a_\mu = 0$, a local Lorentz transformation which leaves a_μ vanishing is allowed because the gravitational field equations are reduced to the Einstein equations which are invariant under local Lorentz transformations.

Appendix B : New general relativity in n+1 dimensional spacetime

Fundamental entity is parallel vector fields $\mathbb{b}^k = (b^k{}_\mu)$. The Greek and the Latin indices run from 0 to n referring to an external general coordinate frame and an internal Lorentz frame. The metric is defined as

$$g_{\mu\nu} = \eta_{km} b^k{}_\mu b^m{}_\nu \quad , \qquad \text{(B.1)}$$

where η_{km} is the metric in the n+1 dimensional Minkowski spacetime: $\eta_{km} = \mathrm{diag}(-1,1,\cdots,1)$. By requiring absolute parallelism $D_\lambda b^k = 0$, the affine connection $\Gamma^\lambda{}_{\mu\nu}$ is given by

$$\Gamma^\lambda{}_{\mu\nu} = b_k{}^\lambda \partial_\nu b^k{}_\mu \; . \qquad \text{(B.2)}$$

The torsion tensor $T^\lambda{}_{\mu\nu}$ are defined by

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \; . \qquad \text{(B.3)}$$

In order to obtain the most general gravitational Lagrangian quadratic in the torsion fields, we should take the irreducible decomposition for the torsion as

$$t_{\lambda\mu\nu} \equiv \frac{1}{2}(T_{\lambda\mu\nu}+T_{\mu\lambda\nu}) + \frac{1}{2n}(g_{\lambda\nu}v_\mu + g_{\mu\nu}v_\lambda - 2g_{\lambda\mu}v_\nu) \quad , \qquad \text{(B.4a)}$$

$$v_\mu \equiv T^\lambda{}_{\lambda\mu} \quad , \qquad \text{(B.4b)}$$

$$a_{\lambda\mu\nu} \equiv \frac{1}{3}(T_{\lambda\mu\nu}+T_{\mu\nu\lambda}+T_{\nu\lambda\mu}) \quad . \qquad \text{(B.4c)}$$

The gravitational action with a cosmological constant Λ is expressed as

$$I_G = \int d^{n+1}x \,\sqrt{-g}\, [\, a_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + a_2(v^\mu v_\mu) - \frac{1}{6}a_3(a^{\lambda\mu\nu}a_{\lambda\mu\nu}) - \Lambda \,] \; . \qquad \text{(B.5)}$$

In order to clarify the difference between general relativity and new general relativity, we rewrite the action (B.5) as

$$I_G = \frac{1}{2\kappa}\int d^{n+1}x\ \sqrt{-g}\ [\ R(\{\}) + 2c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + 2c_2(v^{\mu}v_{\mu}) - \frac{1}{3}c_3(a^{\lambda\mu\nu}a_{\lambda\mu\nu}) - 2\kappa\Lambda\ ]\ . \qquad (B.6)$$

Here the parameters, c_1, c_2 and c_3, are given by

$$c_1 = \kappa a_1 + \frac{1}{3}\ , \quad c_2 = \kappa a_2 - \frac{n-1}{2n}\ , \quad c_3 = \kappa a_3 + \frac{3}{4}\ , \qquad (B.7)$$

the scalar $R(\{\})$ is the Riemann-Chrisfoffel scalar curvature in n+1 dimensional spacetime. The total action is given by

$$I \equiv I_G + I_M\ , \qquad (B.8)$$

where I_M represents the action constructed by the matter Lagrangian in the n+1 dimensional spacetime. As for spinor fields, we should take care of their existence.[45)]

By taking variation of the action I with respect to the parallel vector fields $b^k{}_{\mu}$, we obtain the gravitational field equations,

$$G^{\mu\nu}(\{\}) + 2D_{\lambda}F^{\mu\nu\lambda} + 2v_{\lambda}F^{\mu\nu\lambda} + 2H^{\mu\nu} - g^{\mu\nu}L' = \kappa T^{\mu\nu}\ , \qquad (B.9)$$

where

$$F^{\mu\nu\lambda} \equiv c_1(t^{\mu\nu\lambda}-t^{\mu\lambda\nu}) + c_2(g^{\mu\nu}v^{\lambda}-g^{\mu\lambda}v^{\nu}) - \frac{1}{3}c_3a^{\mu\nu\lambda}\ , \qquad (B.10a)$$

$$D_{\lambda}F^{\mu\nu\lambda} \equiv \partial_{\lambda}F^{\mu\nu\lambda} + \Gamma^{\mu}{}_{\rho\lambda}F^{\rho\nu\lambda} + \Gamma^{\nu}{}_{\rho\lambda}F^{\mu\rho\lambda} + \Gamma^{\lambda}{}_{\rho\lambda}F^{\mu\nu\rho}\ , \qquad (B.10b)$$

$$H^{\mu\nu} \equiv T^{\rho\sigma\mu}F_{\rho\sigma}{}^{\nu} - \frac{1}{2}T^{\nu\rho\sigma}F^{\mu}{}_{\rho\sigma} = H^{\nu\mu}\ , \qquad (B.10c)$$

$$L' \equiv c_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + c_2(v^{\mu}v_{\mu}) - \frac{1}{6}c_3(a^{\lambda\mu\nu}a_{\lambda\mu\nu}) - \kappa\Lambda\ , \qquad (B.10d)$$

$$\sqrt{-g}\ T^{\mu\nu} = \eta^{km}b_k{}^{\mu}\ \delta(\sqrt{-g}\ L_M)/\delta b^m{}_{\nu}\ . \qquad (B.10e)$$

Here the tensor $G^{\mu\nu}(\{\})$ is the Einstein tensor in n+1 dimensional spacetime formed of the Christoffel symbols.

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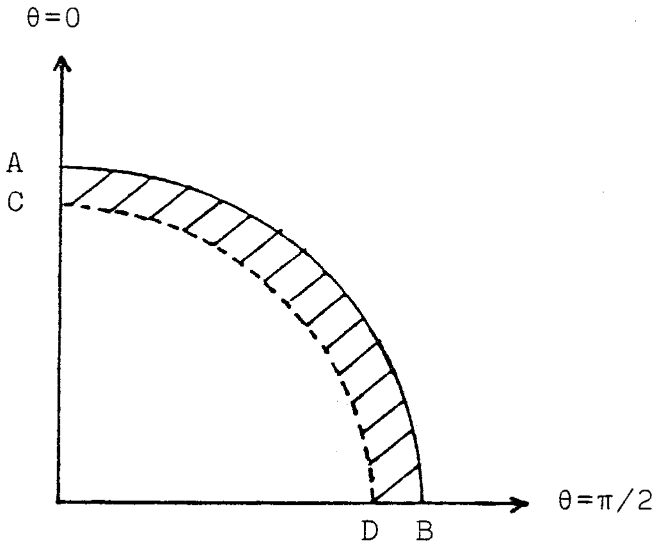
Figure caption

These figures show the region $X^2+Y^2 < Z^2$ referring to (3.33) in terms of the polar coordinate (3.37). The broken lines represent infinite red-shift surfaces. The regions of oblique lines correspond to the regions of $X^2+Y^2 < Z^2$. It should be noted that we are now investigating the outside region of the infinite redshift surface because this coordinate system cannot be used inside the surface.

Figure



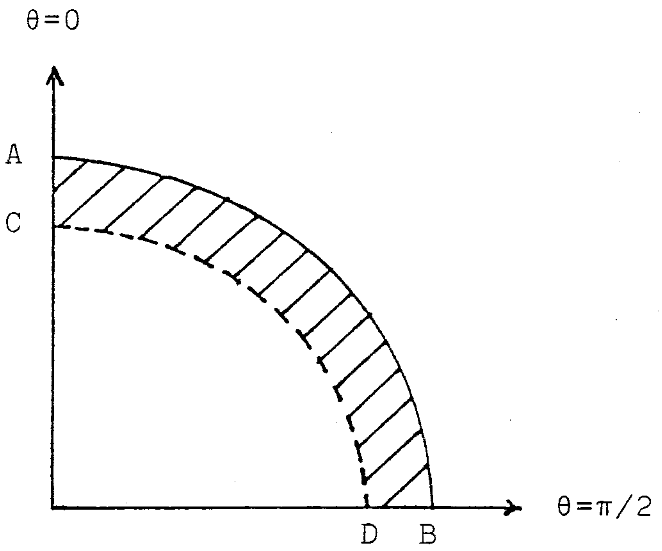
a) $a/Gm = 1/4$

$r_A = 2.24Gm$

$r_B = 2.32Gm$

$r_C = (1+\sqrt{15}/4)Gm$

$r_D = 2Gm$



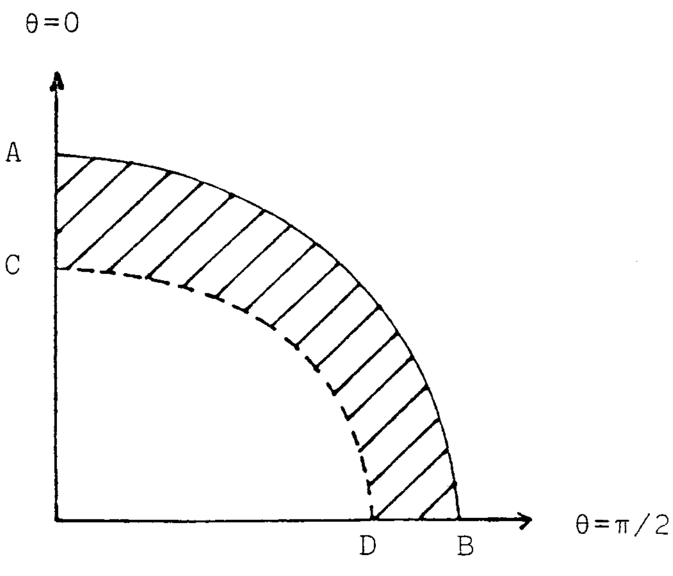
b) $a/Gm = 1/2$

$r_A = 2.32Gm$

$r_B = 2.44Gm$

$r_C = (1+\sqrt{3}/2)Gm$

$r_D = 2m$



c) $a/Gm = 3/4$

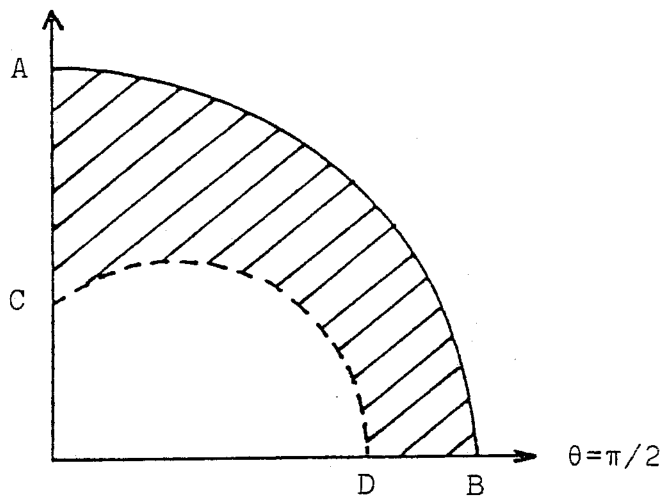
$r_A = 2.42Gm$

$r_E = 2.57Gm$

$r_C = (1+\sqrt{7}/4)Gm$

$r_D = 2Gm$

Figure



d) $a/Gm = 1$

$r_A = 2.54Gm$

$r_B = 2.70Gm$

$r_C = Gm$

$r_D = 2Gm$