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Fusion rules and macroscopic loops from discretized approach to two-dimensional gravity

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Abstract

We investigate the multi-loop correlators and the multi-point functions for all of the scaling operators in unitary minimal conformal models coupled to two-dimensional gravity from the two-matrix model. We show that simple fusion rules for these scaling operators exist, and these are summarized in a compact form as fusion rules for loops. We clarify the role of the boundary operators and discuss its connection to how loops touch each other. We derive a general formula for the n-resolvent correlators, and point out the structure similar to the crossing symmetry of underlying conformal field theory. We discuss the connection of the boundary conditions of the loop correlators to the touching of loops for the case of the four-loop correlators.

Contents

1 Introduction

Quantization of gravity is one of the most important issues in physics. The understanding of two-dimensional quantum gravity, which is the simplest quantum gravity, has experienced great progress through the study of matrix models.[3] It was proposed [2] that the integral over the geometry of two-dimensional surface can be descretized as a sum over randomly triangulated surfaces and such regularized two-dimensional gravity can be realized by hermitian matrix models. Feynman diagrams of the matrix models correspond to the dynamically triangulated surfaces and the continuum limit of the models then describe the theory of two-dimensional gravity.

Due to the double scaling limit [3] the sum of the contributions from all topologys of two-dimensional surface can be treated, and thereby the matrix models have been drawn much attention as a non-perturbative definition of non-critical string theories. Following the discovery of the double scaling limit, many important structures of the models have become clear; for example, the connection to KdV flow [4], the Virasoro and W constraints [4] [5, 6, 7]. Field theory of non-critical strings [9, 10] has been constructed based on the matrix models.

The matrix models include infinite critical points, which are considered to represent certain conformal matters coupled to two-dimensional gravity. The m-th critical point of the one-matrix model corresponds to the $(2m+1, 2)$ minimal conformal model coupled to two-dimensional gravity. The general (p, q) minimal conformal model, where the central charge is $c = 1 - \frac{6(p-q)^2}{pq}$, can be realized as the continuum limit of the discrete system where the degrees of freedom are points on the A(DE) Dynkin diagram [12]. Multi-matrix chain model has been introduced as a model which includes the critical points corresponding to the general (p, q) minimal models coupled to gravity. In this model, q matrices interact as a chain. The two-matrix model [13, 14, 15], which is the simplest multi-matrix chain model, however, turned out to include all (p, q) critical points, which was pointed out in [13, 14] and shown explicitly in [15]. We use the two-matrix model to investigate the unitary minimal model $(m+1, m)$ coupled to two-dimensional gravity.

The emergence of the infinite number of scaling operators is one of the most important properties of the matrix models. Before coupled to gravity, the minimal model has finite number of primary fields. Coupled to gravity, however, infinite number of scaling operators emerge. This phenomenon can be understood as follows. In the Kac table we can divide the primary fields $\Phi_{r,s}$ into those which are

[3] See for example [1] for review.

[4] The corresponding structures in continuum framework have been shown also in [8].

inside the the minimal conformal grid $1 \leq r \leq q-1,\ 1 \leq s \leq p-1$ and those outside, which correspond to the null states. Before dressed by gravity, the fields outside the minimal conformal grid decouple [17] from physical correlators. After gravitational dressing, they cease to decouple [18, 19] and become infinite number of scaling operators. The similar phenomenon has been shown in continuum framework. Through the examination of the BRST cohomology of the coupled system composed of Liouville theory, the ghosts and the minimal matter, infinite physical states were shown to exist [20, 21]. These states have their counterparts in the matrix models as the scaling operators. Some of the scaling operators do not have their counterparts in the BRST cohomology, which we will discuss later.

In ordinary (p, q) minimal conformal model the primary fields satisfy certain fusion rules [17]; three-point function $\langle \Phi_{r_1,s_1} \Phi_{r_2,s_2} \Phi_{r_3,s_3} \rangle$ is non-vanishing only when

$$\begin{aligned} 1 + |r_1 - r_2| \leq r_3 \leq \mathrm{mim}\{r_1 + r_2 - 1, p\}, \quad r_1 + r_2 + r_3 = \mathrm{odd} \\ 1 + |s_1 - s_2| \leq s_3 \leq \mathrm{mim}\{s_1 + s_2 - 1, q\}, \quad s_1 + s_2 + s_3 = \mathrm{odd}\ . \end{aligned} \tag{1.1}$$

It is interesting to examine how the fusion rules change when the matter couples to gravity. In particular, we are interested in the fusion rules for the gravitational descendants $(\sigma_j,\ j = q+1, q+2, \cdots)$, most of which correspond to the operators outside the minimal conformal grid. Before coupled to gravity the corresponding fusion rules do not exist. Such three-point functions were examined from the point of view of the generalized KdV flow in [22] for lower dimensional scaling operators in the case of $(m+1, m)$ unitary matter. It was shown that the gravitational primaries σ_j $(j = 1, \cdots, m-1)$ satisfy fusion rules of BPZ type; for $j_1 + j_2 + j_3 \leq 2m - 1$, $\langle \sigma_{j_1} \sigma_{j_2} \sigma_{j_3} \rangle$ is non-vanishing only when

$$1 + |j_1 - j_2| \leq j_3 \leq j_1 + j_2 - 1\ . \tag{1.2}$$

The fusion rules were also examined in continuum framework [19]. As for the gravitational descendants, however, we think clear results have not been obtained. In this paper we would like to clarify the fusion rules for all of the scaling operators including the gravitational descendants in the case of unitary minimal model. This paper is based on [23, 24, 25, 26].

Macroscopic loop correlators [28, 29], which are the amplitudes of the surfaces with boundaries (loops) of fixed lengths, are the fundamental amplitudes of the matrix models. Although these amplitudes are hard to treat in the continuum framework [32], they are defined quite naturally in the matrix models. It was shown [30, 31] that these correlators have more information than those of local operators and that the latter correlators can be extracted from the former correlators explicitly in the case of $c = 0, 1/2, 1$. They argued there that macroscopic loops could be replaced by a sum of local operators in a certain situation and thereby obtained the correlators of local operators from those of macroscopic loops.

One of the purposes of this paper is to generalize the idea in [30] to the cases of the general unitary minimal models and to clarify the fusion rules for macroscopic loops and all of the scaling operators. First, we derive the three- and n-loop correlators from the two-matrix model at the general unitary critical points [24, 25], and then derive the explicit forms of the correlators of the scaling operators [26]. The main conclusion is that the three-point correlators of all of the scaling operators satisfy certain simple fusion rules [26] and the fusion rules for all of the scaling operators are summarized in a compact form as the fusion rules for three-loop correlators [24].

In matrix models, there are infinite subset of the scaling operators which do not have their counterparts in the BRST cohomology of Liouville theory. In the case of one-matrix model, Martinec, Moore and Seiberg [33] argued that these operators are boundary operators, which correspond to the vertex operators of open string and couple to the boundaries of two-dimensional surface. They proved that one of them is in fact a boundary operator which measures the total loop length. We think, however, the roles of the rest of these operators were not clear. We also clarify the role of these operators and its connection to the touching of loops in the case of general unitary models [26].

We also determine completely the forms of the multi-resolvent correlators, which are the Laplace transform of the multi-loop correlators, and point out that the loop correlators have the structures similar to those of the crossing symmetry of the underlying conformal field theory [25]. In the cases of four- and five-loop, we discuss the connection of the boundary conditions of the loops to the touching of the loops [26].

As another formulation of 2D gravity with matter system, models of string whose target spaces are the Dynkin diagrams have been investigated [34]. We also comment on the connection of our results to those from these models.

The paper is organized as follows. Sect. 2 is devoted to the review of the matrix models and the macroscopic loops. We limit our discussion to the subjects that have direct connections to the subsequent sections.

In sect. 3 we derive the three-loop correlators and extract the three-point functions for all of the scaling operators through expansion of loops in terms of the local operators. We then show that certain simple fusion rules exist for these local operators or loops. We also discuss the role of the boundary operators there.

We derive the formulas for the multi-resolvent correlators in sect. 4, and give the explicit forms of the four- and five-loop correlators. We point out that the structure corresponding to the crossing symmetry of the underlying conformal field theory exists in the multi-loop correlators. We also discuss the connection of the boundary condition of the loops to the touching of the loops.

Sect. 6 is a summary.

2 Conformal field theory coupled to two-dimensional gravity (review)

2.1 Matrix models and two-dimensional gravity

Let us briefly review the matrix models and the connection to the Liouville theory, emphasizing on the notion of the scaling operators and that of the macroscopic loops. We limit our discussion to the subjects which have direct connections to the later sections.

2.1.1 Matrix models and random triangulation

Let us consider the model defined by the path integral with respect to an $N \times N$ hermitian matrix Φ,

$$e^Z = \int \mathrm{d}\Phi e^{-\frac{1}{2}\,\mathrm{Tr}\,\Phi^2 + \frac{g}{\sqrt{N}}\,\mathrm{Tr}\,\Phi^3} \quad , \tag{2.1}$$

where the measure is

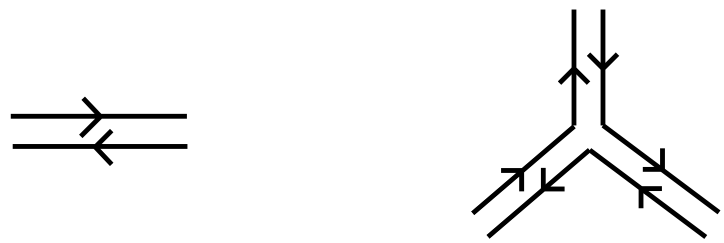
$$\mathrm{d}\Phi = \prod_i \mathrm{d}\Phi^i{}_i \prod_{i<j} \mathrm{dRe}\Phi^i{}_j \mathrm{dIm}\Phi^i{}_j \ . \tag{2.2}$$

The propagator is $\left\langle \Phi^i{}_j \Phi^k{}_l \right\rangle = \delta^i_l \delta^k_j$, and is represented by the double lines in fig. 1. (a). The arrows connect the upper matrix indices to the lower ones. The vertex in the action is represented by fig. 1 (b). Expanding the partition function in term of g, we find that the each Feynmam diagram represents a net on an orientable two-dimensional surface. Taking the dual of such a diagram, the vertices turn into triangles and the dual diagram represents a random triangulation of two-dimensional surface. Therefore, the model specified by eq. (2.1) can be considered to represent a theory of random triangulation of 2D surfaces and is expected to be a theory of 2D quantum gravity when we take continuum limit.

Let us count the power of N associated to each diagram. Changing variables $\Phi \to \Phi/\sqrt{N}$, the action becomes $N\,\mathrm{Tr}\left(-\frac{1}{2}\,\mathrm{Tr}\,\Phi^2 + g\,\mathrm{Tr}\,\Phi^3\right)$. From this form of action it is clear that each vertex contributes a factor of N, each propagator (edge) contributes a factor of N^{-1}, and loop (face) contributes a factor of N due to the index summation associated. Each diagram has thus an overall factor

$$N^{V-E+F} = N^{\chi} = N^{2-2h} \quad , \tag{2.3}$$

where χ and h are the Euler character and the number of genera of the surface associated to the diagram respectively. From (2.3), the partition function can be



(a) (b)

Figure 1: propagator and vertex

expanded as

$$Z(g) = \sum_h N^{2-2h} Z_h(g) \tag{2.4}$$

where $Z_h(g)$ represents the contribution from the surfaces of genus h. In the large N limit, the contribution from the planar surfaces dominate.

2.1.2 Continuum limit

When expanded in the coupling g, for large order n, Z_0 behaves as

$$Z_0(g) \sim \sum_n n^{\Gamma_{str}-3} \Big(\frac{g}{g_*}\Big) \sim (g_* - g)^{2-\Gamma_{str}} \tag{2.5}$$

and the expectation value of the number of vertices (triangles in the dual diagram) is given by

$$\langle n \rangle = \frac{\partial}{\partial g} \ln Z_0(g) \sim \frac{1}{g - g_*} \, . \tag{2.6}$$

The partition function $Z_0(g)$ thus becomes non-analytic and $\langle n \rangle$ diverges when g approaches some critical value g_*. Since $\langle n \rangle$ diverges as $g \to g_*$, it is expected that the contribution from the continuum surface with finite area can be obtained by rescaling the area of the individual triangles to zero accordingly. The contribution from the continuum surface is considered to correspond the non-analytic part of eq. (2.5). Therefore, the behavior of the model (2.1) near the critical point is considered to represent two-dimensional quantum gravity.

2.1.3 Multi-critical points and multi-matrix models

So far, we have considered the model(2.1), which consist of one kind of vertex in the action. As a generalization of this model, let us consider the the model specified by the action

$$S = \frac{N}{\Lambda} \operatorname{Tr} V(\Phi) \tag{2.7}$$

where $V(\Phi)$ is some polynomial of matrix Φ. This model can represent a series of systems of consisting of matter and two-dimensional gravity. By tuning the couplings in the potential, various critical points are obtained. The m-th critical point corresponds to the $(2m+1,2)$ minimal conformal model coupled to two-dimensional gravity.

As another generalization, let us consider the multi-matrix chain model,

$$e^Z = \int \prod^{(\alpha)} \Phi_\alpha \; e^{-S} \; , \tag{2.8}$$

$$S = \sum_{\alpha=1}^{\nu-1} V_\alpha(\Phi^{(\alpha)}) - \sum_{\alpha=1}^{\nu-2} c_\alpha \Phi^{(\alpha)} \Phi^{(\alpha+1)} \; . \tag{2.9}$$

Here the different matrices $\Phi^{(\alpha)}$ represent $\nu-1$ different matter degrees of freedom that can exist at the vertices. Note the couplings c_α in the action (2.9) couple the matrices along a line (chain).

It was pointed out, however, in [13, 14] and shown explicitly in[15] that it is sufficient to consider the two-matrix model in order to generate the most general critical points, which correspond to the $c<1$ minimal conformal models. We will thus use the two-matrix model to examine the minimal models coupled to two-dimensional gravity in this article.

2.2 Scaling operators in matrix models

2.2.1 KdV flows and scaling operators

Consider the two-matrix model with symmetric potential,

$$\mathrm{e}^Z = \int \mathrm{d}\hat{A}\mathrm{d}\hat{B} \; \mathrm{e}^{-\frac{N}{\Lambda} \operatorname{tr}\left(U(\hat{A}) + U(\hat{B}) - \hat{A}\hat{B}\right)} \; , \tag{2.10}$$

where U is an arbitrary polynomial of order m. In this article, we limit our discussion to the two-matrix model with symmetric potential and to the critical points which correspond to the unitary minimal models. In the case of asymmetric potential, some of the boundaries (loops) of the surface would have fractal dimensions different from the usual dimension of length.

Integrating out the "angular" variables, we have [16]

$$e^Z = \int d\lambda_i d\tilde{\lambda}_i \Delta(\lambda)\Delta(\tilde{\lambda})\ e^{-\frac{N}{\Lambda}\sum_i\left(U(\lambda_i)+U(\tilde{\lambda}_i)-\lambda_i\tilde{\lambda}_i\right)} \ . \tag{2.11}$$

Here $\Delta(\lambda)$ is the Vandermonde determinant, λ and $\tilde{\lambda}$ represent the eigenvalues of the matrices $\hat{A}$ and $\hat{B}$ respectively.

We introduce the orthogonal polynomials $|j\rangle = \xi_j(\lambda)$ and $\langle k| = \xi_k(\tilde{\lambda})$ by the orthonormality relation

$$\begin{aligned} \langle j|k\rangle &= \int d\mu\ \xi_j(\tilde{\lambda})\xi_k(\lambda) = \delta_{jk}\ , \\ & \quad d\mu \equiv d\lambda d\tilde{\lambda}\ e^{-\frac{N}{\Lambda}\left(U(\lambda)+U(\tilde{\lambda})-\lambda\tilde{\lambda}\right)} \ . \end{aligned} \tag{2.12}$$

We define matrices A and P by

$$A_{nm} = \langle n|\lambda|m\rangle \tag{2.13}$$

$$P_{nm} = \langle n|\frac{\partial}{\partial\lambda}|m\rangle \ . \tag{2.14}$$

It is obvious from the definition (2.13) and (2.14), that A and P obey the Heisenberg commutation relations

$$[P, A] = 1 \ . \tag{2.15}$$

Now the important fact is that the operators P and A have non-zero matrix elements P_{ij} and A_{ij} only if $|i-j|$ is sufficiently small. Since the bounds are independent of N, in the limit $N \to \infty$, P and A become differential operators (in x, the cosmological constant) of finite order. The continuum scaling limit of the two-matrix model is abstracted to the mathematical problem of finding solution to eq. (2.15). Let us consider the $(m+1, m)$ critical point which corresponds to the $(m+1, m)$ minimal model. After suitable renormalization, A is given by

$$A = D^m + u(x)_{m-2}D^{m-2} + u(x)_{m-3}D^{m-3} + \cdots + u(x)_0 \ , \quad D = \partial_x \ . \tag{2.16}$$

(By a change of basis of the form $A \to f^{-1}(x)Af(x)$, the coefficient of D^{m-1} may be always be set to zero.) and P is given by [4]

$$P = (L^{m+1})_+ \ . \tag{2.17}$$

Here $L \equiv A^{1/m}$ is a pseudodifferential operator satisfying

$$A = L^m \ , \quad L = D + a_1 D^{-1} + a_2 D^{-2} + \cdots \ , \tag{2.18}$$

and $(L^\alpha)_+$ denotes the nonnegative (differential operator) part of L^α. Substituting eqs. (2.16) and (2.17) into eq. (2.15), we find that differential equations for $u(x)_i$. These equations determine $u(x)_i$ up to $m-1$ integration constants t_i $(i = 1, \cdots, m-1)$. The dependence of L on the constants t_i is given by the first $m-1$ generalized KdV flows:

$$\frac{\partial}{\partial t_i} L = \Big[(L^i)_+ ,\ L \Big] \quad . \tag{2.19}$$

In general, the perturbation from the $(m+1, m)$ point is represented by the independent flows in term of commuting operators $\{L^i | i = 1, 2, \cdots \neq 0 \ (mod\ m)\}$:

$$\begin{aligned} \frac{\partial}{\partial t_i} A &= \Big[(L^i)_+ ,\ A \Big] = - \Big[(L^i)_- ,\ A \Big] \quad , \\ \frac{\partial}{\partial t_i} P &= \Big[(L^i)_+ ,\ P \Big] = - \Big[(L^i)_- ,\ P \Big] \quad . \end{aligned} \tag{2.20}$$

The infinite number of directions of the perturbation correspond to flows along RG trajectories between various critical theories, identified with the (p, q) minimal model coupled to two-dimensional gravity. Since the perturbation is realized by adding an infinite number of relevant matter operators dressed by gravity to the original critical action, the correlation functions of the scaling operators are defined by the following relation:

$$\langle \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_n} \rangle = \frac{\partial}{\partial t_{j_1}} \frac{\partial}{\partial t_{j_2}} \cdots \frac{\partial}{\partial t_{j_n}} \log Z \ . \tag{2.21}$$

2.2.2 Correlators from KdV flow

The correlation functions of the scaling operators σ_j on the sphere were calculated in [22] for lower j from the point of view of the KdV flow.

Let us define the following infinite number of commuting operators on the sphere:

$$\frac{1}{n} Q_n = \sum_{j=0}^{\infty} \begin{pmatrix} n-j-1 \\ j-1 \end{pmatrix} \frac{(-\frac{1}{2}u)^j}{j} D^{n-2j} \tag{2.22}$$

with

$$[Q_n, Q_k] = 0 \quad . \tag{2.23}$$

The operators A and P on the sphere are given by

$$\begin{aligned} A &= (Q_m)_+ = L^m = D^m - \frac{1}{2} m u D^{m-2} + \cdots \ , \\ P &= (Q_{m+1})_+ \ , \end{aligned} \tag{2.24}$$

where u is the two-point function of dressed identity operator. Substituting eqs. (2.24) into eq. (2.15), we have

$$\frac{u}{2} = \left(\frac{x}{m+1}\right)^{1/m} . \tag{2.25}$$

The correlation functions of σ_j for lower j can be calculated from the KdV flow. For example, from eqs. (2.20), we obtain the following expression for the one-point function:

$$\frac{\partial}{\partial t_n} u = -2(\text{Res } L^n)' , \tag{2.26}$$

where Res L^n is the coefficient of D^{-1} in L^n and L^n ($n \leq 2m-3$) can be replaced with Q_n due to the relation

$$Q_n - L^n = \frac{n}{m} c_n D^{n-2m} + \mathcal{O}(D^{n-2m-2}) . \tag{2.27}$$

The two-point functions can be calculated from

$$\frac{\partial}{\partial t_n}\frac{\partial}{\partial t_k} u = 2(\text{Res } [L_-^n, L_+^k])' . \tag{2.28}$$

As for explicit results of the correlators, we mention these in sec. 3.5.

2.2.3 Connection to continuum theory

The m-th critical point of one-matrix model corresponds to the $(2m+1,2)$ minimal model coupled to two-dimensional gravity. The scaling operators are naively expected to correspond to the following operators in the $(2m+1,2)$ minimal model coupled to Liouville theory in the continuum framework:

$$\sigma_j \leftrightarrow \int e^{\alpha_j \varphi} \Phi_{1,m-1-j} , \quad j = 0, \cdots, m-2, \tag{2.29}$$

where $\alpha_j = \frac{1}{2}\gamma(m-j)$, $\Phi_{r,s}$ are the primary fields of the corresponding conformal field theory.

This correspondence fails however. In [30], it was argued that the discrepancies were due to contact terms which arise when two operators are at coincident points. They showed explicitly the correct correspondence by the analytic redefinition of coupling constants t_j

$$t_n = C_n^i \hat{t}_i + C_n^{ij} \hat{t}_i \hat{t}_j + \cdots , \tag{2.30}$$

mainly for the case of one-matrix model. The original frame of operators σ_j and couplings t_j is referred to as the KdV frame and the new frame of operators $\widehat{\sigma}_j$ and

$\hat{t}_j$ is referred to as conformal field frame. The wave function of $\widehat{\sigma}_j$ is proportional to the modified Bessel function $K_{\frac{j}{q}}(2\sqrt{\mu}\ell)$ so that it satisfies the (minisuperspace) Wheeler-deWitt equation

$$\left[-\left(\ell\frac{\partial}{\partial\ell}\right)^2 + 4\mu\ell^2 + \left(\frac{j}{q}\right)^2\right]\Psi_j(\ell) = 0\ , \tag{2.31}$$

which is a desirable property.

The BRST cohomology of the coupled system of Liouville theory, ghosts and the (p,q) minimal matter was examined in [20, 21]. It turned out that the BRST cohomology is spanned by infinite operators of the form

$$\tilde{\mathcal{O}}_j\ \mathrm{e}^{\alpha_j\varphi},\quad \frac{\alpha_j}{\gamma} = \frac{p+q-j}{2q}\qquad j \geq 1,\ \neq 0\ (\mathrm{mod}\ p),\ \neq 0\ (\mathrm{mod}\ q)\ , \tag{2.32}$$

where φ is the Liouville field and $\gamma = (\sqrt{25-c} - \sqrt{1-c})/\sqrt{12}$. The operators $\tilde{\mathcal{O}}_j$ are made of ghosts, matter and derivatives of φ. On the other hand, the scaling operators σ_j of matrix model at the (p,q) critical point scale like Liouville operators of the form

$$\tilde{\mathcal{O}}_j\ \mathrm{e}^{\alpha_j\varphi},\quad \frac{\alpha_j}{\gamma} = \frac{p+q-j}{2q}\qquad j \geq 1,\ \neq 0\ (\mathrm{mod}\ q)\ . \tag{2.33}$$

Apart from the discrepancy of the operators with $j = 0$ (mod p), the two calculations are in remarkable agreement. It was argued in [33] that the scaling operators with $j = 0$ (mod p) are boundary operators, which couple to the boundaries of two-dimensional surface and correspond to the vertex operators of open string.

2.3 Macroscopic loops

2.3.1 Macroscopic loops in two-matrix model

In the two-matrix model, the operators

$$\mathrm{Tr}\,\hat{A}^{n_1}\ ,\quad \mathrm{Tr}\,\hat{B}^{n_2} \tag{2.34}$$

create holes with boundaries of lattice lengths n_1 and n_2 respectively. The correlation functions of $\mathrm{Tr}\,\hat{A}^{n_i}$ or $\mathrm{Tr}\,\hat{A}^{n_i}$ are expected to become those of macroscopic loops in the limit $an_i \to \ell_i$ with ℓ_i finite, when the unit lattice length a approaches zero.

It is convenient to consider first the correlators of the resolvents

$$\hat{W}^+(p_i) = \mathrm{Tr}\,\frac{1}{p_i - \hat{A}}\ ,\quad \hat{W}^-(p_i) = \mathrm{Tr}\,\frac{1}{p_i - \hat{B}}\ , \tag{2.35}$$

where p_i is a parameter corresponding to the bare boundary cosmological constant of each loop. Due to the formal expansion

$$\hat{W}^+(p_i) = \sum_{n=0}^{\infty} \frac{\mathrm{Tr}\,\hat{A}^n}{p_i^{n+1}} \ , \tag{2.36}$$

the resolvents include the contributions from loops of any length. The correlators become singular when p_i approach some critical value p_*. Since the contributions from loops of finite continuum length corresponds to those of infinite lattice length, continuum loop correlators are defined as the inverse-Laplace image of non-analytic part of the resolvent correlators with respect to $\zeta_i = (p_i - p_*)/a$.

2.3.2 'Classical' solutions to Heisenberg relation

In later sections we will use extensively the 'classical' solutions to the 'classical' Heisenberg relation. Let us explain these in this subsection 2.3.2.

Since we would like to examine the correlators on the sphere, we are interested only in the planar limit (large N limit). The Heisenberg relation (2.15) turns into

$$[P, A] = \frac{\Lambda}{N} \ , \tag{2.37}$$

after rescaling $P \to \frac{N}{\Lambda} P$. From eq. (2.37), we see that $\frac{\Lambda}{N}$ plays the role of Planck constant. It is thus expected that the corresponding 'classical' functions would be much easier to handle than the operators A and P in the large N limit.

At this point it is useful to change notation for the indices of the matrix elements:

$$A_k(n) = A_{n-k,n} \quad , \quad P_k(n) = P_{n-k,n} \ . \tag{2.38}$$

Here n represents the position of the matrix element on the diagonal, and k is its deviation from it. Then the action of the operators A and P on the orthogonal polynomial basis is described by

$$A\,|n\rangle = \sum_{k=-1}^{m-1} |n-k\rangle\, A_k(n) \ \ , \ \ P\,|n\rangle = \sum_{k=-1}^{(m-1)^2} |n-k\rangle\, P_k(n) \ \ . \tag{2.39}$$

The 'classical' functions are defined by

$$\begin{aligned} A(\omega, s) &= \sum_{k=-1}^{m-1} \mathrm{e}^{k\omega} A_k(n) \\ P(\omega, s) &= \sum_{k=1}^{(m-1)^2} \mathrm{e}^{k\omega} P_k(n) \ , \end{aligned} \tag{2.40}$$

where s is the continuous variable

$$s = \frac{n}{N}\Lambda \tag{2.41}$$

and ω is its conjugate coordinate.

The equation of motion

$$\frac{\Lambda}{N}P_{ij} = \langle i|U'(\lambda)|j\rangle - A^T_{ij} \ , \tag{2.42}$$

which is obtained by doing an integration by parts, reads

$$P(\omega, s) = U'\Big(A(\omega, s - \frac{\Lambda}{N}\frac{\partial}{\partial\omega})\Big)\cdot 1 - A(-\omega + \frac{\Lambda}{N}\frac{\partial}{\partial s}, s)\cdot 1 \ , \tag{2.43}$$

when expressed in term of the classical functions. In the planar limit, eq. (2.43) reads

$$P(\omega, s) = U'\Big(A(\omega, s)\Big) - A(-\omega, s) \ , \tag{2.44}$$

and the Heisenberg commutation relation (2.37) is replaced by the Poisson bracket

$$\{P(\omega, s), A(\omega, s)\} \equiv \frac{\partial P}{\partial s}\frac{\partial A}{\partial\omega} - \frac{\partial P}{\partial\omega}\frac{\partial A}{\partial s} = -1 \ . \tag{2.45}$$

Note that in the large N limit the classical functions A and P depend on Λ only through s, which is easily seen from eqs. (2.43) and (2.44).

Let us find the solution to the classical Heisenberg relation eq. (2.45) near the (m+1,m) critical point, which corresponds to the $(m+1, m)$ unitary minimal conformal model.

At the critical point, one expects the following singular behavior of A and P:

$$A(z) - A_* \sim (1-z)^m \ , \quad P(z) - P_* \sim (1-z)^{m+1} \ . \tag{2.46}$$

From the scaling laws (2.46), the solution to the Heisenberg relation is given by [15]

$$\begin{aligned} A(z, s) - A_* &= 2\eta^m \cosh m\theta \\ P(z, s) - P_* &= 2\eta^{m+1}\cosh(m+1)\theta \\ s - \Lambda_* &= (m+1)\eta^{2m} \ . \end{aligned} \tag{2.47}$$

Here P_*, A_* and Λ_* denote the critical values of the corresponding quantities and the parametrization

$$\omega = 2\eta\cosh\theta \tag{2.48}$$

is used. We will use the classical functions (2.47) extensively to calculate the loop correlators in later sections.

2.3.3 Loops in semi-classical Liouville theory

When we discuss the loop correlators, it turns out to be very helpful to consider these correlators semi-classically in Liouville theory. Let us explain these [30] briefly in this subsection 2.3.3. In the continuum framework, two-dimensional gravity part of the coupled system can be described by Liouville theory based on the action,

$$\begin{aligned} S_L[\varphi;\widehat{g}] &= \frac{1}{8\pi}\int_\Sigma \mathrm{d}^2\xi\sqrt{\widehat{g}}\widehat{g}^{ab}\partial_a\varphi\partial_b\varphi \\ &+ \frac{Q}{8\pi}\left(\int_\Sigma \mathrm{d}^2\xi\sqrt{\widehat{g}}\ \widehat{R}\varphi + \oint_{\partial\Sigma}\mathrm{d}\widehat{s}\ \widehat{k}\varphi\right) \\ &+ \frac{\mu}{8\pi\gamma^2}\int_\Sigma \mathrm{d}^2\xi\sqrt{\widehat{g}}\ \mathrm{e}^{\gamma}\varphi + \frac{\rho}{4\pi\gamma^2}\oint_{\partial\Sigma}\mathrm{d}\widehat{s}\ \mathrm{e}^{\gamma\varphi/2}\ , \end{aligned} \tag{2.49}$$

where $\widehat{g}_{ab}$ is a reference metric and $\widehat{g}_{ab}\ \mathrm{e}^{\gamma\varphi}$ is a physical metric, $\widehat{R}$ and $\widehat{k}$ are respectively the curvature and the extrinsic curvature of the boundary with respect to the reference metric $\widehat{g}_{ab}$. We denote by μ and ρ bulk and boundary cosmological constants respectively. Classically, $Q = 2/\gamma$ where γ is the Liouville coupling constant. Let us consider the correlation function

$$\left\langle \prod_i \mathrm{e}^{\alpha_i\varphi(z_i)} \right\rangle = \int \mathcal{D}\varphi\ \mathrm{e}^{-S_L}\prod_i \mathrm{e}^{\alpha_i\varphi(z_i)}\ . \tag{2.50}$$

We obtain the classical equation of motion:

$$\frac{1}{4\pi}\Delta\varphi - \frac{\mu}{8\gamma}\ \mathrm{e}^{\gamma\varphi} + \sum_i \alpha_i\delta^{(2)}(z-z_i) = 0\ . \tag{2.51}$$

Since the curvature of the physical metric is

$$R[\mathrm{e}^{\gamma\varphi}\widehat{g}] = -\mathrm{e}^{-\gamma\varphi}\Delta(\gamma\varphi)\ , \tag{2.52}$$

eq. (2.51) describes a surface with constant negative curvature and the inserted operators $\mathrm{e}^{\alpha_i\varphi(z_i)}$ play the role of the sources of curvature. Note that in the absence of a boundary a solution exists only when

$$X = \sum_i \alpha_i + \frac{Q}{2}(2h-2) \tag{2.53}$$

is positive, where h is the number of handles. The nature of the surface and hence the nature of associated quantum states depend crucially on the sign of X. When there are boundaries, a classical solution always exists. Let us restrict our attention to the case with a single boundary and discuss whether the boundary can be replaced by local sources of curvature. In this case the nature of the surface depends crucially on the sign of

$$Y = X + \frac{1}{2}Q = \sum_i \alpha_i - \frac{1}{2}Q\chi \ . \tag{2.54}$$

Case 1: Fixed μ, $Y > 0$. When the loop is shrunk to a point, there exists a classical solution with constant negative curvature. A small loop behaves like a local source of curvature $Q/2$.

Case 2: Fixed μ, $Y < 0$. When the loop is shrunk to a point, there is no classical solution with constant negative curvature. We can understand this case better if we constrain the area of the surface to be A.

Case 2-1: Fixed $A \gg \ell^2$, $Y < 0$. The classical solution has positive constant curvature and the small-ℓ limit is smooth and the loop becomes as a puncture with curvature $Q/2$.

Case 2-2: Fixed $A \ll \ell^2$, $Y < 0$. The classical solution has negative constant curvature and the loop cannot be thought of as a local disturbance.

So far we have discussed classically. Semi-classically, Y must be modified by

$$Y = X + \alpha_{\text{mim}} \ , \tag{2.55}$$

where α_{mim} is the curvature associated with the lowest dimension operator $\mathcal{O}_{\text{min}}$ (the dressed identity operator, in the case of the unitary minimal matter) because this is the maximum curvature that can be localized in a point in the quantum theory. Similar observations follow in semi-classical discussion. In case 1 and case 2-1 the loop can be replaced by a sum of local operators and the contribution to the amplitude give rise to non-analytic terms in μ. In case 2-2 the loop cannot be replaced by a sum of local operators and the contribution to the amplitude give rise to analytic terms in μ; the loop length ℓ plays the role of ultraviolet cutoff.

3 Three-loop correlators and fusion rules

In this section we consider the loop correlators in the unitary minimal models $(m+1, m)$ coupled to two dimensional gravity and the physical information we can extract from these. As shown in the case of one-matrix model [30], the loop correlators are expected to have much more information than those of local operators. We calculate the three-loop correlators in the systems stated above, from the two-matrix model with symmetric potential, at the $(m+1, m)$ critical points and show that simple fusion rules exist for the loop correlators and for all of the scaling operators.

3.1 Formula for n-resolvent correlator

Consider the connected part of the n-point correlators of the resolvents, which we introduced in sec. 2.3.1:

$$\hat{W}^+(p_i) \equiv \mathrm{Tr}\,\frac{1}{p_i - \hat{A}} \qquad , \qquad \hat{W}^-(p_j) \equiv \mathrm{Tr}\,\frac{1}{p_j - \hat{B}} \qquad . \tag{3.1}$$

First, let us show briefly the formula for the n-resolvent correlators we obtained in [25][5]. The explicit derivation of the formula will be shown in section 4.1 later. At the $(m+1, m)$ critical point, we obtained the following formula for the n-resolvent correlator:

$$\left(\frac{N}{\Lambda}\right)^{n-2} \left\langle\!\!\left\langle \prod_{i=1}^{n} \mathrm{Tr}\,\frac{1}{p_i - \hat{A}} \right\rangle\!\!\right\rangle = \prod_{i=1}^{n} \left(-\frac{\partial}{\partial(a\zeta_i)}\right) R^{(n)}(\zeta_i, \Lambda_i)|_{\Lambda_i = \Lambda} \quad . \tag{3.2}$$

Here we denote by $\langle\!\langle \cdots \rangle\!\rangle$ the connected part of the averaging with respect to the matrix integrations and $R^{(n)}$ is some function of ζ_i and Λ_i through z_i^* and their derivatives with respect to the bare cosmological constant Λ_i. Note that we introduced independent cosmological constants Λ_i for each loop for the convenience of the calculation. We put $\Lambda_i = \Lambda$ at the end of the calculation in eq. (3.2). The function z_i^* of ζ_i and Λ_i is parametrized as follows,

$$z_i^* = \exp(2\eta_i \cosh\theta_i) \quad , \tag{3.3}$$

where

$$p_i - p_* \equiv a\zeta_i = A(z_i^*; \Lambda_i) - A_* = aM_i \cosh m\theta_i \quad , \quad \eta_i = (aM_i/2)^{1/m} \quad , \tag{3.4}$$

[5] The formula for the multi-loop amplitudes in the case of the general one-matrix model was derived in [29].

$$\Lambda_i - \Lambda_* = -(m+1)\eta_i^{2m} = -a^2\mu_i \ , \quad \left(\frac{M_i}{2}\right)^2 = \frac{\mu_i}{m+1} \quad . \tag{3.5}$$

where p_* and A_* represent the critical values of p_i and $A(z;\Lambda)$ respectively. We denote by ζ_i and μ_i the renormalized boundary and bulk cosmological constants for the corresponding loop respectively.

The origin of the parametrization eq. (3.3) comes from the planar solution to the Heisenberg algebra (2.47). In fact, the function $A(z,\Lambda)$ in eq. (3.4) represents the solution at the $(m+1,m)$ critical point.

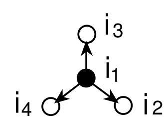
The function $R^{(n)}$ is easily written down for lower n. For $n=2,3$, we have

$$\frac{\partial}{\partial\Lambda}R^{(2)} = \sum_{i=1}^{2}\frac{\partial z_i^*}{\partial\Lambda_i}\prod_{j(\neq i)}^{2}(z_i^* - z_j^*)^{-1} \ , \tag{3.6}$$

$$R^{(3)} = \sum_{i=1}^{3}\frac{\partial z_i^*}{\partial\Lambda_i}\prod_{j(\neq i)}^{3}(z_i^* - z_j^*)^{-1} \quad . \tag{3.7}$$

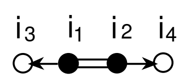
For $n=4,5$, the correlators can be written compactly using graphs as introduced below:

$$R^{(4)} = \sum \frac{\partial}{\partial\Lambda_{i_1}}\Big\{$$



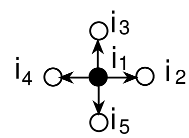
$$\Big\}$$

$$+ \sum \Big\{$$



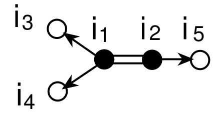
$$\Big\} \ , \tag{3.8}$$

$$R^{(5)} = \sum \left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^2\Big\{$$



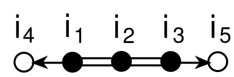
$$\Big\}$$

$$+ \sum \left(\frac{\partial}{\partial\Lambda_{i_1}}\right)\Big\{$$



$$\Big\}$$

$$+ \sum \Big\{$$



$$\Big\} \quad . \tag{3.9}$$

In these figures a double line linking circle i and circle j, a single line having an arrow from circle i to circle j and a solid circle i represent $(z_i^* - z_j^*)^{-2}$, $(z_i^* - z_j^*)^{-1}$ and $\frac{\partial z_i^*}{\partial \Lambda_i}$ respectively. The summations are over all possible graphs that have the same topology specified. Each graph appears just for once in the summation. Note that the links to the external circles are not double lines but the single ones with arrows and that the internal circles are solid circles.

For general n, the function $R^{(n)}$ is expressed in the same way. The rule is as follows. First, we consider all possible graphs which have n circles and $n-1$ links in the same way as in the case $n = 5$. Second, if the internal solid circle i has l_i links in each graph, the graph is operated by $\prod_i \left(\frac{\partial}{\partial \Lambda_i}\right)^{l_{i-2}}$. Then the summation over all graphs gives the expression for $R^{(n)}$.

3.2 Three-loop correlators

3.2.1 Derivation of three-resolvent correlators

As a example, let us calculate explicitly the three-loop correlator, which we examined in [24], in order to understand how we got the formula from the classical solution to Heisenberg relation at the $(m+1, m)$ critical point.

In the second quantized free fermion formalizm (see, for example, [28, 6]), the connected part of the correlator consisting of the product of arbitrary analytic functions $f^{(i)}(\hat{A})$ in two-matrix model can be expressed as

$$\begin{aligned} &\left\langle\!\!\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, f^{(i)}(\hat{A}) \right\rangle\!\!\right\rangle \\ &\quad = \langle\!\langle N | \prod_{i=1}^{n} : \int \mathrm{d}\mu_i \Psi^\dagger(\tilde{\lambda}_i) f^{(i)}(\lambda_i) \Psi(\lambda_i) : | N \rangle\!\rangle \\ &\quad = \langle\!\langle N | \prod_{i=1}^{n} : a_{k_i}^\dagger a_{l_i} : | N \rangle\!\rangle \prod_{i=1}^{n} \int \mathrm{d}\mu_i \xi_{k_i}(\tilde{\lambda}_i) f^{(i)}(\lambda_i) \xi_{k_i}(\lambda_i) \\ &\quad = \langle\!\langle N | \prod_{i=1}^{n} : a_{k_i}^\dagger a_{l_i} : | N \rangle\!\rangle \prod_{i=1}^{n} \langle k_i | f^{(i)}(A) | l_i \rangle \quad , \end{aligned} \tag{3.10}$$

where

$$\Psi(\lambda) = \sum_k a_k \; \xi_k(\lambda) \; , \tag{3.11}$$

$$\Psi^\dagger(\tilde{\lambda}) = \sum_k a_k^\dagger \xi_k(\tilde{\lambda}) \tag{3.12}$$

are second quantized free fermion fields constructed from the orthogonal polynomials

ξ_k and $|N\rangle\rangle$ is a state corresponding to the filled fermi sea,

$$a_k\,|N\rangle\rangle = 0 \;\text{ for }\; k \geq N \tag{3.13}$$

$$a_k^\dagger\,|N\rangle\rangle = 0 \;\text{ for }\; k \leq N-1\;. \tag{3.14}$$

The normal ordering $:\cdots:$ is with respect to $|N\rangle\rangle$. The connected three-point correlator consisting of arbitrary analytic functions $f(\hat{A})$, $g(\hat{A})$, and $h(\hat{A})$ can thus be expressed as

$$\begin{aligned}\left\langle\!\!\left\langle \mathrm{Tr}\,f(\hat{A})\;\;\mathrm{Tr}\,g(\hat{A})\;\;\mathrm{Tr}\,h(\hat{A})\right\rangle\!\!\right\rangle \;&=\; \sum_{i=0}^{N-1}\sum_{k=N}^{\infty}\sum_{l=N}^{\infty}[f(A)]_{ik}\,[g(A)]_{kl}\,[h(A)]_{li} \\ &-\; \sum_{i=0}^{N-1}\sum_{k=N}^{\infty}\sum_{l=0}^{N-1}[f(A)]_{ik}\,[h(A)]_{kl}\,[g(A)]_{li}\;, \end{aligned}\tag{3.15}$$

where $[f(A)]_{ik} \equiv \langle i|f(A)|k\rangle$.

Because we are interested in the case of large N limit only, it is convenient to use the 'classical' function introduced sect. 2.3.2. Note that the 'classical' function depends on the bare cosmological constant Λ only through s when we take $N \to \infty$ limit. It is, therefore, legitimate to introduce $A(z,s) \equiv \lim_{N\to\infty} A(z,s,\Lambda)$:

$$A(z,s,\Lambda) = A(z,s) + O(1/N) \quad . \tag{3.16}$$

Since the matrix elements A_{ij} only near the diagonal are not zero, the matrix element $[f(A)](i)_k \equiv [f(A)]_{i-k,i}$ can be replaced with the coefficient of z^k for the 'classical' function $A(z, s=\Lambda)$,

$$[f(A)](N)_k = \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{k+1}} f\left(A(z,s=\Lambda)\right) + O(1/N) \quad , \tag{3.17}$$

at large N limit.

In the right-hand side of eq. (3.15), the leading terms in $1/N$ of the first term and those of the second term get cancelled. We have to consider the next leading terms. For any integer ϵ, we obtain

$$\begin{aligned}[f(A)](N+\epsilon)_k \;&=\; \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{k+1}} f\left(A(z,s=\Lambda)\right) \\ &+\; \frac{\Lambda\epsilon}{N}\frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{k+1}}\left.\frac{\partial A(z,s)}{\partial s}\right|_{s=\Lambda}\frac{\partial f\left(A(z,\Lambda)\right)}{\partial A} \\ &+\; (\text{the part independent of }\epsilon) + O(1/N^2) \quad . \end{aligned}\tag{3.18}$$

The part independent of ϵ comes from the terms $O(1/N)$ in eq. (3.17). The second term is responsible for the computation in what follows. Using eq. (3.18) and considering the terms $1/N$ in eq. (3.15), we obtain

$$\left\langle\!\!\left\langle \mathrm{Tr}\, f(\hat{A})\ \ \mathrm{Tr}\, g(\hat{A})\ \ \mathrm{Tr}\, h(\hat{A}) \right\rangle\!\!\right\rangle = \frac{\Lambda}{N} \sum_{\delta_1=0}^{\infty} \sum_{\delta_2=0}^{\infty} \sum_{\delta=0}^{\infty}$$
$$\Big\{ (\delta_2 - \delta_1)\ \frac{\partial}{\partial s}[f(A)](N)_{\delta_1+\delta_2+1}\ [g(A)](N)_{\delta-\delta_2}\ [h(A)](N)_{-\delta-\delta_1-1}$$
$$+(\delta+\delta_2+1)\ [f(A)](N)_{\delta_1+\delta_2+1}\ \frac{\partial}{\partial s}[g(A)](N)_{\delta-\delta_2}\ [h(A)](N)_{-\delta-\delta_1-1}$$
$$+\ (\delta-\delta_1)\ [f(A)](N)_{\delta_1+\delta_2+1}\ [g(A)](N)_{\delta-\delta_2}\ \frac{\partial}{\partial s}[h(A)](N)_{-\delta-\delta_1-1} \Big\}$$
$$+O(1/N^2) \quad . \tag{3.19}$$

Here we have used the fact that at large N, the original summations in eq. (3.15) can be safely replaced with the triple summations from zero to infinity.

The summations can be carried out after putting this equation into the form of contour integrals using eq. (3.17). The three-point function eq. (3.15) in the planar limit can thus be expressed in terms of the 'classical' function in the form of contour integrals,

$$\left\langle\!\!\left\langle \mathrm{Tr}\, f(\hat{A})\ \ \mathrm{Tr}\, g(\hat{A})\ \ \mathrm{Tr}\, h(\hat{A}) \right\rangle\!\!\right\rangle = \frac{\Lambda}{N}\frac{1}{(2\pi i)^3} \oint_{|z_1|>|z_2|>|z_3|} \mathrm{d}z_1 \mathrm{d}z_1 \mathrm{d}z_3$$
$$\Big\{ \frac{z_1}{(z_1-z_2)^2(z_1-z_3)^2}\ \frac{\partial}{\partial\Lambda}\Big[f\left(A(z_1)\right)\Big] g\left(A(z_2)\right) h\left(A(z_3)\right)$$
$$+\ \frac{z_2}{(z_2-z_1)^2(z_2-z_3)^2}\ f\left(A(z_1)\right) \frac{\partial}{\partial\Lambda}\Big[\left(A(z_2)\right)\Big] h\left(A(z_3)\right)$$
$$+\ \frac{z_3}{(z_3-z_1)^2(z_3-z_2)^2}\ f\left(A(z_1)\right) g\left(A(z_2)\right) \frac{\partial}{\partial\Lambda}\Big[h\left(A(z_3)\right)\Big] \Big\} \quad , \tag{3.20}$$

where we set $A(z) \equiv A(z, s = \Lambda)$. The condition for the contour paths $|z_1| > |z_2| > |z_3|$ follows from the condition that makes the infinite summations converge.

From the above expression (eq. (3.20)), the three-point resolvent in the planar limit is expressed as

$$\frac{N}{\Lambda_*} \left\langle\!\!\left\langle \mathrm{Tr}\,\frac{1}{p_1-\hat{A}}\ \ \mathrm{Tr}\,\frac{1}{p_2-\hat{A}}\ \ \mathrm{Tr}\,\frac{1}{p_3-\hat{A}} \right\rangle\!\!\right\rangle = \frac{1}{(2\pi i)^3} \oint_{|z_1|>|z_2|>|z_3|} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3$$
$$\Big\{ \frac{z_1}{(z_1-z_2)^2(z_1-z_3)^2}\ \frac{\partial}{\partial\Lambda}\left[\frac{1}{p_1-A(z_1)}\right] \frac{1}{p_2-A(z_2)} \frac{1}{p_3-A(z_3)}$$
$$+\ \frac{z_2}{(z_2-z_1)^2(z_2-z_3)^2} \frac{1}{p_1-A(z)} \frac{\partial}{\partial\Lambda}\left[\frac{1}{p_2-A(z_2)}\right] \frac{1}{p_3-A(z_3)}$$
$$+\ \frac{z_3}{(z_3-z_1)^2(z_3-z_2)^2}\ \frac{1}{p_1-A(z_1)} \frac{1}{p_2-A(z_2)} \frac{\partial}{\partial\Lambda}\left[\frac{1}{p_3-A(z_3)}\right] \Big\}, \tag{3.21}$$

where the contour of z_1 encircles that of z_2 and similarly the contour of z_2 encircles that of z_3. We pick up only the contributions form the poles at $z_i = z_i^*$ of the parts $1/[p_i - A(z_i)]$. The contributions from the poles at $z_i = z_j^*$ $(i \neq j)$ give rise to terms which do not have the inverse Laplace image. We thus discard these terms.

We obtain the following rather simple expression for the three-resolvent correlator,

$$
\begin{aligned}
&\frac{N}{\Lambda_*}\left\langle\!\!\left\langle \mathrm{Tr}\,\frac{1}{p_1-\hat{A}}\ \mathrm{Tr}\,\frac{1}{p_2-\hat{A}}\ \mathrm{Tr}\,\frac{1}{p_3-\hat{A}} \right\rangle\!\!\right\rangle \\
&\quad = -\frac{1}{a^3}\left\{\frac{\partial}{\partial\Lambda_1}\left[\frac{1}{(z_1^*-z_2^*)^2(z_1^*-z_3^*)^2}\frac{\partial z_1^*}{\partial\zeta_1}\right]\frac{\partial z_2^*}{\partial\zeta_2}\frac{\partial z_3^*}{\partial\zeta_3} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\right\} \\
&\quad = -\frac{1}{a^3}\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\frac{\partial}{\partial\zeta_3}\left\{\frac{1}{(z_1^*-z_2^*)(z_1^*-z_3^*)}\frac{\partial z_1^*}{\partial\Lambda_1} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\right\}\Bigg|_{\Lambda_i=\Lambda} .
\end{aligned}
\tag{3.22}
$$

Here, the poles z_i^* are determined through the classical solution to the Heisenberg relation and are parametrized as eq. (3.3).

Eq. (3.22) agrees with the set of eq. (3.2) and eq. (3.7), the formula for three-resolvent correlator. We have shown how we can get the formula for n-resolvent correlators for the case of $n = 3$ explicitly.

3.2.2 Three-loop correlators in terms of loop lengths

Next, let us consider to how to get the expression for three-loop correlators in terms of the loop lengths, performing inverse Laplace transformation with respect to the renormalized boundary cosmological constants ζ_i. We will show, later, that much physical information can be extracted from the three-loop correlator. The generalization to higher loop will be discussed in sect. 4.

First, we will show we can put eq. (3.22) into a form in which the correlator is expressed as a sum of the product of three factors each of which depends only on ζ_i corresponding to individual loop. In order to show this, first note that eq. (3.22) can be written as

$$
\begin{aligned}
&\frac{N}{\Lambda_*}\left\langle \mathrm{Tr}\,\frac{1}{p_1-\hat{A}}\quad \mathrm{Tr}\quad\frac{1}{p_2-\hat{A}}\ \ \mathrm{Tr}\,\frac{1}{p_3-\hat{A}} \right\rangle_c \\
&\qquad\qquad = \frac{1}{a^3 2^2 m(m+1)}\left(\frac{aM}{2}\right)^{-2-1/m}\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\frac{\partial}{\partial\zeta_3}F(\theta_1,\theta_2,\theta_3)\ ,
\end{aligned}
\tag{3.23}
$$

where

$$F(\theta_1,\theta_2,\theta_3) = \frac{1}{(\cosh\theta_1-\cosh\theta_2)(\cosh\theta_1-\cosh\theta_3)}\frac{\sinh(m-1)\theta_1}{\sinh m\theta_1} + (1\leftrightarrow 2) + (1\leftrightarrow 3) \quad . \tag{3.24}$$

Here the following identity is crucial;

$$\frac{1}{\cosh\alpha-\cosh\beta}\left(\frac{\sinh(n-k)\alpha}{\sinh n\alpha}-\frac{\sinh(n-k)\beta}{\sinh n\beta}\right) = -2\sum_{j=1}^{n-k}\sum_{i=1}^{k}\frac{\sinh(n-j-i+1)\alpha}{\sinh n\alpha}\,\frac{\sinh(n-j-k+i)\beta}{\sinh n\beta} \quad . \tag{3.25}$$

Making use of the above identity twice, we find that eq. (3.24) is written as a triple sum where the summand factorizes into three factors associated with individual loops:

$$F(\theta_1,\theta_2,\theta_3) = 4\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}\sum_{i=1}^{k}\frac{\sinh(m-k)\theta_1}{\sinh m\theta_1}\,\frac{\sinh(m-j-i+1)\theta_2}{\sinh m\theta_2}\,\frac{\sinh(m-k-j+i)\theta_3}{\sinh m\theta_3} \,. \tag{3.26}$$

Here, we should specify the definition of continuum amplitudes at large N. Since the leading term of $\left\langle\!\left\langle \prod_{i=1}^{n}\ \hat{W}(p_i)\right\rangle\!\right\rangle$ is of order of $a^{-n}\kappa^{n-2}$, where $\kappa \equiv a^{-2-\frac{1}{m}}(\Lambda_*/N)$, we should renormalize to obtain continuum quantities. The renormalized resolvent is defined as

$$\hat{W}^{ren}(\zeta_i) = \frac{a}{\kappa}\,\hat{W}(p_i) = \frac{a}{\kappa}\,\mathrm{Tr}\,\frac{1}{p_i-\hat{A}} \ , \tag{3.27}$$

and the renormalized expectation is defined as

$$\langle\cdots\cdots\rangle^{ren} = \kappa^2\,\langle\!\langle\cdots\cdots\rangle\!\rangle \ . \tag{3.28}$$

We will omit the superscript *ren* from now on. The continuum three-loop correlator $\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\rangle$ is defined by the inverse Laplace image of the continuum resolvent correlator, that is,

$$\begin{aligned}&\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\hat{W}^+(\zeta_3)\right\rangle \\ &= \int_0^\infty d\ell_1\int_0^\infty d\ell_2\int_0^\infty d\ell_3 e^{-\zeta_1\ell_1}e^{-\zeta_2\ell_2}e^{-\zeta_2\ell_3}\left\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\right\rangle \\ &\equiv \mathcal{L}\left[\left\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\right\rangle\right] \quad ,\end{aligned} \tag{3.29}$$

where $w^{\pm}(\ell)$ is an operator which makes hole with finite boundary (loop) length ℓ. Due to the following formula for the inverse Laplace image

$$\mathcal{L}^{-1}\left[\frac{\partial}{\partial\zeta}\frac{\sinh k\theta}{\sinh m\theta}\right] = -\frac{M\ell}{\pi}\sin\frac{k\pi}{m}\ K_{\frac{k}{m}}(M\ell) \quad , \tag{3.30}$$

where $K_\nu(z)$ is the modified Bessel function, we can obtain the continuum three-loop amplitude in a rather compact form:

$$\begin{aligned}&\left\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\right\rangle \\ &\quad = -\frac{1}{m(m+1)}\left(\frac{M}{2}\right)^{-2-\frac{1}{m}} \\ &\quad\left(\frac{M}{2}\right)^3 \ell_1\ell_2\ell_3 \sum_{k=1}^{m-1}\sum_{j=1}^{m-k}\sum_{i=1}^{k} \widetilde{K}_{\frac{m-k}{m}}(M\ell_1)\ \widetilde{K}_{\frac{m-j-i+1}{m}}(M\ell_2)\ \widetilde{K}_{\frac{m-k-j+i}{m}}(M\ell_3)\ ,\end{aligned} \tag{3.31}$$

where we introduced a notation,

$$\widetilde{K}_p(M\ell) \equiv \frac{\sin\pi|p|}{\pi/2}\ K_p(M\ell) \quad . \tag{3.32}$$

The expression for the summation in eq. (3.31) looks asymmetric with respect to the loop indices. By elementary algebras, we can convert it into a form which have explicit symmetry with respect to the interchange of loops:

$$\begin{aligned}&\left\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\right\rangle \\ &\quad = -\frac{1}{m(m+1)}\left(\frac{M}{2}\right)^{-2-\frac{1}{m}} \\ &\quad\left(\frac{M}{2}\right)^3 \ell_1\ell_2\ell_3 \sum_{\substack{(k_1-1,k_2-1,k_3-1)\\ \in\mathcal{D}_3^{(m)}}} \widetilde{K}_{1-\frac{k_1}{m}}(M\ell_1)\ \widetilde{K}_{1-\frac{k_2}{m}}(M\ell_2)\ \widetilde{K}_{1-\frac{k_3}{m}}(M\ell_3)\ ,\end{aligned} \tag{3.33}$$

Here we have denoted by $\mathcal{D}_3^{(m)}$

$$\begin{aligned}\mathcal{D}_3^{(m)} \quad = \quad & \Bigg\{(a_1,a_2,a_3)\ \Big|\ \sum_{i(\neq j)}^{3} a_i - a_j \geq 0 \text{ for } j = 1\sim 3\ , \\ & \sum_{i=1}^{3} a_i = \text{even} \leq 2(m-2)\ , a_i = 0,1,2,\cdots\Bigg\}\ .\end{aligned} \tag{3.34}$$

Eqs. (3.33) and (3.34) give the final expression for the three loop correlator at the $(m+1,m)$ critical point. It is interesting that the selection rule obtained in eqs. (3.33) and (3.34) agrees exactly with the fusion rules for the diagonal primary fields in the Kac table of underlying conformal field theory of the unitary minimal model $(m+1, m)$ [17]. In fact, the fusion rules for the diagonal primary fields read as

$$\langle \phi_{ii}\ \phi_{jj}\ \phi_{kk} \rangle \neq 0 \quad , \tag{3.35}$$

if and only if $i+j \geq k+1$ and two other permutations and $i+j+k$ (= odd) $\leq 2m-1$ hold. This set of rules is nothing but $\mathcal{D}_3^{(m)}$.

3.2.3 Boundary conditions of loops

A similar expression to eq. (3.33) was obtained in [34], where loop correlators were examined for closed string with one-dimensional discrete target space, that is, the degrees of freedom for matter part are labeled by the points of Dynkin diagram. The matter degrees of freedom are labeled also by the discrete momentum p instead of the discrete target space coordinate x. They examined the loop correlators treating the boundary condition of each loop ℓ_i to be specified by a single momentum p_i. Thus, it follows directly that the three-loop correlator which is specified by three momenta p_1, p_2 and p_3 and loop lengths, is proportional to the expectation value for wave functions for matter part, that is,

$$\tilde{C}_{p_1 p_2 p_3} = \sum_x S^x_{(p_1)} S^x_{(p_2)} S^x_{(p_3)} / S^x \ , \tag{3.36}$$

where $S^x_{(p_i)}$ and S^x are the wave function of a point particle moving on the discrete target space with momentum p_i and that for ground state respectively,

$$\begin{aligned} S_p^{(x)} = \langle x|p \rangle = \left(\frac{2}{h-1} \right)^{1/2} & \sin \pi p x \\ x = 1, 2, \cdots, h-1 & \\ p = 1/h, 2/h, \cdots, (h-1)/h \ . & \end{aligned} \tag{3.37}$$

For example, for the A_{h-1} Dynkin diagram, each momentum takes discrete values of $\frac{1}{h}, \cdots, \frac{h-1}{h}$ and $\tilde{C}_{p_1 p_2 p_3}$ is nonvanishing only when hp_i satisfy the equivalent rule we found in eq. (3.33) and eq. (3.34), that is,

$$(hp_1 - 1, hp_2 - 1, hp_3 - 1) \in \mathcal{D}_3^{(h)} \ . \tag{3.38}$$

The similarity between the three-loop correlator in the case of the closed strings in discrete target space and that in the two-matrix model we found in eqs. (3.33) and (3.34) indicates that the each terms in the sum in eq. (3.33) represents the amplitude with the the loops specified by the momentum $\frac{k_i}{m}$. It appears that we can decompose the loop operator in the two-matrix into parts each of which specified by a momentum $\frac{k}{m}$:

$$\text{“}w^+(\ell) \sim \sum_{k=1}^{m-1} c_k^+ \, w_k(\ell)\text{”} \quad , \tag{3.39}$$

From the selection rules in eq. (3.34), we can deduce the fusion rules for the gravitational descendants as well as for the gravitational primaries. In other words, some fusion rules are satisfied among all of the scaling operators including the gravitational descendants as well. We suggested first that the selection rules in the three-loop correlator correspond to those for the gravitational primaries by examining the limit of small loop length [23]. In [26], we obtained the fusion rules for all of the scaling operators from the three-loop correlator. We will discuss these issues in detail in later subsections after we examine the two-loop amplitude and an expansion of the loop operator in terms of local operators.

3.3 Expansion of loop operators

In [30, 31] it was proposed that a loop operator can be replaced by a sum of local operators if the loop correlator does not diverge when the loop shrink to a point. This was discussed explicitly in the case of the one-matrix model and $c = 1$ case. We apply this idea to the general minimal models coupled to gravity and discuss the correlation functions for the scaling operators. In order to derive the form of the expansion of the loop in terms of the local operators, let us first consider the two- and one-loop amplitudes.

3.3.1 Two-loop correlators from three-loop correlators

Let us derive two- and one-loop correlators from the three loop correlator eq. (3.33). As we shrink one of the three loops, the three-loop correlator should approach the derivative of the two-loop correlator with respect to the cosmological constant. Consider shrinking the third loop $M\ell_3$ in eq. (3.33). Since, for $M\ell \ll 1$, we have

$$\begin{aligned} \frac{M\ell}{2}\widetilde{K}_{1-\frac{k}{m}}(m\ell) \quad &= \frac{M\ell}{2}\, \left\{ I_{-1+\frac{k}{m}}(M\ell) - I_{1-\frac{k}{m}}(M\ell) \right\} \\ &\approx \frac{M\ell}{2}\, I_{-1+\frac{k}{m}}(M\ell) \\ &\approx \left(\frac{M\ell}{2}\right)^{\frac{k}{m}} \frac{1}{\Gamma(\frac{k}{m})} \quad , \end{aligned} \tag{3.40}$$

the leading contribution in the summation in eq. (3.33) comes from the part of $k_3 = 1$ and we have

$$\begin{aligned}&\left\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\right\rangle \\ &\qquad\approx -\frac{1}{m(m+1)}\frac{\ell_3^{\frac{1}{m}}}{\Gamma(1/m)}\,\ell_1\ell_2\sum_{k=1}^{m-1}\widetilde{K}_{1-\frac{k}{m}}(M\ell_1)\,\widetilde{K}_{1-\frac{k}{m}}(M\ell_2)\quad ,\end{aligned} \tag{3.41}$$

for $M\ell_3 \ll 1$. This should be proportional to the derivative of two-loop correlator with respect to μ. In fact, by the explicit calculation similar to the case of three-loop, one can obtain [15, 23]

$$\begin{aligned}&\frac{\partial}{\partial\mu}\left\langle w^+(\ell_1)w^\pm(\ell_2)\right\rangle \\ &\qquad = -\frac{1}{m(m+1)}\,\ell_1\ell_2\sum_{k=1}^{m-1}(\pm)^{k-1}\,\widetilde{K}_{1-\frac{k}{m}}(M\ell_1)\,\widetilde{K}_{1-\frac{k}{m}}(M\ell_2)\quad .\end{aligned} \tag{3.42}$$

It is clear that eq. (3.41) and eq. (3.42) are consistent.

Due to a relation $\frac{\partial}{\partial\mu} = \frac{2}{m+1}\frac{\partial}{M\partial M}$ and a formula of a integral (for $\alpha \neq \beta$),

$$\int^z dzzK_\nu(\alpha z)K_\nu(\beta z) = \frac{-z}{\beta^2-\alpha^2}\Big\{\beta K_\nu(\alpha z)K_{\nu-1}(\beta z) - \alpha K_{\nu-1}(\alpha z)K_\nu(\beta z)\Big\}, \tag{3.43}$$

one can obtain the two-loop correlator (for $\ell_1 \neq \ell_2$):

$$\begin{aligned}&\left\langle w^+(\ell_1)w^\pm(\ell_2)\right\rangle \\ &\quad = \frac{1}{m}\frac{M}{2}\frac{\ell_1\ell_2}{\ell_1^2-\ell_2^2}\sum_{k=1}^{m-1}(\pm)^{k-1}\Big\{\ell_1\widetilde{K}_{\frac{k}{m}}(M\ell_1)\,\widetilde{K}_{1-\frac{k}{m}}(M\ell_2) \\ &\qquad\qquad - \ell_2\widetilde{K}_{1-\frac{k}{m}}(M\ell_1)\,\widetilde{K}_{\frac{k}{m}}(M\ell_2)\Big\} \\ &\quad = \frac{1}{m}\frac{M}{2}\frac{\ell_1\ell_2}{\ell_1+(\pm)^m\ell_2}\sum_{k=1}^{m-1}(\pm)^{k-1}\widetilde{K}_{\frac{k}{m}}(M\ell_1)\,\widetilde{K}_{1-\frac{k}{m}}(M\ell_2)\quad .\end{aligned} \tag{3.44}$$

3.3.2 One-loop amplitudes from two-loop correlators

Shrinking ℓ_2 in eq. (3.44) as well, one should have the derivative of the one-loop amplitude with respect to the cosmological constant. For $M\ell_2 \ll 1$, we have

$$\left\langle w^+(\ell_1)w^\pm(\ell_2)\right\rangle \approx \frac{1}{m}\frac{\ell_2^{\frac{1}{m}}}{\Gamma(1/m)}\left(\frac{M}{2}\right)^{\frac{1}{m}}\widetilde{K}_{\frac{1}{m}}(M\ell_1)\quad . \tag{3.45}$$

In fact by explicit calculation, one can obtain

$$\frac{\partial}{\partial\mu}\left\langle w^\pm(\ell_1)\right\rangle = \frac{1}{m}\left(\frac{M}{2}\right)^{\frac{1}{m}}\widetilde{K}_{\frac{1}{m}}(M\ell_1)\quad . \tag{3.46}$$

Note that this amplitude is nothing but the wave function of the dressed identity operator. Performing the integral with respect to μ, one obtain the one-loop amplitude:

$$\left\langle w^{\pm}(\ell_1)\right\rangle = -\left(1+\frac{1}{m}\right)\,\ell_1^{-1}\,\frac{\sin\pi/m}{\pi/2}\,\left(\frac{M}{2}\right)^{1+\frac{1}{m}}K_{1+\frac{1}{m}}(M\ell_1)\ . \tag{3.47}$$

Note that $\ell_1\,\langle w^{\pm}(\ell_1)\rangle$ is the wave function of a boundary operator $\widehat{\sigma}_{1+m}$ [33] which couples to the boundary of two-dimensional surfaces (i.e. loops),

$$\ell_1\left\langle w^{\pm}(\ell_1)\right\rangle = \left(1+\frac{1}{m}\right)\left(\frac{M}{2}\right)^{1+\frac{1}{m}}\ \widetilde{K}_{1+\frac{1}{m}}(M\ell_1)\ . \tag{3.48}$$

3.3.3 Expansion of loops in local operators

In [30], in the case of one-matrix, it was argued that the loop operator can be expanded in terms of local operators inside the loop correlators, that is, the loop can be replaced with the infinite combination of local operators, except some special cases. Whether this replacement can be done safely or not is connected with whether the corresponding classical solution exits or not in the limit of small length of corresponding loop. This claim is quite natural because, in the one-matrix model, all of the scaling operators are expressed in term of one matrix Φ as

$$\sigma_j = \mathrm{Tr}(1-\Phi)^{j+1/2} = \sum_n a_n(j)n^{-1}\,\mathrm{Tr}\,\Phi^n\ . \tag{3.49}$$

On the other hand, in the two-matrix model, since there are two kinds of matrix $\hat{A}$ and $\hat{B}$, there can exist many kinds of microscopic loops, $\mathrm{Tr}(\hat{A}^{n_1}\hat{B}^{n_2}\hat{A}^{n_3}\cdots)$. In the case of the two-matrix model, thus, the direct connection of the scaling operators to the loops $\mathrm{Tr}\,\hat{A}^{\ell/a}$ or $\mathrm{Tr}\,\hat{B}^{\ell'/a}$ is not clear. At first sight, the expansion of loops in terms of local operators is not legitimate. We think, however, this expansion is possible by the following reason. When one of the loops on two-dimensional surface shrunk to a microscopic loop, the loop represents local deformation of the surface. The microscopic loop can be considered to be replaced by the insertions of local operators. As we will see later, the loop correlators except one-loop case are continuous when the length of one of the loops approaches zero, so that we expect that a macroscopic loop can also be replaced by a sum of local operators.

In the following, we find the form of the expansion of the loop operator. First, let us represent the two-loop correlator in term of off shell renormalizable wave function [30, 31],

$$(E\sinh\pi E)^{1/2}\ K_{iE}(M\ell)\ . \tag{3.50}$$

We show that the two-loop correlators are expressed as

$$\left\langle w^+(\ell_1) w^\pm(\ell_2) \right\rangle = \sum_{k=1}^{m-1} \frac{1}{2m} (\pm)^{k-1} \left(\frac{\sin \pi \frac{k}{m}}{\pi/2} \right)^2 \int_0^\infty \mathrm{d}E \frac{E \sinh \pi E}{\cosh \pi E - \cos \pi \frac{k}{m}} \; K_{iE}(M\ell_1) \; K_{iE}(M\ell_2) \tag{3.51}$$

From eq. (3.42), it is reasonable to assume

$$\text{“} w^\pm(\ell) = \sum_{k=1}^{m-1} (\pm)^{k-1} \frac{\sin \pi \frac{k}{m}}{\pi/2} \; w_k(\ell) \text{”} \tag{3.52}$$

in the case of two- and three-loop correlators, where we have introduced loop operators $w_k(\ell)$ which represent loops with some distinct matter boundary condition (see sect. 3.2.3). From eq. (3.42), we have

$$\left\langle w^+(\ell_1) w^\pm(\ell_2) \right\rangle = \sum_{k=1}^{m-1} (\pm)^{k-1} \left(\frac{\sin \pi \frac{k}{m}}{\pi/2} \right)^2 \; \langle w_k(\ell_1) w_k(\ell_2) \rangle \tag{3.53}$$

and

$$\frac{\partial}{\partial M} \langle w_k(\ell_1) w_k(\ell_2) \rangle = - \frac{1}{m} \frac{M}{2} \ell_1 \ell_2 \; K_{1-\frac{k}{m}}(M\ell_1) \; K_{1-\frac{k}{m}}(M\ell_2) \; . \tag{3.54}$$

Making use of a formula

$$K_\nu(z) \; K_\nu(w) = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{t} \; K_\nu \left(\frac{zw}{t} \right) \; \exp \left(- \frac{t}{2} - \frac{z^2 + w^2}{2t} \right) \tag{3.55}$$

and replacing t with tM^2, we have

$$\begin{aligned} & M\ell_1 \ell_2 \; K_{1-\frac{k}{m}}(M\ell_1) \; K_{1-\frac{k}{m}}(M\ell_2) \\ & \quad = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{t} \; K_{1-\frac{k}{m}} \left(\frac{\ell_1 \ell_2}{t} \right) \; \exp \left(- \frac{tM^2}{2} - \frac{\ell_1^2 + \ell_2^2}{2t} \right) \quad . \end{aligned} \tag{3.56}$$

Carrying out the integral with respect to M, and from eq. (3.54), we have

$$\langle w_k(\ell_1) w_k(\ell_2) \rangle = \frac{1}{4m} \int_0^\infty \frac{\mathrm{d}t}{t} \; \frac{\ell_1 \ell_2}{t} K_{1-\frac{k}{m}} \left(\frac{\ell_1 \ell_2}{t} \right) \; \exp \left(- \frac{tM^2}{2} - \frac{\ell_1^2 + \ell_2^2}{2t} \right) \; . \tag{3.57}$$

Due to a formula,

$$z K_{1-|p|}(z) = \int_0^\infty \mathrm{d}E \frac{E \sinh \pi E}{\cosh \pi E - \cos \pi p} \; K_{iE}(z) \;\; , \tag{3.58}$$

the right hand side of eq. (3.57) turns into

$$\frac{1}{4m}\int_0^\infty \frac{\mathrm{d}t}{t} \int_0^\infty \mathrm{d}E \frac{E\sinh\pi E}{\cosh\pi E-\cos\pi p}\; K_{iE}\left(\frac{\ell_1\ell_2}{t}\right)\exp\left(-\frac{tM^2}{2}-\frac{\ell_1^2+\ell_2^2}{2t}\right) \; . \qquad (3.59)$$

Using a formula eq. (3.55) again, eq. (3.59) turns out to be

$$\frac{1}{2m}\int_0^\infty \mathrm{d}E \frac{E\sinh\pi E}{\cosh\pi E-\cos\pi\frac{k}{m}}\; K_{iE}(M\ell_1)\; K_{iE}(M\ell_2)\; . \qquad (3.60)$$

Putting eq. (3.60) and eq. (3.53) together, we have proved eq. (3.51).

Let us go back to eq. (3.51) and perform the E-integral. The integral can be carried out by deforming the contour. The residues for poles

$$E=\pm i(\frac{k}{m}+2n) \;\; , \;\; n=0,\pm1,\pm2,\cdots \;\; , \qquad (3.61)$$

contribute to the integral and, after all, we obtain the following expansion for the two-loop correlators (for $\ell_1<\ell_2$)

$$\langle w_k(\ell_1)w_k(\ell_2)\rangle = \frac{1}{m}\sum_{n=-\infty}^{\infty}\Big(\frac{k}{m}+2n\Big)\left(\frac{\sin\pi\frac{k}{m}}{\pi/2}\right)^{-1} I_{|\frac{k}{m}+2n|}(M\ell_1)\; K_{\frac{k}{m}+2n}(M\ell_2) \qquad (3.62)$$

and

$$\Big\langle w^+(\ell_1)w^\pm(\ell_2)\Big\rangle = \frac{1}{m}\sum_{k=1}^{m-1}\sum_{n=-\infty}^{\infty}(\pm)^{k-1}\Big|\frac{k}{m}+2n\Big|\; I_{|\frac{k}{m}+2n|}(M\ell_1)\; \widetilde{K}_{\frac{k}{m}+2n}(M\ell_2)\; . \qquad (3.63)$$

Since the two-loop correlators eq. (3.44) or eq. (3.63) do not diverge when the length of one of the loops approaches zero, we expect that one of the loops can be replaced by an infinite combination of local operators. From the consideration of the minisuperspace Wheeler-deWitt equation and the scaling behavior, we expect that the wave function of the scaling operator $\widehat{\sigma}_j$ is proportional to $M^{j/m}K_{j/m}(M\ell)$. Eq. (3.46) and eq. (3.48) indicate that the following normalization of the wave function

$$\begin{aligned}\Big\langle \widehat{\sigma}_j\; w^+(\ell)\Big\rangle &= \frac{j}{m}\left(\frac{M}{2}\right)^{\frac{j}{m}}\frac{\sin\frac{j}{m}\pi}{\pi/2}\; K_{\frac{j}{m}}(M\ell) \\ &= \frac{j}{m}\left(\frac{M}{2}\right)^{\frac{j}{m}}\widetilde{K}_{\frac{j}{m}}(M\ell)\;\; , \;\; j\geq 1\; \neq 0 \;(\mathrm{mod}\; m)\;\; . \qquad (3.64)\end{aligned}$$

would be reasonable. Note that the normalization factor $\sin\frac{j}{m}\pi$ in eq. (3.64) is consistent because there are no scaling operators $\widehat{\sigma}_j$ for $j=0 \pmod m$ in the matrix model. Comparing eq. (3.64) with eq. (3.63), we expect the following expansions of the loop operators in term of the local operators:

$$\text{“}\ w^{\pm}(\ell) = \frac{1}{m}\sum_{k=1}^{m-1}\sum_{n=-\infty}^{\infty}(\pm)^{k-1}\left(\frac{M}{2}\right)^{-|\frac{k}{m}+2n|} I_{|\frac{k}{m}+2n|}(M\ell)\ \widehat{\sigma}_{|k+2mn|}\ \text{”}\ . \tag{3.65}$$

These expansions are the generalizations of those in the case of the one-matrix model [30] to the case of arbitrary unitary minimal model coupled to two-dimensional gravity.

Since the loop correlators are symmetric under the interchange of two kinds of loops, that is, $\langle w^+(\ell_1)\ w^+(\ell_2)\rangle = \langle w^-(\ell_1)\ w^-(\ell_2)\rangle$, the wave functions of the scaling operators with respect to loop $w^-(\ell)$ are read as

$$\left\langle \widehat{\sigma}_j\ w^-(\ell)\right\rangle = (-1)^{j-1}\left\langle \widehat{\sigma}_j\ w^+(\ell)\right\rangle \ . \tag{3.66}$$

The wave functions with respect to the loop $w_k(\ell)$ are

$$\langle \widehat{\sigma}_{|k+2mn|} w_{k'}(\ell)\rangle = \delta_{kk'}\left(\frac{k}{m}+2n\right)\left(\frac{M}{2}\right)^{|\frac{k}{m}+2n|} K_{|\frac{k}{m}+2n|}(M\ell)\ . \tag{3.67}$$

3.4 Relation to the multi-matrix model

Let us comment on the relation between loops in the two-matrix model and those in the multi-matrix chain models. The lowest critical point of the $(m-1)$-matrix chain model represents also the $(m+1,m)$ minimal model, which corresponds to A_{m-1} Dynkin diagram, coupled to two-dimensional gravity. From the observation,

$$\begin{aligned} |x=1\rangle &= \sum_{k=1}^{m-1} |p=\tfrac{k}{m}\rangle \left(\tfrac{2}{m-1}\right)^{1/2}\ \sin\pi\tfrac{k}{m}\ , & (3.68)\\ |x=m-1\rangle &= \sum_{k=1}^{m-1} |p=\tfrac{k}{m}\rangle \left(\tfrac{2}{m-1}\right)^{1/2}\ \sin\pi\tfrac{k}{m}(m-1) \\ &= \sum_{k=1}^{m-1} |p=\tfrac{k}{m}\rangle \left(\tfrac{2}{m-1}\right)^{1/2}\ (-1)^{k-1}\sin\pi\tfrac{k}{m}\ , & (3.69)\end{aligned}$$

where $\langle x|p\rangle$ is the wave function introduced in eq. (3.38), we think $w^+(\ell)$ and $w^-(\ell)$ should correspond $|x=1\rangle$ and $|x=m-1\rangle$ respectively. The loop operator $w^{(x)}(\ell)$ created by the x-th matrix $\hat{A}^{(x)}$ of the $(m-1)$-matrix chain model, thus, corresponds to

$$|x\rangle = \sum_{k=1}^{m-1} |p=\tfrac{k}{m}\rangle \left(\tfrac{2}{m-1}\right)^{1/2}\ \sin\pi\tfrac{k}{m}x\ , \tag{3.70}$$

and would be represented accordingly as

$$w^{(x)}(\ell) = \sum_{k=1}^{m-1} \frac{\sin \pi \frac{k}{m} x}{\pi/2}\, w_k(\ell) \quad , \quad x = 1, \cdots, m-1 \quad . \tag{3.71}$$

We think this relation is valid at least for loop correlators with less than four loops. Using the relation, we can construct the loop correlators of the multi-matrix models from those of the two-matrix model.

3.5 Three-point functions and fusion rules

3.5.1 One- and two-point functions

Let us consider the correlators of the scaling operators. We can extract these from loop correlators due to the relation eq. (3.65).

Since the one-loop amplitude diverges when the loop length approaches to zero, this amplitude include the contribution which is not represented by the local operators. Putting the one-loop amplitude into

$$\begin{aligned} \left\langle w^{+}(\ell) \right\rangle &= -\left(\frac{M}{2}\right)^{2+\frac{1}{m}} \left(\widetilde{K}_{2+\frac{1}{m}}(M\ell) - \widetilde{K}_{\frac{1}{m}}(M\ell) \right) \\ &= \left(\frac{M}{2}\right)^{2+\frac{1}{m}} \left(I_{2+\frac{1}{m}}(M\ell) - I_{-2-\frac{1}{m}}(M\ell) - I_{\frac{1}{m}}(M\ell) + I_{-\frac{1}{m}}(M\ell) \right) \ , \end{aligned} \tag{3.72}$$

and extracting the parts proportional to I_ν ($\nu > 0$), which parts would be considered as the contributions from local operators, we can obtain the one-point functions of the scaling operators:

$$\langle \widehat{\sigma}_1 \rangle = -m \left(\frac{M}{2}\right)^{2+\frac{2}{m}} \quad , \tag{3.73}$$

$$\langle \widehat{\sigma}_{1+2m} \rangle = m \left(\frac{M}{2}\right)^{4+\frac{4}{m}} \quad , \tag{3.74}$$

$$\langle \widehat{\sigma}_j \rangle = 0 \ , \quad j \neq 1, 1+2m \quad . \tag{3.75}$$

Let us turn to the two-point functions. Substituting eq. (3.65) into eq. (3.64), we obtain the two-point functions:

$$\langle \widehat{\sigma}_i \widehat{\sigma}_j \rangle = \delta_{ij}\, j \left(\frac{M}{2}\right)^{2j/m} \quad , \quad i, j \neq 0 \ (\mathrm{mod}\ m) \quad . \tag{3.76}$$

Note that we obtain diagonal two-point functions.

3.5.2 Three-point functions

As for three-point functions, using the formula,

$$
\begin{aligned}
zK_{1-|p|}(z) &= \int_0^\infty \mathrm{d}E \frac{E\sinh\pi E}{\cosh\pi E-\cos\pi p}\, K_{iE}(z) \\
&= \pi\sum_{n=-\infty}^{\infty}\frac{|p+2n|}{\sin\pi|p+2n|} I_{|p+2n|}(z) \ ,
\end{aligned}
\tag{3.77}
$$

we first expand the three-loop correlator eq. (3.33) as

$$
\begin{aligned}
&\left\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\right\rangle = \frac{-1}{m(m+1)}\left(\frac{M}{2}\right)^{-2-\frac{1}{m}} \sum_{\mathcal{D}_3}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3-\infty}^{\infty} \\
&\left(\tfrac{k_1}{m}+2n_1\right)\left(\tfrac{k_2}{m}+2n_2\right)\left(\tfrac{k_3}{m}+2n_3\right) I_{|\frac{k_1}{m}+2n_1|}(M\ell_1) I_{|\frac{k_2}{m}+2n_2|}(M\ell_2) I_{|\frac{k_3}{m}+2n_3|}(M\ell_3) \ .
\end{aligned}
\tag{3.78}
$$

Comparing eq. (3.78) with eq. (3.65), we can extract the three-point functions [26]:

$$
\begin{aligned}
&\left\langle \widehat{\sigma}_{|k_1+2mn_1|}\widehat{\sigma}_{|k_2+2mn_2|}\widehat{\sigma}_{|k_3+2mn_3|}\right\rangle \\
&= C_{k_1k_2k_3}\frac{-1}{m(m+1)}\prod_{i=1}^{3}(k_i+2mn_i)\left(\frac{M}{2}\right)^{-2-\frac{1}{m}+\sum_{i=1}^{3}\frac{1}{m}|k_i+2mn_i|} \ ,
\end{aligned}
\tag{3.79}
$$

where

$$
C_{k_1k_2k_3} = \begin{cases} 1 \ , & (k_1-1,k_2-1,k_3-1)\in\mathcal{D}_3^{(m)} \\ 0 \ , & \text{otherwise} \end{cases} \ . \tag{3.80}
$$

For $n_i = 0$, eq. (3.79) is nothing but the correlator of the gravitational primaries. For the gravitational primaries, eq. (3.76) and eq. (3.79) agree with the correlators obtained in [22] from the generalized KdV flow up to a factor -2. Note that we obtain, here, the correlators of the gravitational descendants as well.

In [19], the fusion rules for the gravitational primaries were examined in continuum framework. Note that we have found here the fusion rules for the gravitational descendants as well as for the gravitational primaries. These fusion rules are similar to those for the gravitational primaries due to the factor $C_{k_1k_2k_3}$ in eq. (3.79). Introducing the equivalence classes $[\widehat{\sigma}_k]$ by the equivalence relation

$$
\widehat{\sigma}_k \sim \widehat{\sigma}_{|k+2mn|} \ , \quad n\in\mathbb{Z} \ , \tag{3.81}
$$

we can consider the fusion rules in eq. (3.79) as fusion rules among $[\widehat{\sigma}_k]$ ($k = 1,\cdots,m-1$). Note, here, that the class $[\widehat{\sigma}_k]$ does not correspond to the set which consist of the gravitational primary $\mathcal{O}_k$ and its gravitational descendants $\sigma_l(\mathcal{O}_k)$, $l = 1,2,\cdots$ in [22] introduced from the viewpoint of KdV flow. The three-loop correlator eq. (3.33) represents the fusion rules for all of the scaling operators including the gravitational descendants in a compact form.

3.6 Further on the fusion rules

Let us examine the fusion rules in eq. (3.79) further and consider the relation of the scaling operators to the primary fields in the corresponding conformal field theory.

In the (p, q) minimal conformal model, the primary field Φ_{rs} has the conformal dimension

$$\Delta_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq} \quad , \tag{3.82}$$

where r and s are positive integers. Since

$$\Delta_{r,s} = \Delta_{r+q,s+p} = \Delta_{q-r,p-s} \quad , \tag{3.83}$$

the corresponding primary fields can be regarded as the same one. The integers r and s can thus be restricted in the range

$$\begin{cases} 1 \leq r \leq q - 1 \\ 1 \leq s \ \neq 0 \ (\mathrm{mod}\ p) \\ pr < qs \end{cases} \tag{3.84}$$

(see fig. 2). In fig. 2, the primary fields in the region $(\!(2)\!)$ or $(\!(2)\!)'$ are the secondary fields of those in the region $(\!(1)\!)$. In general, the fields in the region $(\!(n + 1)\!)$ or $(\!(n + 1)\!)'$ are the secondaries of the fields in $(\!(n)\!)$ or $(\!(n)\!)'$. Since the secondary fields correspond to null vectors, those fields decouple. One can thus construct consistent conformal field theory which include only the primary fields in the region $(\!(1)\!)$ (i.e. inside the minimal table), that is, the minimal model (p, q) [17]. Coupled to Liouville theory, however, the fields outside the the minimal table fail to decouple [18] and infinite physical states emerge accordingly. These states are considered to correspond to the primaries outside the minimal table. This correspondence is implied by the BRST cohomology [20, 21] of the coupled system.

Denoting the gravitational dimension of the dressed operator for $\Phi_{r,s}$ by $\Delta^G_{r,s} = 1 - \frac{\alpha_{r,s}}{\gamma}$, in the minimal model coupled to Liouville theory, the following relation was shown [20, 21],

$$\frac{\alpha_{r,s}}{\gamma} = \frac{p + q - |pr - qs|}{2q} \quad , \tag{3.85}$$

where r and s take the values in the range eq. (3.84). On the other hand, in the matrix model, the corresponding relation for the scaling operator $\widehat{\sigma}_j$ is

$$\frac{\alpha_j}{\gamma} = \frac{p + q - j}{2q} \quad . \tag{3.86}$$

From eq. (3.85) and eq. (3.86), we should take as

$$j = |pr - qs| \ \ , \ \ j = 1, 2, \cdots \neq 0 \ (\mathrm{mod}\ q) \quad , \tag{3.87}$$

for $\widehat{\sigma}_j$.

Consider now the relation of $\widehat{\sigma}_{|k+2mn|}$ to the primary field $\Phi_{r,s}$ of the unitary $(m+1,m)$ minimal model. Let us first compare the two sets

$$S_k = \left\{ |k+2nm| \;\mid\; n \in \mathbb{Z} \right\} \;, \tag{3.88}$$

and

$$\begin{aligned} \left\{ \; |pr - qs| \; \right\} \;&=\; \left\{ \; |(m+1)r - ms| \; \right\} \\ &=\; \left\{ \; r' + (s-r-1)m \; \right\} \;, \end{aligned} \tag{3.89}$$

where r and s are positive integers in the range

$$\begin{cases} 1 \le r \le m-1 \\ 1 \le s \\ r+1 \le s \end{cases} \tag{3.90}$$

and $r' \equiv m - r$. Note that we include $s = 0 \pmod{m+1}$ here. Dissolving the set S_k into two sets as

$$S_k = S_k^+ \oplus S_k^- \;, \tag{3.91}$$

where

$$\begin{aligned} S_k^+ \;&=\; \left\{ k + 2nm \;\mid\; n = 0,1,2,\cdots \right\} \;, \\ S_k^- \;&=\; \left\{ (m-k) + (2n'+1)m \;\mid\; n' = 0,1,2,\cdots \right\} \;, \end{aligned} \tag{3.92}$$

and comparing eq. (3.92) and eq. (3.89), we can express the sets S_k^+ and S_k^- in terms of $|(m+1)r - ms|$ as

$$\begin{aligned} S_k^+ \;&=\; \left\{ \; |(m+1)r - ms| \;\mid\; r' = k, \; s - r = 2n+1, \; n = 0,1,\cdots \right\} , \\ S_k^- \;&=\; \left\{ \; |(m+1)r - ms| \;\mid\; r = k, \; s - r = 2n'+2, \; n' = 0,1,\cdots \right\} . \end{aligned} \tag{3.93}$$

From eq. (3.93), the following correspondence is obtained:

$$\begin{aligned} \widehat{\sigma}_{|k+2mn|} \; (n \ge 0) \;&\leftrightarrow\; \Phi_{m-k,\, r+2n+1} \; (n \ge 0) \\ \widehat{\sigma}_{|k+2m(-1-n')|} \; (n' \ge 0) \;&\leftrightarrow\; \Phi_{k,\, r+2n'+2} \; (n' \ge 0) \;, \end{aligned} \tag{3.94}$$

where $s \ne 0 \pmod{m+1}$.

In [33], it was suggested that the scaling operators $\widehat{\sigma}_j, \; j = 0 \pmod{m+1}$, should be identified as the boundary operators which couple to the boundaries of two-dimensional surface. These scaling operators do not have their counterparts in the BRST cohomology of the system coupled to Liouville theory. From

$$|(m+1)r - ms| = (s-r)(m+1) - s = 0 \pmod{m+1} \;, \tag{3.95}$$

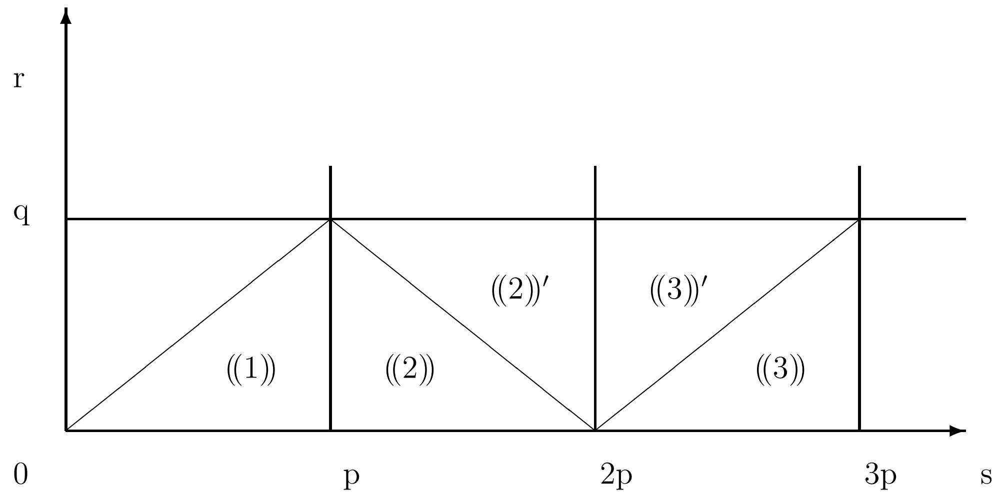


Figure 2: the range of (r, s)

$s = 0$ (mod $m+1$) in the range eq. (3.90) should correspond to the boundary operators.

As an example, we depicted the scaling operators on the r-s plane for the case of $m = 4$ in fig. 3.

Concerning the scaling operators inside the minimal table, the fusion rules involving non-diagonal operators dose not agree with the fusion rules of the unitary minimal model;

$$\begin{aligned}&\langle \Phi_{r_1,\,s_1}\Phi_{r_2,\,s_2}\Phi_{r_3,\,s_3}\rangle \neq 0 \ \text{ iff} \\ &(r_1-1, r_2-1, r_3-1) \in \mathcal{D}_3^{(m)} \ \text{ and } \ (s_1-1, s_2-1, s_3-1) \in \mathcal{D}_3^{(m+1)}. \end{aligned} \tag{3.96}$$

For example, in the Ising model ($m = 3$) the three point function for the energy operators vanish,

$$\langle \Phi_{1,3}\Phi_{1,3}\Phi_{1,3}\rangle = 0 \ . \tag{3.97}$$

Coupled to gravity, however, the corresponding three point function does not vanish.

3.7 Boundary operators

3.7.1 Boundary operators and touching of loops

In [33] it was proposed that the scaling operators which do not occur in the BRST cohomology of Liouville theory are boundary operators and one of them, which is

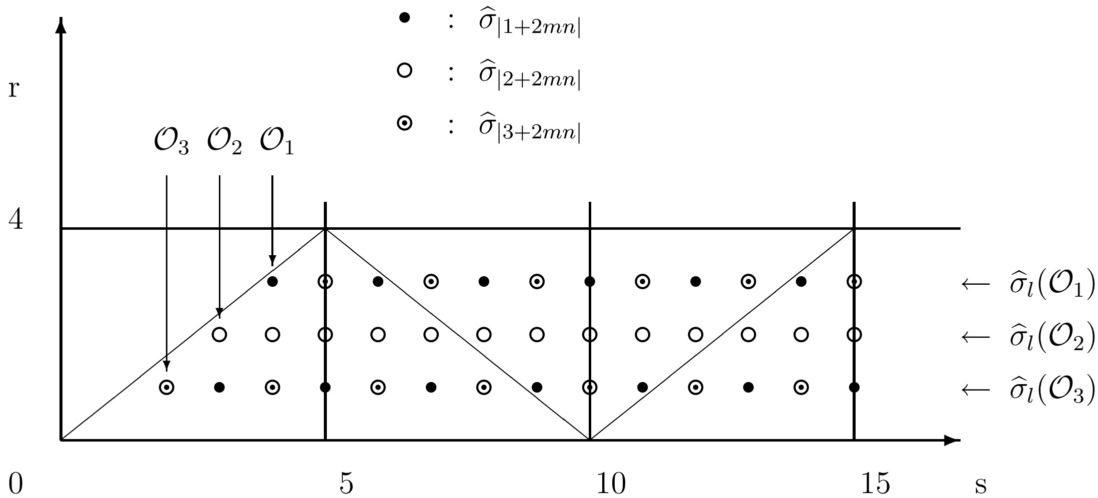


Figure 3: the scaling operators $\widehat{\sigma}_{|k+2mn|}$ in the $(5,4)$ minimal model coupled to 2D gravity

$\widehat{\sigma}_3 = \widehat{\sigma}_1(\mathcal{O}_1)$ in the case of pure gravity, was in fact proven to be a boundary operator for the one-matrix model and the Ising model case. We would like to examine the role of the operators $\widehat{\sigma}_{n(m+1)}$, $n = 1, 2, \cdots \neq 0 \pmod m$ as well as $\widehat{\sigma}_{m+1}$ for general unitary minimal models.

Let us denote these operators by

$$\widehat{\mathcal{B}}_n = \widehat{\sigma}_{n(m+1)}, \quad n = 1, 2, \cdots \neq 0 \pmod m \; . \tag{3.98}$$

In the matrix models the loop amplitudes contain the contribution from the configuration with loops touching each other. In two-loop case, let us consider the configuration in which the two loops touch each other on n points. When we shrink one of the loops to a microscopic loop, the other loop splits into n loops, which are stuck each other through the microscopic loop (see figs.4, 5 and 6). Since the microscopic loop represents a sum of the scaling operators, the wave functions of some scaling operators contain the contribution from the surfaces with split loop.

We now show that the boundary operators indeed represent these configurations. From eqs. (3.64) and (3.47), the wave function of $\widehat{\mathcal{B}}_n$ and the one-loop amplitude are

$$\left\langle \widehat{\mathcal{B}}_n w^+(\ell) \right\rangle = n(1 + \frac{1}{m}) \left(\frac{M}{2}\right)^{n(1+\frac{1}{m})} \widetilde{K}_{n(1+\frac{1}{m})}(M\ell) \; , \tag{3.99}$$

$$\left\langle w^+(\ell) \right\rangle = (1 + \frac{1}{m}) \; \ell^{-1} \left(\frac{M}{2}\right)^{1+\frac{1}{m}} \widetilde{K}_{1+\frac{1}{m}}(M\ell) \; . \tag{3.100}$$

In the space of Laplace transformed coordinates, we have

$$\mathcal{L}\left[\ell^{-1}\langle\widehat{\mathcal{B}}_n w^+(\ell)\rangle\right] = -\left(\frac{M}{2}\right)^{n(1+\frac{1}{m})} 2\cosh n(m+1)\theta\ , \tag{3.101}$$

$$\mathcal{L}\left[\langle w^+(\ell)\rangle\right] = -\left(\frac{M}{2}\right)^{1+\frac{1}{m}} 2\cosh(m+1)\theta\ , \tag{3.102}$$

where we have used the relation

$$\mathcal{L}\left[-\ell^{-1}|\nu|\widetilde{K}_\nu(M\ell)\right] = 2\cosh m\nu\theta\ . \tag{3.103}$$

Note here that $w^+(\ell)$ represents a loop with a marked point and $\ell^{-1}w^+(\ell)$ represents a loop without a marked point. Since $\cosh n(m+1)\theta$ can be expressed as a polynomial of $\cosh(m+1)\theta$,

$$\begin{aligned} 2\cosh n(m+1)\theta &= 2\ T_n\Big(\cosh(m+1)\theta\Big) \\ &\equiv \sum_{r=0}^{[n/2]} c_r^{(n)}\Big[2\cosh(m+1)\theta\Big]^{n-2r}\ , \\ c_r^{(n)} &= \frac{(-1)^r n}{n-r}\binom{n-r}{r} \end{aligned} \tag{3.104}$$

where T_n is the Chebeyshev polynomial, we obtain the following relation:

$$\mathcal{L}\left[-\ell^{-1}\langle\widehat{\mathcal{B}}_n w^+(\ell)\rangle\right] = \sum_{r=0}^{[(n-1)/2]} c_r^{(n)}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}\Big\{\mathcal{L}\left[-\langle w^+(\ell)\rangle\right]\Big\}^{n-2r}\ . \tag{3.105}$$

In the space of loop lengths, the above relation means that the wave function of $\widehat{\mathcal{B}}_n$ is equivalent to a sum of the convolutions of disk amplitudes:

$$\Big\langle\widehat{\mathcal{B}}_n w^+(\ell)\Big\rangle = -\ell\sum_{r=0}^{[(n-1)/2]} c_r^{(n)}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}(-1)^{n-2r}\left[\odot\mathcal{A}_1^+\right]^{n-2r}(\ell)\ . \tag{3.106}$$

Here we introduced a notation $\mathcal{A}_1^+ \equiv \langle w^+(\ell)\rangle$, and $\left[\odot\mathcal{A}_1^+\right]^s(\ell)$ denotes the convolution of s $\mathcal{A}_1^+(\ell)$'s, for example

$$\left[\odot\mathcal{A}_1^+\right]^2(\ell) = \int_0^\infty\int_0^\infty \mathrm{d}\ell_1\mathrm{d}\ell_2\ \delta(\ell_1+\ell_2-\ell)\mathcal{A}_1^+(\ell_1)\mathcal{A}_1^+(\ell_2)\ . \tag{3.107}$$

From eq. (3.106) we can conclude that the operator $\widehat{\mathcal{B}}_n$ couple to the point to which s ($\leq n$) parts of the loop are stuck each other in the case of one-loop amplitude. When there are more than one loop, we infer that the operator couples to the point to which s parts of several loops are stuck each other; the operator will not recognize that it is touching different loops this time.

Using the following relation

$$\left[2\cosh x\right]^n = \sum_{r=0}^{[(n-1)/2]} \binom{n}{r} 2\cosh(n-2r)x, \quad \text{(up to constant)}\ , \tag{3.108}$$

we also obtain

$$\ell\left[\odot\mathcal{A}_1^+\right]^n(\ell) = (-1)^{n+1}\sum_{r=0}^{[(n-1)/2]}\binom{n}{r}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}\left\langle\widehat{\mathcal{B}}_{n-2r}w^+(\ell)\right\rangle\ . \tag{3.109}$$

Here we drop the constant term in eq. (3.108) when we carry out the inverse Laplace transformation. From eq. (3.109), we see that the boundary operator coupled to the point on which n parts of loops are touching each other is given by

$$\mathcal{B}_n = (-1)^{n+1}\sum_{r=0}^{[(n-1)/2]}\binom{n}{r}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}\widehat{\mathcal{B}}_{n-2r}\ . \tag{3.110}$$

Now let us consider the boundary operators when there are two loops on two-dimensional surface. As for $\mathcal{B}_1$, we expect that $\langle w^+(\ell_1)w^+(\ell_2)\mathcal{B}_1\rangle$ should be proportional to $(\ell_1+\ell_2)\langle w^+(\ell_1)w^+(\ell_2)\rangle$. Let us confirm this in the following.

From the three loop correlator (3.33), the expansion of loop operator (3.65) and the wave function of $\widehat{\sigma}_{|k+2mn|}$ (3.64), we obtain the following correlator with two loops and a local operator:

$$\left\langle w^+(\ell_1)w^+(\ell_2)\widehat{\sigma}_{|k_3+2mn_3|}\right\rangle = \frac{-1}{m+1}\sum_{k_1,k_2}C_{k_1k_2k_3}\left(\frac{M}{2}\right)^{-\frac{1}{m}+|\frac{k_3}{m}+2n_3|}$$
$$\times\ell_1\ell_2(\frac{k_3}{m}+2n_3)\widetilde{K}_{1-\frac{k_1}{m}}(M\ell_1)\widetilde{K}_{1-\frac{k_2}{m}}(M\ell_2)\ . \tag{3.111}$$

Consider the amplitude for $\mathcal{B}_1 = \widehat{\mathcal{B}}_1 = \widehat{\sigma}_{m+1} = \widehat{\sigma}_{|\frac{m-1}{m}-2|}$. Since $C_{k_1,k_2,m-1}$ is nonvanishing only for the case of $k_1+k_2=m$, we obtain the following amplitude:

$$\left\langle w^+(\ell_1)w^+(\ell_2)\mathcal{B}_1\right\rangle = \frac{1}{m}\sum_{k}^{m-1}\left(\frac{M}{2}\right)\ell_1\ell_2\widetilde{K}_{\frac{k}{m}}(M\ell_1)\widetilde{K}_{1-\frac{k}{m}}(M\ell_2)\ . \tag{3.112}$$

Comparing eqs. (3.112) to (3.44) we obtain the desired relation:

$$\left\langle w^+(\ell_1)w^+(\ell_2)\mathcal{B}_1\right\rangle = (\ell_1+\ell_2)\left\langle w^+(\ell_1)w^+(\ell_2)\right\rangle\ . \tag{3.113}$$

Next, let us consider $\mathcal{B}_2$. Since we infer that the insertion of $\mathcal{B}_2$ should play the role of connecting two parts of loops together, we expect the following relation:

$$\left\langle w^+(\ell_1)w^+(\ell_2)\mathcal{B}_2\right\rangle = 2\ell_1\int_0^{\ell_1}d\ell_1'\ \left\langle w^+(\ell_1')w^+(\ell_2)\right\rangle\left\langle w^+(\ell_1-\ell_1')\right\rangle$$
$$+2\ell_2\int_0^{\ell_2}d\ell_2'\ \left\langle w^+(\ell_1)w^+(\ell_2')\right\rangle\left\langle w^+(\ell_2-\ell_2')\right\rangle$$
$$+2\ell_1\ell_2\left\langle w^+(\ell_1+\ell_2)\right\rangle\ . \tag{3.114}$$

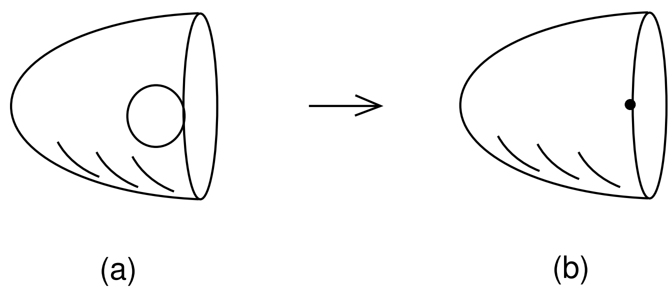


Figure 4: (a): A surface with two loops touching each other on a point. (b): When one of the loops shrinks to a microscopic loop the microscopic loop is equivalent to the insertion of the operator denoted by the dot on the loop.

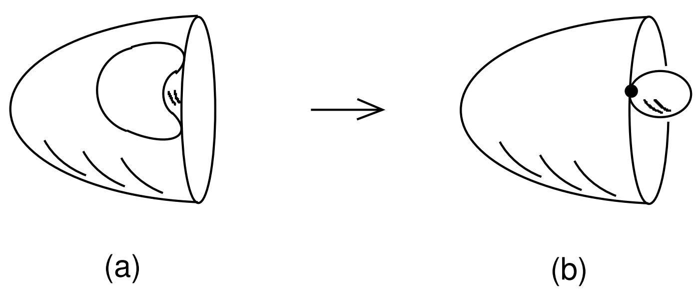


Figure 5: The case of a surface with two loops touching each other on two points.

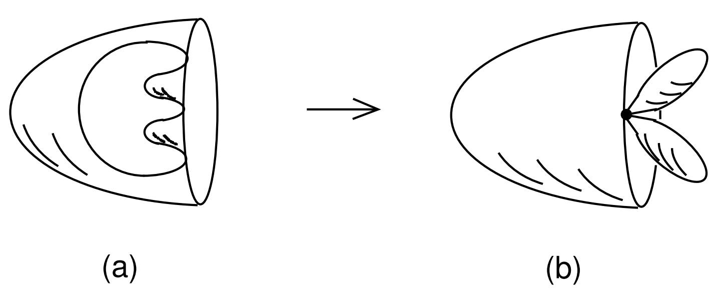


Figure 6: The case of a surface with two loops touching each other on three points.

The third term in the right hand side of eq. (3.114) represents the contribution from the surfaces with loops $w^+(\ell_1)$ and $w^+(\ell_2)$ touching with each other on a point. Let us confirm the relation (3.114) in the following. In this case, it is convenient to consider in the space of Laplace transformed coordinates ζ_i. In this space eq. (3.111) reads as

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\ \widehat{\sigma}_{|k_3+2mn_3|}\right\rangle = \frac{-1}{m+1}\left(\frac{M}{2}\right)^{-\frac{1}{m}-2+|\frac{k_3}{m}+2n_3|}(\frac{k_3}{m}+2n_3) \times\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\sum_{k_1,k_2} C_{k_1k_2k_3}\ \frac{\sinh(m-k_1)\theta_1}{\sinh m\theta_1}\ \frac{\sinh(m-k_2)\theta_2}{\sinh m\theta_2}\right\}. \tag{3.115}$$

Due to the relation

$$\sum_{k_1,k_2} C_{k_1k_2k_3}\ \frac{\sinh(m-k_1)\theta_1}{\sinh m\theta_1}\ \frac{\sinh(m-k_2)\theta_2}{\sinh m\theta_2} = \frac{-1}{2(\cosh\theta_1-\cosh\theta_2)}\left(\frac{\sinh(m-k_3)\theta_1}{\sinh m\theta_1}-\frac{\sinh(m-k_3)\theta_2}{\sinh m\theta_2}\right), \tag{3.116}$$

we have

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\ \widehat{\sigma}_{|k_3+2mn_3|}\right\rangle = \frac{1}{2(m+1)}\left(\frac{M}{2}\right)^{-\frac{1}{m}-2+|\frac{k_3}{m}+2n_3|}(\frac{k_3}{m}+2n_3) \times\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1-\cosh\theta_2}\left(\frac{\sinh(m-k_3)\theta_1}{\sinh m\theta_1}-\frac{\sinh(m-k_3)\theta_2}{\sinh m\theta_2}\right)\right\}. \tag{3.117}$$

Since we should take $k_3 = 2$ for $\mathcal{B}_2 = -\widehat{\sigma}_{2(m+1)}$ (for $m \geq 3$), we obtain the amplitude for $\mathcal{B}_2$

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\mathcal{B}_2\right\rangle = \frac{-1}{m}\left(\frac{M}{2}\right)^{\frac{1}{m}}\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1-\cosh\theta_2}\left(\frac{\sinh(m-2)\theta_1}{\sinh m\theta_1}-\frac{\sinh(m-2)\theta_2}{\sinh m\theta_2}\right)\right\}. \tag{3.118}$$

On the other hand, from the amplitudes

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\right\rangle = \frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\ln\frac{\cosh\theta_1-\cosh\theta_2}{\cosh m\theta_1-\cosh m\theta_2} = \frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1-\cosh\theta_2}\frac{\sinh\theta_1}{mM\sinh m\theta_1}-\frac{1}{\zeta_1-\zeta_2}\right\}, \tag{3.119}$$

$$\left\langle \hat{W}^+(\zeta)\right\rangle = -\left(\frac{M}{2}\right)^{1+\frac{1}{m}} 2\cosh(m+1)\theta\ , \tag{3.120}$$

we obtain the following relation

$$
\begin{aligned}
&-2\frac{\partial}{\partial\zeta_1}\left\{\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\right\rangle \left\langle \hat{W}^+(\zeta_1)\right\rangle\right\} + (1\leftrightarrow 2) \\
&-2\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{\left\langle \hat{W}^+(\zeta_1)\right\rangle - \left\langle \hat{W}^+(\zeta_2)\right\rangle}{\zeta_1-\zeta_2}\right\} \\
&= \frac{2}{m}\left(\frac{M}{2}\right)^{\frac{1}{m}}\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1-\cosh\theta_2}\left(\frac{\sinh\theta_1\cosh(m+1)\theta_1}{\sinh m\theta_1} - (1\leftrightarrow 2)\right)\right\}.
\end{aligned}
\tag{3.121}
$$

One can easily show that the right hand side of eq. (3.121) agrees with that of eq. (3.118). Putting eqs. (3.121) and (3.118) together and performing the inverse Laplace transformation, we obtain the desired relation eq. (3.114).

We have shown that the operator $\mathcal{B}_2$ connects two parts of loops together in the case with two loops. We infer that similar phenomena occur in general; the operator $\mathcal{B}_n$ would connect n parts of loops together in the case with any number of loops.

3.7.2 Connection to the Schwinger-Dyson equations

We can observe close relationship between the boundary operators and the Schwinger-Dyson equations proposed in [10]. Continuum limit of the Schwinger-Dyson equations for loops in the two- and multi-matrix models were proposed in [10] under some assumptions. It was shown [10] that these equations contain W_3 constraints, which were derived explicitly in [7]. The integrability of these equations were shown in [11]. These facts justify the proposed Schwinger-Dyson equations.

Let us consider the connection of the boundary operators with the Schwinger-Dyson equations. For the $(m+1, m)$ minimal models, the following Schwinger-Dyson equations were proposed in [10]:

$$
\begin{aligned}
&\int_0^\ell d\ell' \left\langle w^{(1)}(\ell')w^{(1)}\left(\ell-\ell';[\mathcal{H}(\sigma)]^j\right)w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_n)\right\rangle' \\
&+g\sum_i \ell_i \left\langle w^{(1)}\left(\ell+\ell_i;[\mathcal{H}(\sigma)]^j\right)w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_{i-1})w^{(1)}(\ell_{i+1})\cdots w^{(1)}(\ell_n)\right\rangle' \\
&+\left\langle w^{(1)}\left(\ell;[\mathcal{H}(\sigma)]^{j+1}\right)w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_n)\right\rangle' \approx 0\ , \\
&\qquad\qquad \text{for}\ \ j=0,\cdots,m-2\ ,
\end{aligned}
\tag{3.122}
$$

and

$$
\left\langle w^{(1)}\left(\ell;[\mathcal{H}(\sigma)]^{m-1}\right)w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_n)\right\rangle' \approx 0\ . \tag{3.123}
$$

Here $\langle \cdots \rangle'$ represent loop correlators that are not necessarily connected, and $w^{(1)}(\ell)$ represents a loop operator corresponding to a loop created by the matrix $\hat{A}^{(1)}$ in the multi-matrix model. The operator $\mathcal{H}(\sigma)$ describes an operator which changes the 'spin' on a loop locally from 1 to 2. Also $w^{(1)}(\ell; [\mathcal{H}(\sigma)]^j)$ describes a loop with $[\mathcal{H}(\sigma)]^j$ inserted. The symbol $\approx$ means that as a function of ℓ, the quantity has its support at $\ell = 0$.

From eq. (3.122) for $j = 0$ and $n = 1$, we have the relation

$$\begin{aligned}
&\ell_1 \left\langle w^{(1)}(\ell_1; \mathcal{H}(\sigma))\, w^{(1)}(\ell_2) \right\rangle' + \ell_2 \left\langle w^{(1)}(\ell_1) w^{(1)}(\ell_2; \mathcal{H}(\sigma)) \right\rangle' \\
&+\ell_1 \int_0^{\ell_1} d\ell_1' \left\langle w^{(1)}(\ell_1') w^{(1)}(\ell_1 - \ell_1') w^{(1)}(\ell_2) \right\rangle' \\
&+\ell_2 \int_0^{\ell_2} d\ell_2' \left\langle w^{(1)}(\ell_1) w^{(1)}(\ell_2') w^{(1)}(\ell_2 - \ell_2') \right\rangle' \\
&+2g\ell_1\ell_2 \left\langle w^{(1)}(\ell_1 + \ell_2) \right\rangle' \approx 0 \, . \qquad (3.124)
\end{aligned}$$

The planar part of the above relation agrees with eq. (3.114). Note that the loop amplitudes in eq. (3.114) represent connected correlators.

This agreement implies that $\mathcal{H}$ would correspond to $\widehat{\mathcal{B}}_2$. Taking into account the fact that $\widehat{\mathcal{B}}_n$ ($n = 0 \bmod m$) do not exist and eq. (3.123), it is legitimate to consider that the amplitude (for $j = 1, \cdots, m$)

$$\left\langle w^+(\ell_1) \cdots w^+(\ell_n) \widehat{\mathcal{B}}_j \right\rangle \qquad (3.125)$$

corresponds to the connected part of the amplitude

$$\sum_{i=1}^{n} \oint d\sigma_i \left\langle w^{(1)}(\ell_1) \cdots w^{(1)}(\ell_{i-1}) w^{(1)} \left(\ell_i; [\mathcal{H}(\sigma_i)]^{j-1} \right) w^{(1)}(\ell_{i+1}) \cdots w^{(1)}(\ell_n) \right\rangle' \, . \qquad (3.126)$$

4 Multi-loop correlators

In this section, we generalize the discussion in sect. 3 to the cases of higher-loop. First, we derive the formula of the n-resolvent correlators, which we quoted in sect. 3.1, and point out that the structure corresponding to the crossing symmetry of the underlying conformal field theory can be seen in the loop correlators. We then discuss the four-loop correlator in detail.

4.1 The derivation of the n-resolvent correlators

Consider in the two-matrix model the connected part of the correlator consisting of the product of n-resolvents.

$$\left\langle\!\!\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, \frac{1}{p_i - \hat{A}} \right\rangle\!\!\right\rangle \quad . \tag{4.1}$$

It should be noted that this expression is at most $\left(\frac{1}{N}\right)^{n-2}$ due to the large N factorization of the correlator consisting of the product of singlet operators. In the second quantized notation, eq. (4.1) is expressible as

$$\begin{aligned}
&\langle\!\langle N| \prod_{i=1}^{n} : \int \mathrm{d}\mu_i \Psi^\dagger(\tilde{\lambda}_i) \frac{1}{p_i - \lambda_i} \Psi(\lambda_i) : |N\rangle\!\rangle \\
&= \langle\!\langle N| \prod_{i=1}^{n} : a^\dagger_{k_i} a_{j_i} : |N\rangle\!\rangle \prod_{i=1}^{n} \int \mathrm{d}\mu_i \xi_{k_i}(\tilde{\lambda}_i) \frac{1}{p_i - \lambda_i} \xi_{k_i}(\lambda_i) \\
&= \langle\!\langle N| \prod_{i=1}^{n} : a^\dagger_{k_i} a_{j_i} : |N\rangle\!\rangle \prod_{i=1}^{n} \langle k_i| \frac{1}{p_i - A_i} |l_i\rangle \quad ,
\end{aligned} \tag{4.2}$$

The normal ordering : ... : is with respect to the filled sea $|N\rangle\!\rangle$. We introduce a notation

$$\left[\frac{1}{p-A}\right](z_i; \Lambda_i, \Lambda, N) \equiv \sum_{\delta} z_i^\delta < j_i - \delta \mid \frac{1}{p-A} \mid j_i > \tag{4.3}$$

$$\Lambda_i = j_i \Lambda / N = \Lambda + \Lambda \tilde{j}_i / N \quad . \tag{4.4}$$

The evaluation of $\langle\!\langle N| \prod_{i=1}^{n} : a^\dagger_{k_i} a_{j_i} : |N\rangle\!\rangle$ by the Wick theorem provides $(n-1)!$ terms of the following structure: each term is given by the product of n-Kronecker delta's multiplied both by a sign factor and by the product of n-step functions to ensure that the summations over the n-indices $\tilde{j}_1, \tilde{j}_2 \cdots$ and $\tilde{j}_n$ are bounded either from below (≥ 0) or from above(≤ -1). We denote this product by $\Theta(\tilde{j}_1, \tilde{j}_2, \cdots \tilde{j}_n; \sigma)$.

These $(n-1)!$ terms are in one-to-one correspondence with the circular permutations of n integers $1, \cdots, n$, which we denote by $\mathcal{S}_n$. The σ is an element of $\mathcal{S}_n$. For large N, we find

$$\left(\frac{N}{\Lambda}\right)^{n-2} \left\langle\!\!\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, \frac{1}{p_i - \hat{A}} \right\rangle\!\!\right\rangle \tag{4.5}$$

$$= \sum_{\tilde{j}_1, \tilde{j}_2, \cdots \tilde{j}_n} \sum_{\sigma \in \mathcal{S}_n} \Theta(\tilde{j}_1, \tilde{j}_2, \cdots \tilde{j}_n; \sigma) sgn(\sigma) \left(\prod_{j=1}^{n} \oint \frac{dz_j}{2\pi i} \right) \prod_{k=1}^{n} \frac{1}{z_k} \left(\frac{z_{\sigma(k)}}{z_k}\right)^{\tilde{j}_k}$$
$$\times \frac{1}{(n-2)!} \left(\sum_{i=1}^{n} \tilde{j}_i \frac{\partial}{\partial \Lambda_i} \right)^{n-2} \prod_{i'=1}^{n} \frac{1}{p_{i'} - A(z_{i'}; \Lambda_{i'})} |_{\Lambda_{i'} = \Lambda} + \mathcal{O}\left(1/N\right) \quad . \tag{4.6}$$

Note that in the large N limit, we can use $\frac{1}{p_i - A(z_i;\Lambda_i)}$ in place of $\left[\frac{1}{p_i - A}\right](z_i; \Lambda_i, \Lambda, N)$ according to the same reason as stated in the case of three-loop correlator. The $sgn(\sigma)$ denotes the signature associated with the permutation σ.

Let us define

$$m! D_m(z, z') \equiv \frac{1}{z} \sum_{\tilde{j} \geq 0} \tilde{j}^m \left(z'/z\right)^{\tilde{j}} = -\frac{1}{z} \sum_{\tilde{j} \leq -1} \tilde{j}^m \left(z'/z\right)^{\tilde{j}} \quad , \quad m = 0, \cdots \quad . \tag{4.7}$$

In the continuum limit we will be focusing from now on, it is sufficient to use

$$D_m(z, z') \approx \frac{1}{(z - z')^{m+1}} \equiv D_m(z - z') \quad . \tag{4.8}$$

Let $sgn_i(\sigma)$ be $+1$ or -1, depending upon whether the restriction on the summation over $\tilde{j}_i$ is bounded from below or from above respectively. It is not difficult to show

$$sgn(\sigma) \prod_{i=1}^{n} sgn_i(\sigma) = -1 \quad , \tag{4.9}$$

for any σ and n. The summations over $\tilde{j}_1, \tilde{j}_2 \cdots$ and $\tilde{j}_n$ can then be performed for all σ at once, leaving with this minus sign.

Now we turn to the integrations over z_i $(i = 1 \sim n)$. The convergence on the geometric series leads to the successively ordered integrations of $z_i' s$ for each σ. We assume here that only the poles at $z_i = z_i^*$ give rise to terms with physical significance. This is the case for three-loop correlators. By simply picking up a pole of z_i at $\frac{1}{p_i - A(z_i;\Lambda_i)}$ for $i = 1 \sim n$ and using

$$\oint \frac{dz_i}{2\pi i} f\left(\cdots z_i, \cdots\right) \left(\frac{\partial}{\partial \Lambda_i}\right)^{\ell} \left(\frac{1}{p_i - [A]\left(z_i; \Lambda_i\right)}\right)$$
$$= -\frac{\partial}{\partial(a\zeta_i)} \left(\frac{\partial}{\partial \Lambda_i}\right)^{\ell} \int^{z_i^*} dz_i f\left(\cdots z_i, \cdots\right) \quad , \quad \ell = 0, 1, \cdots \quad , \tag{4.10}$$

we find that eq. (4.5) is written as

$$\left(\frac{N}{\Lambda}\right)^{n-2} \left\langle\!\!\left\langle \prod_{i=1}^{n} \mathrm{Tr}\,\frac{1}{p_i - \hat{A}} \right\rangle\!\!\right\rangle = \prod_{i=1}^{n}\left(-\frac{\partial}{\partial(a\zeta_i)}\right) R^{(n)}|_{\Lambda_i=\Lambda} \quad , \tag{4.11}$$

where

$$\begin{aligned}
R^{(n)} \;\equiv\;& \sum_{i_1}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-2} \int\cdots\int \sum_{\sigma\in\mathcal{S}_n} D_{n-2}([i_1-\sigma(i_1)]) \prod_{j(\neq i_1)} D_0([j-\sigma(j)]) \\
&+ \sum_{\substack{m_1+m_2\\=n-2}} \sum_{(i_1,i_2)} \Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{m_1}\Big(\frac{\partial}{\partial\Lambda_{i_2}}\Big)^{m-2} \int\cdots\int D_{m_1}([i_1-\sigma(i_1)])D_{m_2}([i_2-\sigma(i_2)]) \\
&\quad\times \prod_{j(\neq i_1,i_2)} D_0([j-\sigma(j)]) \\
&+ \sum_{\substack{m_1+m_2+m_3\\=n-2}} \sum_{(i_1,i_2,i_3)} \Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{m_1}\Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{m_2}\Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{m_3} \int\cdots\int D_{m_1}([i_1-\sigma(i_1)]) \\
&\quad\times D_{m_2}([i_2-\sigma(i_2)])D_{m_3}([i_3-\sigma(i_3)]) \prod_{j(\neq i_1,i_2)} D_0([j-\sigma(j)]) \\
&+ \cdots \\
&+ \sum_{(i_1,i_2,\cdots,i_{n-2})} \Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)\Big(\frac{\partial}{\partial\Lambda_{i_2}}\Big)\cdots\Big(\frac{\partial}{\partial\Lambda_{i_{n-2}}}\Big) \int\cdots\int \\
&\quad\times \prod_{j=1}^{n-2} D_1([i_j-\sigma(i_j)]) \prod_{j(\neq i_1,i_2,\cdots,i_{n-2})} D_0([j-\sigma(j)]) \quad .
\end{aligned} \tag{4.12}$$

The integrals in the equation above are with respect to z_i^*'s and we adopt a notation

$$[i] \equiv z_i^* \quad , \quad [i-j] \equiv z_i^* - z_j^* \quad . \tag{4.13}$$

This expression is in one to one correspondence with the expansion of $\left(\sum_{i=1}^{n} x_i\right)^{n-2}$. The number of terms appearing is equal to the number of partitions of $(n-2)$ objects into parts.

In order to put eqs. (4.11), (4.12) in a simpler form, let us introduce

$$\begin{aligned}
&\begin{pmatrix} m_1, & m_2, & \cdots, & m_n \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix}_n \\
&\quad\equiv -\sum_{\sigma\in\mathcal{S}_n} D_{m_1}([i_1-\sigma(i_1)])D_{m_2}([i_2-\sigma(i_2)])\cdots D_{m_n}([i_k-\sigma(i_n)]) \\
&\quad= -\sum_{\sigma\in\mathcal{S}_n} \frac{1}{[i_1-\sigma(i_1)]^{m_1+1}}\,\frac{1}{[i_2-\sigma(i_2)]^{m_2+1}}\cdots\frac{1}{[i_n-\sigma(i_n)]^{m_n+1}} \quad .
\end{aligned} \tag{4.14}$$

In particular,

$$\begin{pmatrix} n-2, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & 0 \end{pmatrix}_n \equiv -\sum_{\sigma\in\mathcal{S}_n} D_{n-2}([i_1-\sigma(i_1)]) \prod_{j(\neq i_1)} D_0([j-\sigma(j)])$$

$$\begin{pmatrix} n-3, & 1, & 0, & \cdots, & 0 \\ i_1, & i_2, & i_3, & \cdots, & 0 \end{pmatrix}_n \equiv -\sum_{\sigma\in\mathcal{S}_n} D_{n-3}([i_1-\sigma(i_1)]) D_1([i_2-\sigma(i_2)]) \times \prod_{j(\neq i_1,i_2)} D_0([j-\sigma(j)])$$

$$\text{e.t.c} \tag{4.15}$$

In the Appendix B, we prove that

$$\begin{pmatrix} m_1, & m_2, & \cdots, & m_n \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix}_n = 0 \quad \text{if} \quad \sum_\ell m_\ell \leq n-3 \quad , \tag{4.16}$$

as well as

$$\begin{aligned} &\begin{pmatrix} m_1, & m_2, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & i_k, & i_{k+1}, & \cdots, & 0 \end{pmatrix}_n \\ = &\sum_{\ell=1}^{k} \frac{1}{[i_\ell - i_n]^2} \begin{pmatrix} m_1, & \cdots, & m_\ell - 1, & \cdots, & m_k, & 0, & \cdots \\ i_1, & \cdots, & i_\ell, & \cdots, & i_k, & \cdots, & \cdots \end{pmatrix}_{n-1} \\ \text{if} \quad &\sum_\ell m_\ell = n-2 \quad . \end{aligned} \tag{4.17}$$

In particular,

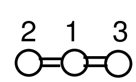
$$\begin{pmatrix} n-2, & 0, & \cdots \\ i_1, & \cdots, & \cdots \end{pmatrix}_n = \frac{1}{[i_1-i_n]^2} \begin{pmatrix} n-3, & 0, & \cdots \\ i_1, & \cdots, & \cdots \end{pmatrix}_{n-1} = \frac{1}{\prod_{j(\neq i_1)}^{n} [i_1-j]^2} \tag{4.18}$$

and

$$\begin{aligned} \begin{pmatrix} n-3, & 1, & \cdots \\ i_1, & i_2 & \cdots \end{pmatrix}_n &= \frac{1}{[i_1-i_n]^2} \begin{pmatrix} n-4, & 1, & \cdots \\ i_1, & i_2, & \cdots \end{pmatrix}_{n-1} \\ &+ \frac{1}{[i_2-i_n]^2} \begin{pmatrix} n-3, & 0, & \cdots \\ i_1, & i_2, & \cdots \end{pmatrix}_{n-1} \quad . \end{aligned} \tag{4.19}$$

Let us introduce graphs in which the factor $1/[i-j]^2$ is represented by a double line linking circle i and circle j to handle the quantities defined by eq. (4.14) more easily. For example, for n=3,

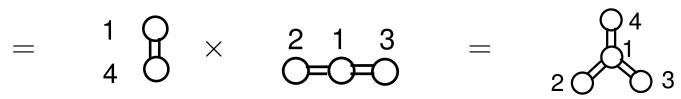
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3 = \frac{1}{[z_1^* - z_2^*]^2} \frac{1}{[z_1^* - z_3^*]^2} \equiv$$



$$. \tag{4.20}$$

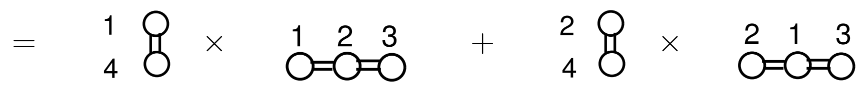
Using the recursion relation eq. (4.17) and eq. (4.16) we have, for n=4,

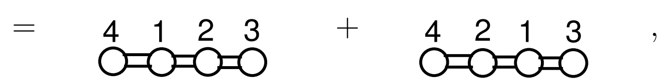
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4 = \frac{1}{[z_1^* - z_4^*]^2} \begin{pmatrix} 2-1 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3$$



, (4.21)

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4 = \frac{1}{[z_1^* - z_4^*]^2} \begin{pmatrix} 1-1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3 + \frac{1}{[z_2^* - z_4^*]^2} \begin{pmatrix} 1 & 1-1 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3$$

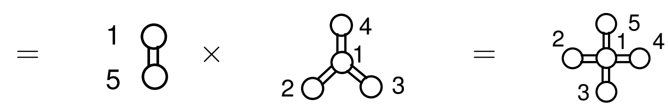




, (4.22)

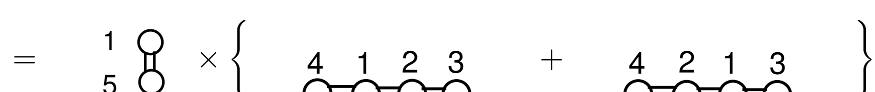
and, for n=5,

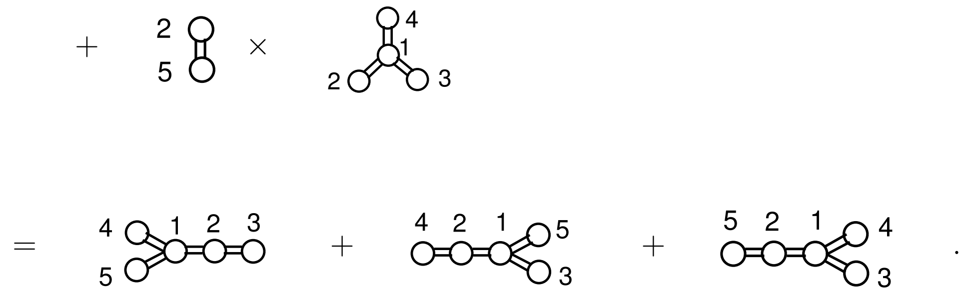
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}_5 = \frac{1}{[z_1^* - z_5^*]^2} \begin{pmatrix} 3-1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4$$



, (4.23)

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}_5 = \frac{1}{[z_1^* - z_5^*]^2} \begin{pmatrix} 2-1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4 + \frac{1}{[z_2^* - z_5^*]^2} \begin{pmatrix} 2 & 1-1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4$$





(4.24)

From the examples above, it is clear that the graphs for the general case can be written down quite easily.

In terms of the quantities defined by eq. (4.14), we obtain a formula for the n-point resolvent :

$$\left(\frac{N}{\Lambda}\right)^{n-2}\left\langle\!\!\left\langle\prod_{i=1}^{n}\mathrm{Tr}\,\frac{1}{p_i-\hat{A}}\right\rangle\!\!\right\rangle=\prod_{i=1}^{n}\left(-\frac{\partial}{\partial(a\zeta_i)}\right)R^{(n)}|_{\Lambda_i=\Lambda}\quad, \tag{4.25}$$

where

$$\begin{aligned}
&R^{(n)}(z_1^*,\cdots.z_n^*)\\
&\quad=\sum_{i_1}^{n}\Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{n-2}\int\cdots\int\begin{pmatrix} n-2, & 0, & \cdots, & 0\\ i_1, & i_2, & \cdots, & i_n\end{pmatrix}_n\\
&\quad+\sum_{m_1+m_2=n-2}\ \sum_{(i_1,i_2)}\Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{m_1}\Big(\frac{\partial}{\partial\Lambda_{i_2}}\Big)^{m_2}\int\cdots\int\begin{pmatrix} m_1, & m_2, & 0, & \cdots, & 0\\ i_1, & i_2, & i_3, & \cdots, & i_n\end{pmatrix}_n\\
&\quad+\sum_{\substack{m_1+m_2+m_3\\=n-2}}\ \sum_{(i_1,i_2,i_3)}\Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)^{m_1}\Big(\frac{\partial}{\partial\Lambda_{i_2}}\Big)^{m_2}\Big(\frac{\partial}{\partial\Lambda_{i_3}}\Big)^{m_3}\int\cdots\int\\
&\qquad\qquad\qquad\qquad\begin{pmatrix} m_1, & m_2, & m_3, & 0, & \cdots, & 0\\ i_1, & i_2, & i_3, & i_4, & \cdots, & i_n\end{pmatrix}_n\\
&\quad+\cdots\\
&\quad+\sum_{(i_1,\cdots,i_{n-2})}\Big(\frac{\partial}{\partial\Lambda_{i_1}}\Big)\cdots\Big(\frac{\partial}{\partial\Lambda_{i_{n-2}}}\Big)\int\cdots\int\begin{pmatrix} 1, & \cdots, & 1, & 0, & 0\\ i_1, & \cdots, & i_{n-2}, & i_{n-1}, & i_n\end{pmatrix}_n.
\end{aligned}\tag{4.26}$$

Here $m_\ell \geq 1$ and the summation $(i_1,\cdots,i_k)$ denotes a set of k unequal integers from $1,2,\cdots,n$ and $i_{k+1},\cdots,i_n$ in the array represents the remaining integers.

From eq. (4.25), eq. (4.26), we can derive the rule which states how to construct the n-resolvent correlator graphically. This rule was shown in sect. 3.1.

4.2 The selection rule in summation and crossing symmetry

In the previous section, we have derive the formula for the n-resolvent correlators, which were functions of ζ_i and μ. In order to obtain the loop correlators, we have to carry out the inverse Laplace transformation with respect to ζ_i further. In the processes, the following functions will play a fundamental role.

$$\begin{aligned} P_n(\theta_1, \theta_2, \cdots \theta_n) &\equiv \sum_{i_1}^{n} \left(\frac{\partial}{\partial \Lambda_{i_1}}\right) \int \cdots \int \begin{pmatrix} n-2, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix} \\ &= \sum_{i=1}^{n} \frac{\partial z_i^*}{\partial \Lambda_i} \frac{1}{\prod_{j(\neq i)}^{n} [i-j]} \quad . \end{aligned} \tag{4.27}$$

where

$$z_i^* = \exp(2\eta_i \cosh\theta_i) \quad , \tag{4.28}$$

$$p_i - p_* = a\zeta_i = 2\eta_i^m \cosh m\theta_i = aM_i \cosh m\theta_i \quad , \quad \eta_i = (aM_i/2)^{1/m} \tag{4.29}$$

$$\Lambda_i - \Lambda_* = -(m+1)\eta_i^{2m} = -a^2\mu_i = -(m+1)(aM_i/2)^2 \quad , \tag{4.30}$$

and

$$\left.\frac{\partial \theta_i}{\partial \Lambda}\right|_{\zeta_i} = -\frac{1}{\eta}\frac{\partial \eta}{\partial \Lambda}\frac{\cosh m\theta_i}{\sinh m\theta_i} \quad , \quad \frac{\partial z_i^*}{\partial \Lambda_i} = 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\frac{\sinh(m-1)\theta_i}{\sinh m\theta_i} \quad . \tag{4.31}$$

The important point here is that the above functions can be written as a sum of the product of n factors each of which is a function only of the corresponding θ_i and μ, so that we can obtain the inverse-Laplace images of $P_n(\theta_1, \cdots, \theta_n)$ immediately. We conjecture that n-resolvent correlator can be expressed in terms of $P_{n'}$ $(n' \leq n)$. In fact, as we will show later explicitly, it is true for the four- and the five-resolvent correlator.

A key manipulation we will use is the partial fraction

$$\frac{1}{[i-j][i-k]} = \frac{1}{[i-k][k-j]} + \frac{1}{[i-j][j-k]} \quad . \tag{4.32}$$

One can associate a line from i to j with $\frac{1}{[i-j]}$. The following identity is responsible for expressing P_n as a sum of the product of n factors each of which depends only on θ_i:

$$\begin{aligned} I_k(\alpha, \beta; m) &\equiv \frac{1}{\cosh\alpha - \cosh\beta}\left(\frac{\sinh(m-k)\alpha}{\sinh m\alpha} - \frac{\sinh(m-k)\beta}{\sinh m\beta}\right) \\ &= -2\sum_{j=1}^{m-k}\sum_{i=1}^{k} \frac{\sinh(m-j-i+1)\alpha}{\sinh m\alpha}\,\frac{\sinh(m-j-k+i)\beta}{\sinh m\beta} \, . \end{aligned} \tag{4.33}$$

Let us first work out the cases $n = 2, 3, 4$ to get a feeling. For $n = 2$,

$$
\begin{aligned}
& P_2(\theta_1, \theta_2) = \frac{\frac{\partial z_1^*}{\partial \Lambda_1} - \frac{\partial z_2^*}{\partial \Lambda_2}}{[1-2]} = 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\frac{1}{2\eta} I_{k_1=1}(\theta_1, \theta_2; m) \\
= \;& 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)\sum_{j=1}^{m-k_1}\sum_{i_1=1}^{k_1}\frac{\sinh(m-j_1-i_1+1)\theta_1}{\sinh m\theta_1}\frac{\sinh(m-j_1+i_1-k_1)\theta_2}{\sinh m\theta_2} \\
= \;& 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)\sum_{j_1=1}^{m-1}\frac{\sinh(m-j_1)\theta_1}{\sinh m\theta_1}\frac{\sinh(m-j_1)\theta_2}{\sinh m\theta_2}\;, \qquad (4.34)
\end{aligned}
$$

where $i_1 = k_1 = 1$. For $n = 3$, we use eq. (4.32) for the term containing $\frac{\partial z_2^*}{\partial \Lambda_2}$ to create a link $[1-3]$, which is originally absent. This relates P_3 to P_2. We find

$$
\begin{aligned}
& P_3(\theta_1, \theta_2, \theta_3) \\
= \;& 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\frac{I_1(\theta_2, \theta_1; m) - I_1(\theta_2, \theta_3; m)}{[1-3]} \\
= \;& 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)^2\left(\sum_{j_1=1}^{m-k_1}\sum_{i_1=1}^{k_1=1}\right)\left(\sum_{j_2=1}^{m-k_2}\sum_{i_2=1}^{k_2}\right) \qquad (4.35) \\
& \frac{\sinh(m-j_2-i_2+1)\theta_1}{\sinh m\theta_1}\frac{\sinh(m-j_1+i_1-k_1)\theta_2}{\sinh m\theta_2}\frac{\sinh(m-j_2+i_2-k_2)\theta_3}{\sinh m\theta_3}\;.
\end{aligned}
$$

Here $k_2 = j_1 + i_1 - 1 = j_1$.

This can be repeated for arbitrary n. In the case $n = 4$, we use the partial fraction for the two terms containing $\frac{\partial z_2^*}{\partial \Lambda_2}$ and $\frac{\partial z_3^*}{\partial \Lambda_3}$ to create a link $[1-4]$, which is originally absent. This enables us to relate the case $n = 4$ to the case $n = 3$. In general, P_n is related to P_{n-1} by using the partial fraction for the terms containing $\frac{\partial z_2^*}{\partial \Lambda_2} \sim \frac{\partial z_{n-1}^*}{\partial \Lambda_{n-1}}$ to create a link $[1-n]$. We obtain

$$
\begin{aligned}
& P_n(\theta_1, \theta_2, \cdots \theta_n) = -\left(\frac{\partial \eta^2}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)^n\left(\prod_{\ell=1}^{n-1}\sum_{j_\ell=1}^{m-k_\ell}\sum_{i_\ell=1}^{k_\ell}\right) \\
& \left(\prod_{\ell'=1}^{n-1}\frac{\sinh(m-j_{\ell'}+i_{\ell'}-k_{\ell'})\theta_{\ell'+1}}{\sinh m\theta_{\ell'+1}}\right)\frac{\sinh(m-j_{n-1}-i_{n-1}+1)\theta_1}{\sinh m\theta_1}\;, \qquad (4.36)
\end{aligned}
$$

where $k_\ell = j_{\ell-1} + i_{\ell-1} - 1, \quad$ for $\ell = 2, 3, \cdots, (n-1)$. Eq. (4.36) expresses the $P_n(\theta_1, \theta_2, \cdots \theta_n)$ as a sum of the n-products of the factor $\dfrac{\sinh(m-k)\theta_i}{\sinh m\theta_i}$. Owing to this property, one can perform the inverse Laplace transform immediately.

Let us now discuss the restrictions on the summations of $2n-3$ integers j_1, i_2, j_2, $\cdots i_{n-1}, j_{n-1}$ in eq. (4.36). We write these as a set:

$$
\mathcal{F}_n(j_1, i_2, j_2 \cdots i_{n-1}, j_{n-1})
$$

$$
\begin{aligned}
&\equiv \{(j_1, i_2, j_2, \cdots i_{n-1}, j_{n-1}) \mid 1 \leq i_\ell \leq k_\ell,\ 1 \leq j_\ell \leq m - k_\ell,\ \text{for } \ell = 1, 2, \cdots n-1\} \\
&= \mathcal{F}_2(i_1 = 1, j_1; k_1 = 1) \prod_{\ell=2}^{n-1} \cap \mathcal{F}_2(i_\ell, j_\ell; k_\ell) \ , \qquad (4.37)
\end{aligned}
$$

where

$$
\mathcal{F}_2(i_\ell, j_\ell; k_\ell) \equiv \{(i_\ell, j_\ell) \mid 1 \leq i_\ell \leq k_\ell,\ 1 \leq j_\ell \leq m - k_\ell, \ \ \text{with } k_\ell \ \ \text{fixed}\} \ . \qquad (4.38)
$$

We will show that these restrictions on the sums are in fact in one-to-one correspondence with the fusion rules of the unitary minimal models for the diagonal primaries. Let us begin with the case $n = 3$. Define

$$
\begin{array}{lll}
p_1 \equiv j_1 + k_1 - i_1 \ , & p_2 \equiv j_2 + k_2 - i_2 \ , & q_3 \equiv j_2 + i_2 - 1 \ , \\
a_{12} \equiv p_1 - 1 \ , & a_{23} \equiv p_2 - 1 \ , & a_{31} \equiv q_3 - 1 \ ,
\end{array} \qquad (4.39)
$$

The inequalities on i_2, j_2 are found to be equivalent to the following four inequalities:

$$
\begin{aligned}
a_{12} + a_{23} - a_{31} &= 2(k_2 - i_2) \geq 0 \ . \\
a_{12} - a_{23} + a_{31} &= 2(i_2 - 1) \geq 0 \ . \\
-a_{12} + a_{23} + a_{31} &= 2(j_2 - 1) \geq 0 \\
a_{12} + a_{23} + a_{31} &= 2(j_2 + k_2 - 2) \leq 2(m-2) \ . \qquad (4.40)
\end{aligned}
$$

From the third and the fourth equation of eq. (4.40), the inequality $a_{12} \leq m - 2$ follows, which is a condition for $\mathcal{F}_2(i_1 = 1, j_1; k_1 = 1)$. Defining a set

$$
\begin{aligned}
\mathcal{D}_3(a_1, a_2, a_3) \ \equiv \ & \{(a_1, a_2, a_3) \mid \sum_{i(\neq j)}^{3} a_i - a_j \geq 0 \ \text{ for } \ i = 1 \sim 3 \ \ , \\
& \sum_{i=1}^{3} a_i = \text{even} \leq 2(m-2)\} \ , \qquad (4.41)
\end{aligned}
$$

we state eq. (4.40) as

$$
\mathcal{F}_3(j_1, i_2, j_2) = \mathcal{D}_3(a_{12}, a_{23}, a_{31}) \ . \qquad (4.42)
$$

We also write

$$
\mathcal{F}_2(j_1) \equiv \mathcal{F}_2(i_1 = 1, j_1; k_1 = 1) \equiv \mathcal{D}_2(a_{12}) \ . \qquad (4.43)
$$

for the case $n = 2$.

Eq. (4.41) is nothing but the condition that a triangle be formed which is made out of a_1, a_2 and a_3 and whose circumference is less than or equal to $2(m-2)$. It is also the selection rule for the three point function of the diagonal primaries in m-th minimal unitary conformal field theory [17].

For the case $n = 4$, introduce $p_3 \equiv j_3 + k_3 - i_3$, $a_{34} \equiv p_3 - 1$, $q_4 \equiv j_3 + i_3 - 1$, $a_{41} \equiv q_4 - 1$. We find

$$\mathcal{F}_2(i_3, j_3; k_3) = \mathcal{D}_3(a_{31}, a_{34}, a_{41}) \tag{4.44}$$

The restrictions on the sum in the case $n = 4$ can be understood as gluing the two triangles:

$$\begin{aligned} \mathcal{F}_4(j_1, i_2, j_2, i_3, j_3) &= \mathcal{D}_3(a_{12}, a_{23}, a_{31}) \cap \mathcal{D}_3(a_{34}, a_{41}, a_{31}) \\ &\equiv \mathcal{D}_4(a_{12}, a_{23}, a_{34}, a_{41}; a_{31}) \ . \end{aligned} \tag{4.45}$$

The allowed integers on a_{31} are naturally interpreted as permissible quantum numbers flowing through an intermediate channel. As one can imagine, eq. (4.45) is not the only way to represent the restriction: one can also represent it as

$$\begin{aligned} &\mathcal{F}_4(j_1, i_2, j_2, i_3, j_3) = \mathcal{D}_3(a_{12}, a_{24}, a_{41}) \cap \mathcal{D}_3(a_{23}, a_{34}, a_{24}) \\ \equiv\ &\mathcal{D}_4(a_{12}, a_{23}, a_{34}, a_{41}; a_{24}) \ , \end{aligned} \tag{4.46}$$

which embodies the crossing symmetric property of the amplitude.

The restrictions in the general case n are understood as attaching a triangle to the case $(n-1)$. To see this, define

$$\begin{matrix} p_\ell = j_\ell + k_\ell - i_\ell \ , & q_\ell = j_{\ell-1} + i_{\ell-1} - 1 \ , \\ a_{\ell,1} = q_\ell - 1 \ , & a_{\ell,\ell+1} = p_\ell - 1 \ , \end{matrix} \quad \text{for} \quad \ell = 1, 2, \cdots \mathrm{n} \ . \tag{4.47}$$

Using $1 \le i_{n-1} \le k_{n-1}, \quad 1 \le j_{n-1} \le m - k_{n-1}$, we derive

$$\begin{aligned} a_{n-1,n} + a_{n,1} - a_{n-1,1} &= 2(j_{n-1} - 1) \ge 0 \ , \\ a_{n-1,n} - a_{n,1} + a_{n-1,1} &= 2(k_{n-1} - i_{n-1}) \ge 0 \ , \\ -a_{n-1,n} + a_{n,1} + a_{n-1,1} &= 2(i_{n-1} - 1) \ge 0 \ , \\ a_{n-1,n} + a_{n,1} + a_{n-1,1} &= 2(j_{n-1} + k_{n-1} - 2) \le 2(m-2) \ . \end{aligned} \tag{4.48}$$

The restriction on i_{n-1} and j_{n-1} are, therefore, $\mathcal{D}_3(a_{n-1,n}, a_{n,1}, a_{n-1,1})$, which is what we wanted to see. All in all, we find

$$\begin{aligned} &\mathcal{F}_n(j_1, i_2, j_2, \cdots i_{n-1}, j_{n-1}) \\ =\ &\mathcal{D}_3(a_{n-1,n}, a_{n,1}, a_{n-1,1}) \cap \mathcal{F}_{n-1}(j_1, i_2, j_2, \cdots i_{n-2}, j_{n-2}) \\ =\ &\mathcal{D}_3(a_{n-1,n}, a_{n,1}, a_{n-1,1}) \cap \mathcal{D}_{n-1}(a_{1,2}, a_{2,3}, \cdots a_{n-2,n-1}, a_{n-1,1}; a_{3,1}, a_{4,1}, \cdots a_{n-2,1}) \\ \equiv\ &\mathcal{D}_n(a_{1,2}, a_{2,3}, \cdots a_{n-1,n}, a_{n,1}; a_{3,1}, a_{4,1}, \cdots a_{n-1,1}) \end{aligned} \tag{4.49}$$

From now on, a shortened notation $\mathcal{D}_n(a_{1,2}, a_{2,3}, \cdots a_{n-1,n}, a_{n,1})$ is understood to represent $\mathcal{D}_n(a_{1,2}, a_{2,3}, \cdots a_{n-1,n}, a_{n,1}; a_{3,1}, a_{4,1}, \cdots a_{n-1,1})$.

Putting eq (4.36) and eq. (4.49) together, we obtain a formula

$$
\begin{aligned}
& P_n(\theta_1, \theta_2, \cdots \theta_n) \\
= & -\left(\frac{\partial \eta^2}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)^n \sum_{\mathcal{D}_n}\left(\prod_{j=2}^{n} \frac{\sinh(m-k_j-1)\theta_j}{\sinh m\theta_j}\right)\frac{\sinh(m-k_1-1)\theta_1}{\sinh m\theta_1} \quad ,
\end{aligned} \tag{4.50}
$$

where $\mathcal{D}_n$ means $\mathcal{D}_n(k_1-1, \cdots, k_n-1)$. Once again, the fact that the different divisions of $\mathcal{D}_n$ into $n-2$ triangles are embodied by this single expression is precisely the statement of the old duality.

The object $P_n(\theta_1, \theta_2, \cdots \theta_n)$ is equipped with θ_j and k_j for $j = 1, 2, \cdots, n$ and any $\mathcal{D}_3(k_1-1, k_2-1, k_3-1)$ obeys the rule of the triangle specified above. It is, therefore, natural to visualize this as a vertex which connects n external legs corresponding to n loops. The vertex can be regarded as a dual graph of an n-gon that corresponds $\mathcal{D}_n(k_1-1, \cdots, k_n-1)$.

Due to the formula (3.30), we can obtain the inverse-Laplace image of eq. (4.50) with respect to ζ_i $(i = 1, \sim n)$ immediately:

$$
\begin{aligned}
\mathcal{L}^{-1}\left[P_n(\theta_1, \cdots, \theta_n)\right] &= -\Big(\frac{\partial \eta^2}{\partial \Lambda}\Big)\Big(\frac{-1}{\eta}\Big)^n\Big(\frac{M}{2}\Big)^n \sum_{\mathcal{D}_n}\Big[\prod_i \widetilde{K}_{1-\frac{k_i}{m}}(M\ell_i)\Big] \\
&= (-1)^n \frac{1}{m(m+1)}\Big(\frac{aM}{2}\Big)^{-2+(2-n)\frac{1}{m}}\Big(\frac{M}{2}\Big)^n \sum_{\mathcal{D}_n}\Big[\prod_i \widetilde{K}_{1-\frac{k_i}{m}}(M\ell_i)\Big] \quad .
\end{aligned} \tag{4.51}
$$

4.3 Four-loop correlators

In this subsection and in the next one, we show how to perform the inverse Laplace transformation of the resolvent correlators to get loop correlators in terms of loop lengths in the case of $n = 4$, 5. It is necessary to put

$$
R^{(n)}(\theta_1, \cdots, \theta_n) \equiv R^{(n)}(z_1^*, \cdots, z_n^*)|_{\Lambda_i=\Lambda} \tag{4.52}
$$

in a manageable form to the inverse Laplace transform. Let us recall that $P_n(\theta_1, \cdots, \theta_n)$ can be inverse Laplace transformed immediately. If $R^{(n)}(\theta_1, \cdots, \theta_n)$ is expressed as a polynomial of $P_j(\theta_1, \cdots, \theta_n)$ and their derivatives with respect to Λ, the inverse Laplace transform can be carried out immediately. In fact, this turns out be true for $n = 4$ and 5, which we will show explicitly in the following.

The important point we will use in the following is the fact that when one of the loops shrinks and the loop length goes to zero, the n-loop amplitude must become proportional to the derivative of the $(n-1)$-loop amplitude with respect to the cosmological constant. We represent this fact by

$$\left\langle w^+(\ell_1)\cdots w^+(\ell_n)\right\rangle \rightarrow \ \propto \frac{\partial}{\partial\mu}\left\langle w^+(\ell_1)\cdots w^+(\ell_{n-1})\right\rangle \quad , \tag{4.53}$$

when n-th loop shrinks. Then in the limit, $R^{(n)}(\theta_1,\cdots,\theta_n)$ must satisfy the following relation:

$$\mathcal{L}^{-1}\left[R^{(n)}(\theta_1,\cdots,\theta_n)\right] \rightarrow \ \propto \mathcal{L}^{-1}\left[\frac{\partial}{\partial\Lambda}R^{(n)}(\theta_1,\cdots,\theta_{n-1})\right] \tag{4.54}$$

This relation restricts the possible form of $R^{(n)}(\theta_1,\cdots,\theta_n)$. We should note here that the inverse Laplace image of $P_n(\theta_1,\cdots,\theta_n)$ satisfy the following relation

$$\mathcal{L}^{-1}\left[P_n(\theta_1,\cdots,\theta_n)\right] \rightarrow \ \propto \mathcal{L}^{-1}\left[P_{n-1}(\theta_1,\cdots,\theta_{n-1})\right] \quad . \tag{4.55}$$

This fact follows from substituting

$$\widetilde{K}_{1-\frac{k_n}{m}}(M\ell_n) \approx \frac{1}{\Gamma(\frac{k_n}{m})}\left(\frac{M\ell_n}{2}\right)^{\frac{k_n}{m}-1} \quad , \quad M\ell_n \ll 1 \tag{4.56}$$

in eq. (4.51) and picking up only the $k_n = 1$ parts.

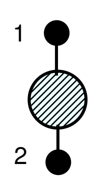
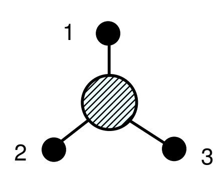
Because we want $R^{(n)}(\theta_1,\cdots,\theta_n)$ to be expressed as a polynomial of P_j and their derivatives with respect to Λ, we need here to introduce a notation

$$\begin{aligned} &\left[\mathcal{S}P_{1,\cdots,i_1}P_{j_2,\cdots,j_2+i_2-1}\cdots P_{n-i_\ell+1,\cdots,n}\right](\theta_1,\theta_2,\cdots\theta_n) \equiv \\ &\frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} P_{i_1}(\theta_{\sigma(1)},\cdots,\theta_{\sigma(i_1)})P_{i_2}(\theta_{\sigma(j_2)},\cdots,\theta_{\sigma(j_2+i_2-1)})\cdots P_{i_\ell}(\theta_{\sigma(n-i_\ell+1)},\cdots,\theta_{\sigma(n)}) \ , \end{aligned} \tag{4.57}$$

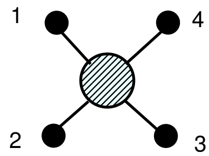
where $\mathcal{P}_n$ represents the permutations of $(1,2,\cdots,n)$. To be more specific, for example

$$\begin{aligned} \left[\mathcal{S}P_{123}P_{234}\right](\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{1}{4!}\sum_{\sigma\in\mathcal{P}_4} P_3(\theta_{\sigma(1)},\theta_{\sigma(2)},\theta_{\sigma(3)})P_3(\theta_{\sigma(2)},\theta_{\sigma(3)},\theta_{\sigma(4)}) \\ \left[\mathcal{S}P_{1234}P_{34}\right](\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{1}{4!}\sum_{\sigma\in\mathcal{P}_4} P_4(\theta_{\sigma(1)},\theta_{\sigma(2)},\theta_{\sigma(3)}\theta_{\sigma(4)})P_2(\theta_{\sigma(3)},\theta_{\sigma(4)}) \ . \end{aligned} \tag{4.58}$$

It is convenient to represent $P_n(\theta_1,\cdots,\theta_n)$ by an n-vertex which connects n external legs. For example for $n = 2$, 3 and 4

$P_2(\theta_1,\theta_2) \equiv$  , $P_3(\theta_1,\theta_2,\theta_3) \equiv$ 

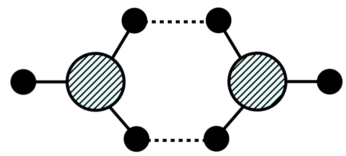
$$P_4(\theta_1, \theta_2, \theta_3, \theta_4) \equiv$$



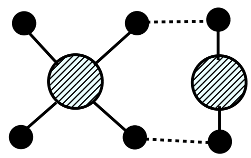
(4.59)

The n-vertex can be regarded as a dual graph of the n-gon (polygon) which corresponds to $\mathcal{D}_n$. In terms of these vertices, let us express eq. (4.58) as follows.

$$\left[\mathcal{S}P_{123}P_{234}\right](\theta_1, \theta_2, \theta_3, \theta_4) \quad \equiv$$

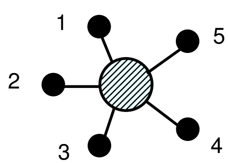


$$\left[\mathcal{S}P_{1234}P_{34}\right](\theta_1, \theta_2, \theta_3, \theta_4) \quad \equiv$$

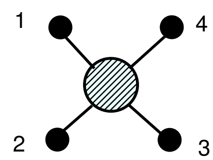


(4.60)

The relation eq. (4.55) can be represented, for example, as



$$\rightarrow \quad \propto$$



(4.61)

in the case of n=5.

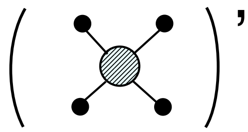
Now we are concerned with the case of n=4 first. Let us recall that for n=3

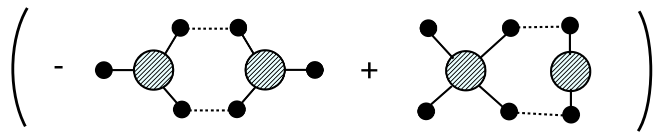
$$R^{(3)}(\theta_1, \theta_2, \theta_3) \propto P_3(\theta_1, \theta_2, \theta_3) \quad . \tag{4.62}$$

$R^{(4)}(\theta_1, \theta_2, \theta_3, \theta_4)$ must include a term which becomes proportional to $P_3(\theta_1, \theta_2, \theta_3)$ in the limit $M\ell_4 \rightarrow 0$, which is $P_4(\theta_1, \theta_2, \theta_3, \theta_4)$. $R^{(4)}(\theta_1, \theta_2, \theta_3, \theta_4)$ may also include terms which vanish in this limit. Such terms must consist of the product of two multi-vertices which have 6 external legs in total.

By explicit computation, we find

$$\frac{-1}{2!} R^{(4)}(\theta_1, \theta_2, \theta_3, \theta_4) \mid = \frac{\partial}{\partial \Lambda} P_4(\theta_1, \theta_2, \theta_3, \theta_4) - \left[\mathcal{S}P_{123}P_{234}\right](\theta_1, \theta_2, \theta_3, \theta_4)$$
$$+ \left[\mathcal{S}P_{1234}P_{34}\right](\theta_1, \theta_2, \theta_3, \theta_4)$$

$=$ 

$+$  , (4.63)

where the prime represents the differentiation with respect to Λ.

By performing the inverse-Laplace transformation and renormalizing, we obtain the complete answer for the macroscopic four loop correlator: [6]

$$\left\langle w^+(\ell_1)\cdots w^+(\ell_4)\right\rangle = \mathcal{A}_4^{fusion}(\ell_1,\cdots\ell_4) + \mathcal{A}_4^{residual}(\ell_1,\cdots\ell_4)\ \ , \tag{4.64}$$

where

$$\begin{aligned}&\mathcal{A}_4^{fusion}(\ell_1,\cdots,\ell_4)\\ &\quad = -\frac{1}{m(m+1)}\prod_{j=1}^{4}\ell_j\ \frac{\partial}{\partial\mu}\left[\Big(\frac{M}{2}\Big)^{2(1-\frac{1}{m})}\sum_{\mathcal{D}_4}\prod_{j=1}^{4}\widetilde{K}_{1-\frac{k_j}{m}}(M\ell_j)\right]\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}\mathcal{A}_4^{residual}(\ell_1\cdots\ell_4) = \Big[\frac{1}{m(m+1)}\Big]^2\prod_{j=1}^{4}\ell_j\ \Big(\frac{M}{2}\Big)^{2(1-\frac{1}{m})}&\\ \left(\frac{1}{4!}\sum_{\sigma\in\mathcal{P}_4}\right)\Bigg\{-\sum_{\mathcal{D}_3}\sum_{\mathcal{D}'_3}\prod_{j=1}^{4}\Big[B_j^{123}\odot B_j'^{234}\Big]\Big(M\ell_{\sigma(j)}\Big)&\\ +\sum_{\mathcal{D}_4}\sum_{\mathcal{D}'_2}\prod_{j=1}^{4}\Big[B_j^{1234}\odot B_j'^{34}\Big]\Big(M\ell_{\sigma(j)}\Big)\Bigg\}\ .&\end{aligned} \tag{4.66}$$

Here

$$B_j^{123} = (\widetilde{K}_{1-k_1/m},\widetilde{K}_{1-k_2/m},\widetilde{K}_{1-k_3/m},\delta), \tag{4.67}$$
$$B_j'^{234} = (\delta,\widetilde{K}_{1-k'_2/m},\widetilde{K}_{1-k'_3/m},\widetilde{K}_{1-k'_4/m}), \tag{4.68}$$
$$B_j^{1234} = (\widetilde{K}_{1-k_1/m},\widetilde{K}_{1-k_2/m},\widetilde{K}_{1-k_3/m},\widetilde{K}_{1-k_4/m}), \tag{4.69}$$
$$B_j'^{34} = (\delta,\delta,\widetilde{K}_{1-k'_3/m},\widetilde{K}_{1-k'_4/m})\ \ . \tag{4.70}$$

[6]This was briefly reported in [27].

We have introduced $\mathcal{D}_3 \equiv \mathcal{D}_3(k_1-1, k_2-1, k_3-1)$, $\mathcal{D}'_3 \equiv \mathcal{D}_3(k'_1-1, k'_2-1, k'_3-1)$, $\mathcal{D}_4 \equiv \mathcal{D}_4(k_1-1, k_2-1, k_3-1, k_4-1)$ and $\mathcal{D}'_2 \equiv \mathcal{D}_2(k'_1-1, k'_2-1)$ and have defined the convolution $[A \odot B](M\ell)$ with respect to ℓ by

$$[A \odot B]\,(M\ell') \equiv \int_0^{\ell} \mathrm{d}\ell' A(M\ell')B(M(\ell-\ell')) \quad , \tag{4.71}$$

in particular

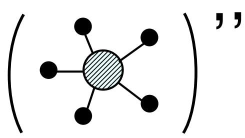
$$[A \odot \delta]\,(M\ell) \equiv \int_0^{\ell} \mathrm{d}\ell A(M\ell')\delta(\ell-\ell') = A(M\ell) \quad . \tag{4.72}$$

The important point to note is that $\mathcal{A}_4^{residual}$ seems to represent the contribution from loops with mixed momenta, that is, a single loop seems to have two distinct parts each of which have different momentum. If two of the loops ℓ_1 and ℓ_2 touch each other on two point, the two-dimensional surface break into two surfaces and the loops ℓ_1 and ℓ_2 also split into two pieces respectively. We infer that this configuration of two-dimensional surface may have connection with $\mathcal{A}_4^{residual}$. For $m = 2$ (pure gravity), $\mathcal{A}_4^{residual}$ vanishes. This fact is consistent with the above consideration because in pure gravity there is no loop with mixed boundary condition.

4.4 Five-loop correlators

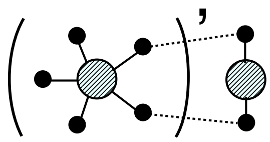
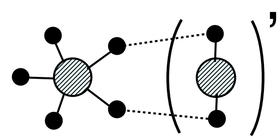
Let us now turn to the n=5 case. $R^{(5)}(\theta_1, \cdots, \theta_5)$ include a term which is proportional to

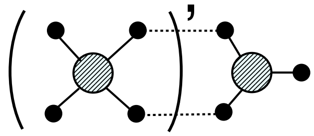
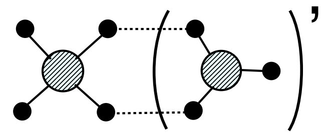
$$\frac{\partial^2}{\partial\Lambda^2} P_5(\theta_1, \cdots, \theta_5) = \tag{4.73}$$



corresponding to the first term in eq. (4.63). Corresponding to the second term in eq. (4.63), $R^{(5)}(\theta_1, \cdots, \theta_5)$ must include a term which consists of the product of two multi-vertices with 7 external legs in total. The possible form is

$$a\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{12345})P_{45}\Big] + b\Big[\mathcal{S}P_{12345}\frac{\partial}{\partial\Lambda}(P_{45})\Big]$$
$$+c\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{1234})P_{345}\Big] + d\Big[\mathcal{S}P_{1234}\frac{\partial}{\partial\Lambda}(P_{345})\Big]$$

$= a$  $+ b$ 

$+c$  $+\, d$  . (4.74)

In the limit $M\ell_5 \to 0$, it becomes

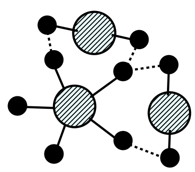
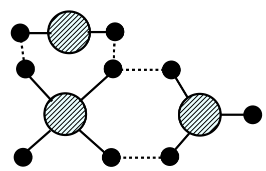
$$(3a+c)\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{1234})P_{34}\Big] + (3b+d)\Big[\mathcal{S}P_{1234}\frac{\partial}{\partial\Lambda}(P_{34})\Big]$$
$$+2c\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{123})P_{234}\Big] + 2d\Big[\mathcal{S}P_{123}\frac{\partial}{\partial\Lambda}(P_{234})\Big] \quad . \tag{4.75}$$

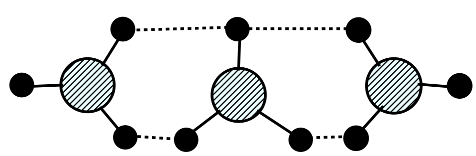
We require this expression to be proportional to the Λ-derivative of the second term in eq .(4.63). We find

$$a = (1-c)/3, \quad b = (2+c)/3, \quad d = -1-c \quad . \tag{4.76}$$

$R^{(5)}(\theta_1, \cdots, \theta_5)$ may include terms which vanish in the limit under consideration as well. They must consist of the products of three multi-vertices with 9 external legs in total. As one of the such terms we have

$$[\mathcal{S}P_{1234}P_{34}P_{45}] - 2[\mathcal{S}P_{1234}P_{23}P_{345}] + [\mathcal{S}P_{123}P_{124}P_{235}]$$

$=$  $-\, 2$ 

$+$  . (4.77)

(There are some other combinations which satisfy the conditions.) $R^{(5)}(\theta_1, \cdots, \theta_5)$ must be expressed as a linear combination of the above three types of terms eq. (4.73), eq. (4.74) and eq. (4.77) if the assumption under consideration is true. By explicit calculation, we have found in fact that $R^{(5)}(\theta_1, \cdots, \theta_5)$ can be expressed as a linear combination of eq. (4.73), eq. (4.74) and eq. (4.77) :

$$R^{(5)}(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) = \left(\frac{\partial}{\partial\Lambda}\right)^2 P_5(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$$
$$+ \quad \left[\mathcal{S}P_{12345}\left(2\frac{\overrightarrow{\partial}}{\partial\Lambda} + 3\frac{\overleftarrow{\partial}}{\partial\Lambda}\right)P_{45}\right](\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$$

$$
\begin{aligned}
- & \left[\mathcal{S} P_{1234} \left(\frac{\overrightarrow{\partial}}{\partial \Lambda} + 4 \frac{\overleftarrow{\partial}}{\partial \Lambda} \right) P_{345} \right] (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \\
+ & \left[(2 P_{1234} P_{34} P_{45} - 4 P_{123} P_{23} P_{345} + 2 P_{123} P_{124} P_{235}) \right] (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \ . \quad (4.78)
\end{aligned}
$$

Following the same procedure as obtaining eq. (4.64), we find the complete answer for the five loop amplitude:

$$
\begin{aligned}
\Big\langle w^+(\ell_1) \cdots w^+(\ell_5) \Big\rangle &= \mathcal{A}_5^{fusion}(\ell_1 \cdots \ell_5) + \mathcal{A}_5^{residual-1}(\ell_1 \cdots \ell_5) \\
&+ \mathcal{A}_5^{residual-2}(\ell_1 \cdots \ell_5) \ , \qquad (4.79)
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathcal{A}_5^{fusion}(\ell_1, \cdots, \ell_5) \\
& \quad = -\frac{1}{m(m+1)} \prod_{j=1}^{5} \ell_j \left(\frac{\partial}{\partial \mu} \right)^2 \left[\left(\frac{M}{2} \right)^{3(1-\frac{1}{m})} \sum_{\mathcal{D}_5} \prod_{j=1}^{5} \widetilde{K}_{1-\frac{k_j}{m}}(M \ell_j) \right] \ , \qquad (4.80)
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{A}_5^{residual-1}(\ell_1 \cdots \ell_5) \\
& = \Big[\frac{1}{m(m+1)} \Big]^2 \prod_{j=1}^{5} \ell_j \left(\frac{1}{5!} \sum_{\sigma \in \mathcal{P}_5} \right) \\
& \times \Bigg\{ \Bigg\{ \left(\frac{\partial}{\partial \mu} \Big(\frac{M}{2} \Big)^{3(1-\frac{1}{m})} \right)_R + 3 \left(\frac{\partial}{\partial \mu} \right)_L \Big(\frac{M}{2} \Big)^{3(1-\frac{1}{m})} \Bigg\} \\
& \qquad \sum_{\mathcal{D}_5} \sum_{\mathcal{D}'_2} \prod_{j=1}^{5} \left[B_j^{12345} \odot B_j'^{45} \right] (M \ell_{\sigma(j)}) \\
& - \Bigg\{ \Big(\frac{M}{2} \Big)^{2(1-\frac{1}{m})} \left(\frac{\partial}{\partial \mu} \Big(\frac{M}{2} \Big)^{1-\frac{1}{m}} \right)_R + 4 \left(\frac{\partial}{\partial \mu} \Big(\frac{M}{2} \Big)^{2(1-\frac{1}{m})} \right)_L \Big(\frac{M}{2} \Big)^{1-\frac{1}{m}} \Bigg\} \\
& \qquad \sum_{\mathcal{D}_4} \sum_{\mathcal{D}'_3} \prod_{j=1}^{5} \left[B_j^{1234} \odot B_j'^{345} \right] (M \ell_{\sigma(j)}) \Bigg\} \qquad (4.81)
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{A}_5^{residual-2}(\ell_1 \cdots \ell_5) \\
& = \Big[\frac{1}{m(m+1)} \Big]^3 \prod_{j=1}^{5} \ell_j \left(\frac{1}{5!} \sum_{\sigma \in \mathcal{P}_5} \right) \Big(\frac{M}{2} \Big)^{3(1-\frac{1}{m})} \\
& \times \Bigg\{ -2 \sum_{\mathcal{D}_5} \sum_{\mathcal{D}'_2} \sum_{\mathcal{D}''_2} \prod_{j=1}^{5} \left[B_j^{12345} \odot B_j'^{34} \odot B_j''^{45} \right] (M \ell_{\sigma(j)}) \\
& \quad + 4 \sum_{\mathcal{D}_4} \sum_{\mathcal{D}'_2} \sum_{\mathcal{D}''_3} \prod_{j=1}^{5} \left[B_j^{1234} \odot B_j'^{23} \odot B_j''^{345} \right] (M \ell_{\sigma(j)}) \\
& \quad - 2 \sum_{\mathcal{D}_3} \sum_{\mathcal{D}'_3} \sum_{\mathcal{D}''_3} \prod_{j=1}^{5} \left[B_j^{123} \odot B_j'^{124} \odot B_j''^{235} \right] (M \ell_{\sigma(j)}) \Bigg\} \ . \qquad (4.82)
\end{aligned}
$$

Here $\left(\frac{\partial}{\partial\mu}\right)_{L,(R)}$ means the derivative acting only on the left(right) part of the convolutions. The rest of the notations here are similar to those of the $n=4$ case and will be self-explanatory.

We conjecture that $R^{(n)}$ can be represented as a polynomial of $P_j,\ j \le n$ and their μ derivatives: the final answer would then be obtained by convolutions of various B_j's and their derivatives. If the conjecture is true in fact, the power counting argument tells us that the j-th term of $R^{(n)}$ turns out to be represented by a figure which consists of the products of j multi-vertices with $n+2j$ external legs in total. We hope that, for higher loops, $R^{(n)}(\theta_1,\cdots,\theta_n)$ can be put in principle in a form such as eqs. (4.64), (4.79) in the same manner as we have determined the four and five loops from the lower ones. From eqs. (4.64) $\sim$ (4.65) and eqs. (4.79) $\sim$ (4.80), we guess that the multi-loop correlators, in general, would include the following term corresponding $P_n(\theta_1,\cdots,\theta_n)$:

$$\begin{aligned}&\mathcal{A}_n^{fusion}(\ell_1,\cdots,\ell_n)\\&\quad=-\frac{1}{m(m+1)}\prod_{j=1}^{n}\ell_j\left(\frac{\partial}{\partial\mu}\right)^{n-3}\left[\left(\frac{M}{2}\right)^{-(n-2)(1-\frac{1}{m})}\sum_{\mathcal{D}_n}\prod_{j=1}^{n}\widetilde{K}_{1-\frac{k_j}{m}}(M\ell_j)\right]\ .\end{aligned} \tag{4.83}$$

4.5 Four-point functions from loop correlators

The four-loop correlators we found in eqs. (4.64) $\sim$ (4.66) do not diverge when the loop lengths approach zero, so that we expect that the loop operator can be replaced by the sum of the local operators in this case. Let us derive the four-point functions of the scaling operators, applying the expansion of loop to the four loop correlators.

First, consider the part $\mathcal{A}_4^{fusion}(\ell_1,\ell_2,\ell_3,\ell_4)$ and expand this in terms of the modified Bessel functions $I_\nu(\ell_i)$. From eq. (4.65) we have

$$\begin{aligned}&\mathcal{A}_4^{fusion}(\ell_1,\ell_2,\ell_3,\ell_4)\\&\quad=\frac{1}{m^2(m+1)}\left(\frac{M}{2}\right)^{-4-\frac{2}{m}}\sum_{\mathcal{D}_4}\prod_{j=1}^{4}\frac{M\ell_i}{2}\ \widetilde{K}_{1-\frac{k_j}{m}}(M\ell_j)\\&\qquad-\frac{2}{m(m+1)^2}\left(\frac{M}{2}\right)^{-2-\frac{2}{m}}\frac{1}{M}\frac{\partial}{\partial M}\left\{\sum_{\mathcal{D}_4}\prod_{j=1}^{4}\frac{M\ell_i}{2}\ \widetilde{K}_{1-\frac{k_j}{m}}(M\ell_j)\right\}\ .\end{aligned} \tag{4.84}$$

Since we can prove the relations

$$\frac{z}{2}\widetilde{K}_{1-p}(z)=\sum_{n=-\infty}^{\infty}(p+2n)\ I_{|p+2n|}(z)\ , \tag{4.85}$$

and

$$z\frac{\partial}{\partial z}\left\{\frac{z}{2}\widetilde{K}_{1-p}(z)\right\} = \sum_{n=-\infty}^{\infty}(p+2n)\Big(p+2n(p+n)\Big)\ I_{|p+2n|}(z)\ , \tag{4.86}$$

for $0<p<1$, we obtain the following expression for $\mathcal{A}_4^{fusion}$:

$$\begin{aligned}\mathcal{A}_4^{fusion}(\ell_1,\ell_2,\ell_3,\ell_4) &= \frac{1}{m(m+1)^2}\left(\frac{M}{2}\right)^{-4-\frac{1}{m}}\\ &\times\sum_{\mathcal{D}_4}\prod_{j=1}^{4}\left(\sum_{n_j=-\infty}^{\infty}\right)\left\{(1+\frac{1}{m})-\frac{1}{2}\sum_{j=1}^{4}\left[\frac{k_j}{m}+2n_j(\frac{k_j}{m}+n_j)\right]\right\}\\ &\times\prod_{j=1}^{4}\left\{(\frac{k_j}{m}+2n_j)\ I_{|\frac{k_j}{m}+2n_j|}(M\ell_j)\right\}\ .\end{aligned} \tag{4.87}$$

Comparing eq. (4.87) to the expansion of loop operators (3.65), we obtain the following contribution of the four-point functions from $\mathcal{A}_4^{fusion}(\ell_1,\cdots,\ell_4)$:

$$\begin{aligned}\left\langle\prod_{j=1}^{4}\widehat{\sigma}_{|k_i+2mn_i|}\right\rangle^{fusion} &= C_{k_1k_2k_3k_4}\frac{1}{2m^2(m+1)^2}\left(\frac{M}{2}\right)^{-4-\frac{1}{m}+\sum_{i=1}^{4}|\frac{k_i}{m}+2n_i|}\\ &\times\ \left\{2(m+1)-\sum_{j=1}^{4}\Big[k_j+2n_j(k_j+n_jm)\Big]\right\}\\ &\times\ \prod_{j=1}^{4}(k_j+2n_jm)\ ,\end{aligned} \tag{4.88}$$

where

$$C_{k_1k_2k_3k_4} = \sum_{k'=1}^{m-1}C_{k_1k_2k'}C_{k'k_3k_4}\ . \tag{4.89}$$

Next, let us consider $\mathcal{A}_4^{residual}(\ell_1,\cdots,\ell_4)$ part. We can prove that the convolution of two modified Bessel functions $\widetilde{K}_p(M\ell)$ and $\widetilde{K}_{p'}(M\ell)$ is expanded in terms of $I_\nu(M\ell)$ as

$$\begin{aligned}\ell\ \Big[\widetilde{K}_p\odot\widetilde{K}_{p'}\Big](M\ell) &= \left(\frac{2}{M}\right)^2\sum_{n=-\infty}^{\infty}\Big\{n(2n+p+p')\ I_{|2n+p+p'|}(M\ell)\\ &\qquad -n(2n+p-p')\ I_{|2n+p-p'|}(M\ell)\Big\}\ .\end{aligned} \tag{4.90}$$

The above relation is easily derived in the space of Laplace transformed coordinate using the relations

$$\mathcal{L}[\ I_p(M\ell)] = \frac{1}{M}\frac{e^{-pm\theta}}{\sinh m\theta},\ \ \text{for } p>-1\ , \tag{4.91}$$

and

$$\begin{aligned}\mathcal{L}\left[\,\widetilde{K}_p(M\ell)\right] &= \mathcal{L}\left[\, I_{-p}(M\ell) - I_p(M\ell)\right] \\ &= \frac{2}{M}\frac{\sinh pm\theta}{\sinh m\theta}, \quad \text{for } p > -1 \;, \end{aligned} \tag{4.92}$$

because the convolution in ℓ-space corresponds to the product in ζ-space.

From eqs. (4.85) and (4.90) we obtain the following expression for the residual part of the four loop correlator:

$$\begin{aligned}\mathcal{A}_4^{residual}(\ell_1,\cdots,\ell_4) &= \frac{1}{m^2(m+1)^2}\left(\frac{M}{2}\right)^{-4-\frac{2}{m}}\frac{1}{4!}\sum_{\sigma\in\mathcal{P}_4} \\ &\times\prod_{j=1}^{4}\left\{\sum_{k_j}\sum_{n_j=-\infty}^{\infty}\right\}\sum_{k_1'}\sum_{k_2'}(-C_{k_1k_2k_3}C_{k_1'k_2'k_4}+C_{k_1k_2k_3k_4}C_{k_1'k_2'}) \\ &\times\sum_{+-}(\pm)n_1\Big(2n_1+(1-\frac{k_1}{m})\pm(1-\frac{k_1'}{m})\Big)\; I_{|(2n_1+(1-\frac{k_1}{m})\pm(1-\frac{k_1'}{m})|}(M\ell_{\sigma(1)}) \\ &\times\sum_{+-}(\pm)n_2\Big(2n_2+(1-\frac{k_2}{m})\pm(1-\frac{k_2'}{m})\Big)\; I_{|(2n_2+(1-\frac{k_2}{m})\pm(1-\frac{k_2'}{m})|}(M\ell_{\sigma(2)}) \\ &\times(2n_3+\frac{k_3}{m})\; I_{|(2n_3+\frac{k_3}{m})|}(M\ell_{\sigma(3)})\;(2n_4+\frac{k_4}{m})\; I_{|(2n_4+\frac{k_4}{m})|}(M\ell_{\sigma(4)})\;,\end{aligned} \tag{4.93}$$

where $C_{kk'} = \delta_{kk'}$.

From the above expression and the expansion of the loop operator (3.65), we can obtain the contribution of the four-point functions from $\mathcal{A}_4^{residual}$ part. The explicit expression, however, would be complicated.

Let us comment on the role of the coefficients $(-C_{k_1k_2k_3}C_{k_1'k_2'k_4}+C_{k_1k_2k_3k_4}C_{k_1'k_2'})$ in eq. (4.93). At first sight, it appears that we would have $I_\nu(M\ell_i)$ with integer order in eq. (4.93). For example, in the case of $k_1 = k_1'$ or $k_1 + k_1' = m$, we have $I_\nu(M\ell_1)$ with integer order. Terms including $I_\nu(M\ell_i)$ with integer order as a factor cannot be explained from the viewpoint of the local operators. These terms, however, are cancelled due to the coefficients $(-C_{k_1k_2k_3}C_{k_1'k_2'k_4}+C_{k_1k_2k_3k_4}C_{k_1'k_2'})$, so that we do not have $I_\nu(M\ell_i)$ with integer order in eq. (4.93) after all.

5 Summary and discussion

In this paper we have investigated the correlators of macroscopic loops and those of local operators in the unitary minimal models coupled to two-dimensional gravity using the two-matrix model. We calculated the general multi-resolvent correlators, and examined one- to five-loop correlators explicitly.

From these loop correlators we obtained the correlator of the scaling operators by applying the idea [30] that the macroscopic loop can be replaced by a sum of local operators, to the case of the two-matrix model. We found that there exist the fusion rules for the three-loop correlators, which are similar to those for the three-point functions of the gravitational primaries. From the three-loop correlators, we deduced the three-point functions of the scaling operators, and found the gravitational descendants as well as the gravitational primaries satisfy the fusion rules of the same kind. These fusion rules for the loops can be considered to express those for all of the scaling operators in a compact form.

At the $(m+1, m)$ critical point in two-matrix models, the scaling operators $\widehat{\sigma}_j,\ j = 0 \pmod{\mathrm{m}+1}$ have no counterparts in the BRST cohomology of Liouville theory coupled to the corresponding conformal matter. In [33],these operators were argued to be boundary operators which couple to loops in the case of the one-matrix model. It was also shown explicitly that one of them, corresponding to $\widehat{\sigma}_{m+1}$ in the case of the unitary matter, is a operator which measures the total length of the loops.

We examined the role of the rest of these operators. We showed, in some examples, that the operator $\mathcal{B}_n$ couples to the points to which n parts of several loops are stuck each other. In other words, the operator $\mathcal{B}_n$ connects n parts of loops together. We think these operators play an important role concerning the touching of the macroscopic loops. The emergence of these operators in matrix models can easily be understood from the viewpoint of macroscopic loops and their expansion in terms of local operators.

In sect. 4 we examined the property of the multi-loop correlators. We pointed out that the structure similar to those of the crossing symmetry in the underlying conformal field theory can be seen in the loop correlators. This structure appears in the selection rules for the summations in the expression of loop correlators.

We calculated explicitly four- and five-loop correlators. From the expression of these correlators, we inferred that these include the contribution from the loops with boundary condition specified by more than one momentum. We guess this property can be understood as follows. When two loops touch each other on two points, each loop breaks into two pieces. Since a single loop breaks into two pieces, the broken pieces can have distinct momenta. The configuration probably have non-vanishing contribution to the amplitude in the case of the four- and five-loop amplitudes. Note that matter degrees of freedom are fixed only inside the loops when we calculate the loop correlators. Each loop therefore represents a superposition of loops with various momenta. Distinct lattice elements on a single loop can have distinct momenta.

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Appendix A

Here we collect some formulae concerning the modified Bessel functions. $I_\nu(z)$ and $K_\nu(z)$ are linearly independent solutions of the Bessel equation

$$\left[\left(z\frac{\partial}{\partial z}\right)^2 - z^2 - \nu^2\right] Z_\nu(z) = 0 \ . \tag{A.1}$$

$I_\mu(z)$ can be expanded as

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{1}{k!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n} , \tag{A.2}$$

and $K_\nu(z)$ is defined as

$$K_\nu(z) = \frac{\pi/2}{\sin \nu\pi}\left[I_{-\nu}(z) - I_\nu(z)\right] \ . \tag{A.3}$$

We collect another useful formulae in the following.

$$-\frac{2\nu}{z}K_\nu(z) = K_{\nu-1}(z) - K_{\nu+1}(z) \tag{A.4}$$

$$\left(\frac{d}{zdz}\right)^n [z^\nu K_\nu(z)] = (-1)^n z^{\nu-n} K_{\nu-n}(z) \tag{A.5}$$

$$K_\nu(z)K_\nu(\zeta) = \frac{1}{2}\int_0^\infty \frac{dt}{t} K_\nu\left(\frac{z\zeta}{t}\right) \exp\left(-\frac{t}{2} - \frac{z^2+\zeta^2}{2t}\right)$$
$$[Re\ z > 0, \quad Re\ \zeta > 0, \quad |\arg(z+\zeta)| < \pi/4] \tag{A.6}$$

Introducing the Laplace transformation of a function $f(\ell)$ of loop length ℓ by

$$\mathcal{L}\Big[f(\ell)\Big] = \int_0^\infty \mathrm{d}l\ e^{-\zeta\ell} f(x) , \tag{A.7}$$

we have the following relations on the Laplace transformations of the Bessel functions:

$$\mathcal{L}\Big[\ I_p(M\ell)\Big] = \frac{1}{M}\frac{e^{-pm\theta}}{\sinh m\theta}, \quad p > -1, \tag{A.8}$$

$$\mathcal{L}\Big[\widetilde{K}_p(M\ell)\Big] = \frac{2}{M}\frac{\sinh |p| m\theta}{\sinh m\theta} , \tag{A.9}$$

$$\mathcal{L}\Big[-\ell^{-1}|p|\widetilde{K}_p(M\ell)\Big] = 2\cosh pm\theta, \tag{A.10}$$

where ζ is parametrized as

$$\zeta = M\cosh m\theta , \tag{A.11}$$

and we introduced a notation

$$\widetilde{K}_p(M\ell) = \frac{\sin \pi|p|}{\pi/2} K_p(M\ell) \ . \tag{A.12}$$

Appendix B

In this appendix, we prove the recursion relations for

$$\begin{pmatrix} m_1, & m_2, & m_3, & \cdots \\ i_1, & i_2, & i_3, & \cdots \end{pmatrix}_n \qquad \text{with} \qquad \sum_\ell m_\ell \le n-2 \tag{B.1}$$

introduced in the text. The proof goes by mathematical inductions.

We will first prove the simplest case

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ i_1, & i_2 & \cdots, & i_n \end{pmatrix}_n = \frac{1}{\prod\limits_{j(\neq i_1)}^{n} [i_1 - j]^2} \quad , \qquad \text{for} \quad m = n-2 \tag{B.2}$$

and

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ i_1, & i_2 & \cdots, & i_n \end{pmatrix}_n = 0 \quad , \qquad \text{for} \quad m \le n-3 \quad . \tag{B.3}$$

Assume that eq. (B.2) and eq. (B.3) are true at n. Without loss of generality, i_1 can be taken to be 1. Let us consider the left hand side of eq. (B.2) or eq. (B.3) in which n is replaced by $n+1$. To compute them we observe that the elements of $\mathcal{S}_{n+1}$ are generated by associating n different ways of inserting $[n+1]$ with each element $\sigma \in \mathcal{S}_n$. In the case where $[n+1]$ is inserted in between $[1]$ and $[\sigma(1)]$, this contribution is equal to

$$-\sum_{\sigma\in\mathcal{S}_n} \frac{1}{[1-\sigma(1)]^m} \frac{[1-\sigma(1)]^m}{[1-(n+1)]^{m+1}[(n+1)-\sigma(1)]} \prod_{j=2}^{n} \frac{1}{[j-\sigma(j)]} \quad . \tag{B.4}$$

The contributions from the sum of the remaining $n-1$ insertions are found to be equal to

$$-\sum_{\sigma\in\mathcal{S}_n} \frac{1}{[1-\sigma(1)]^m} \frac{1}{[\sigma(1)-(n+1)][1-(n+1)]} \prod_{j=2}^{n} \frac{1}{[j-\sigma(j)]} \quad . \tag{B.5}$$

Here we have used

$$\frac{1}{[j-(n+1)][(n+1)-m]} = \frac{1}{[j-m]} \left(\frac{1}{[j-(n+1)]} - \frac{1}{[m-(n+1)]} \right) \quad . \tag{B.6}$$

Note also that $\prod\limits_{j=1}^{n-1} 1/[\sigma^j(1) - \sigma^{j+1}(1)] = \prod\limits_{j=2}^{n} 1/[j-\sigma(j)]$. Putting eqs. (B.4) and (B.5) together, we find

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ 1, & 2, & \cdots, & n+1 \end{pmatrix}_{n+1} =$$

$$
- \sum_{\sigma \in \mathcal{S}_n} \frac{1}{[1-\sigma(1)]^m} \frac{\left\{1 - \left(\frac{[1-\sigma(1)]}{[1-(n+1)]}\right)^m\right\}}{[\sigma(1)-(n+1)][1-(n+1)]} \prod_{j=2}^{n} \frac{1}{[j-\sigma(j)]} \quad . \tag{B.7}
$$

Factorizing the expression inside the bracket $\{\ \cdots\ \}$, we have

$$
\begin{pmatrix} m, & 0, & \cdots, & 0 \\ 1, & 2, & \cdots, & n+1 \end{pmatrix}_{n+1} = \sum_{l=1}^{m-1} \begin{pmatrix} m-l, & 0, & \cdots, & 0 \\ 1, & 2, & \cdots, & n \end{pmatrix}_n \frac{1}{[1-(n+1)]^{1+l}} \quad . \tag{B.8}
$$

Then from the assumption, eq. (B.2) and eq. (B.3) are also satisfied when n is replaced by $n+1$. On the other hand for $n=3$ eq. (B.2) and eq. (B.3) are clearly true, so we have proven the relations.

Now we turn to the more general case the proof of which is a straightforward generalization of the one given above. To derive the relations

$$
\begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & \cdots & i_k, & i_{k+1}, & \cdots, & i_n \end{pmatrix}_n = 0 \quad , \qquad \text{for} \quad \sum_{\ell=1}^{k} m_\ell \le n-3 \quad , \tag{B.9}
$$

and

$$
\begin{aligned}
&\begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & \cdots & i_k, & i_{k+1}, & \cdots, & i_n \end{pmatrix}_n \\
&\quad = \sum_{j=1}^{k} \begin{pmatrix} m_1, & \cdots, & m_j - 1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & \cdots, & i_j, & \cdots, & i_k, & i_{k+1}, & \cdots, & i_{n-1} \end{pmatrix}_{n-1} \frac{1}{[j-n]^2} \quad , \\
&\qquad\qquad \text{for} \quad \sum_{\ell=1}^{k} m_\ell = n-2 \quad .
\end{aligned} \tag{B.10}
$$

Let us assume eq. (B.9) at n .

We take $i_\ell = \ell, \quad \ell = 1 \sim k$ without loss of generality. The way in which the elements of $\mathcal{S}_{n+1}$ are generated is the same as the one given above. In the case where $[n+1]$ is inserted in between $[\ell]$ and $[\sigma(\ell)]$ $\ell = 1 \sim k$, the contribution is

$$
\begin{aligned}
&- \sum_{\sigma \in \mathcal{S}_n} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[\ell-\sigma(\ell)]^{m_\ell}} \frac{[\ell-\sigma(\ell)]^{m_\ell}}{[\ell-(n+1)]^{m_\ell+1}[(n+1)-\sigma(\ell)]} \\
&\quad \times \frac{1}{[(\ell+1)-\sigma(\ell+1)]^{m_{\ell+1}+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_k+1}} \prod_{j(\neq 1,2,\cdots k)}^{n-1} \frac{1}{[j-\sigma(j)]} \quad .
\end{aligned} \tag{B.11}
$$

The contributions from the sum of the remaining insertions are

$$
\begin{aligned}
&- \sum_{\sigma \in \mathcal{S}_n} \sum_{\ell(\neq p_2,\cdots p_k)}^{n-1} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_{k+1}}} \\
&\times \prod_{j(\neq p_2,\cdots p_k)}^{n-1} \frac{1}{[\sigma^j(1)-\sigma^{j+1}(1)]} \left(\frac{1}{[\sigma^\ell(1)-(n+1)]} - \frac{1}{[\sigma^{\ell+1}(1)-(n+1)]} \right) \quad .
\end{aligned} \tag{B.12}
$$

Here p_ℓ $\ell = 1 \sim k$ are such that $\sigma^{p_\ell}(1) = \ell$. Using eq. (B.6) again, we find that this equals

$$- \sum_{\sigma\in\mathcal{S}_n} \sum_{\ell=1}^{k} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[\ell-\sigma(\ell)]^{m_\ell}} \frac{1}{[\sigma(\ell)-(n+1)][\ell-(n+1)]}$$
$$\frac{1}{[(\ell+1)-\sigma(\ell+1)]^{m_{\ell+1}+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_k+1}} \prod_{j(\neq 1,2,\cdots k)}^{n-1} \frac{1}{[j-\sigma[j]]} \quad . \tag{B.13}$$

Putting eqs. (B.11) and (B.13) together, we find

$$\begin{aligned} &\begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ 1, & \cdots & k, & k+1, & \cdots, & n+1 \end{pmatrix}_{n+1} \\ = \quad & -\sum_{\ell}^{k} \sum_{\sigma\in\mathcal{S}_n} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[\ell-\sigma(\ell)]^{m_\ell}} \frac{\left\{1-\left(\frac{[\ell-\sigma(\ell)]}{[\ell-(n+1)]}\right)^{m_\ell}\right\}}{[\sigma(\ell)-n][\ell-n]} \\ \times \quad & \frac{1}{[(\ell+1)-\sigma(\ell+1)]^{m_{\ell+1}+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_k+1}} \prod_{j(\neq 1,2,\cdots k)}^{n-1} \frac{1}{[j-\sigma(j)]} \, . \end{aligned} \tag{B.14}$$

Factorizing the expression inside the bracket, we have

$$\begin{aligned} &\begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ 1, & \cdots & k, & k+1 & \cdots, & n+1 \end{pmatrix}_{n+1} \\ &\quad = \sum_{j=1}^{k} \sum_{l=1}^{m_j} \begin{pmatrix} m_1, & \cdots, & m_j - l, & \cdots, & m_k, & 0, & \cdots, & 0 \\ 1, & \cdots & j, & \cdots, & k, & k+1 & \cdots, & n \end{pmatrix}_n \frac{1}{[j-n]^{1+l}} \quad . \end{aligned} \tag{B.15}$$

Then from the assumption eq. (B.9) at n, eq. (B.9) in which n is replaced by $n+1$ is also satisfied. On the other hand for $n = 3$ eq. (B.9) is clearly satisfied, so we have proven eq. (B.9) and eq. (B.10) .

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