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<th>SK1 (Z [G]) of finite solvable groups which act linearly and freely on spheres</th>
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1. Introduction

Let $G$ be a finite group, $\mathbb{Z}$ the ring of integers and $\mathbb{Q}$ the ring of rational numbers. For $R = \mathbb{Z}$ or $\mathbb{Q}$, $R[G]$ denotes the group ring of $G$ over $R$. Put $GL(R[G]) = \lim_{\rightarrow} GL_n(R[G])$ and $E(R[G]) = [GL(R[G]), GL(R[G])]$ the commutator subgroup of $GL(R[G])$. Then $K_1(R[G])$ denotes the quotient group $GL(R[G])/E(R[G])$. The natural inclusion map $i : GL(\mathbb{Z}[G]) \rightarrow GL(\mathbb{Q}[G])$ gives rise to a group homomorphism $i_* : K_1(\mathbb{Z}[G]) \rightarrow K_1(\mathbb{Q}[G])$. Then $SK_1(\mathbb{Z}[G])$ is defined by setting

$$SK_1(\mathbb{Z}[G]) = \ker i_*.$$

In [9], C. T. C. Wall showed that $SK_1(\mathbb{Z}[G])$ is isomorphic to the torsion subgroup of the Whitehead group $Wh(G)$ of $G$. Since it can be shown that

$$SK_1(\mathbb{Z}[G]) = \ker(\text{Res} : Wh(G) \rightarrow \oplus Wh(C)),$$

for $C \in c$.
$SK_1(Z[G])$ gives information which cannot be obtained by
restricting $\text{Wh}(G)$ to $\oplus_{C \in c} \text{Wh}(C)$, where $c$ is the class of all
cyclic subgroups of $G$.

Incidentally, Whitehead group plays a role not only in
studying simple homotopy equivalences of finite CW complexes,
but also in classifying manifolds. The $s$-cobordism theorem says
that if $M$ and $N$ are smooth closed $n$-dimensional manifolds,
where $n \geq 5$, and if $W$ is a compact $(n+1)$-dimensional manifold
such that $\partial W = M \sqcup N$, and such that the inclusions $M \to W$
and $N \to W$ are simple homotopy equivalences, then $W$ is
diffeomorphic to $M \times [0,1]$ (see [5]).

For a finite group $G$, $SK_1(Z[G])$ has been calculated by
several authors. Let $\mathbb{Z}_m$ be a cyclic group of order $m$.
At first, it was shown by Bass, Milnor, and Serre ([1]) that
$SK_1(Z[G]) = 0$ if $G$ is cyclic or if $G \cong (\mathbb{Z}_2)^n$ for some $n$.
Also, it was shown by T. Y. Lam ([3]) that $SK_1(Z[G]) = 0$ if $G
\cong \mathbb{Z}_p^n \times \mathbb{Z}_p$ for any prime $p$ and any $n$. Later, it was shown
by R. Oliver ([8]) that for a finite abelian group $G$, $SK_1(Z[G])$
$= 0$ if and only if either $G \cong (\mathbb{Z}_2)^n$, or each Sylow subgroup of $G$
has the form $\mathbb{Z}_p^n$ or $\mathbb{Z}_p^n \times \mathbb{Z}_p$. As far as non-abelian groups
are concerned, it was shown in [2], [4], [6] and [7] that
$SK_1(Z[G])$ vanishes if $G$ is a dihedral group.

The purpose of this paper is to determine $SK_1(Z[G])$ for
finite solvable groups $G$ which act linearly and freely on
spheres. As in [10, Theorem 6.1.11], there are 4 types for such
kinds of groups. For the convenience of the reader, the table of

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these groups are cited in Appendix. In order to state our main theorem, we prepare the following notations.

Let $G_1$, $G_2$, $G_3$ and $G_4$ denote the groups of type I, II, III and IV respectively mentioned in the table in Appendix. Let $(a_1, a_2, \ldots, a_\lambda)$ denote the greatest common divisor of integers $(a_1, a_2, \ldots, a_\lambda)$, and let $m, n, r, l, k, u, v$ and $d$ be the integers appeared in the definition of $G_1$, $G_2$, $G_3$ and $G_4$. For positive integers $\alpha, \beta, \gamma$ and $\delta$, put

\[ M_\beta = (r^\beta - 1, m), \]
\[ D(\alpha) = \{ x \in \mathbb{N} \mid x \text{ is a divisor of } \alpha \}, \]
\[ D(\alpha, \beta) = \{ x \in D(\alpha) \mid x \text{ can be divided by } \beta \}, \]
\[ D(\alpha)_{\gamma} = \{ x \in D(\alpha) \mid x\gamma \equiv 0 \pmod{\delta} \}. \]

If $d$ is an even integer, we put $d' = d/2$, and put

\[ t(2) = \#((\alpha, \beta) \mid \beta \in D(v)^{k-1}, \alpha \in D(M_{2^u\beta}), \]
\[ (\alpha + aM_{2^u\beta})(l - 1, r^{n/4} - 1) \equiv 0 \pmod{m} \]

for some integer $a$ with $0 \leq a < m/M_{2^u\beta}$

\[ - \# \bigcup_{0 \leq b < d} D(m)^{\lambda} \]
\[ \lambda = 0, 1 \]

\[ \bigcup_{\lambda = 1} (l-1, r^{n/4} - 1, 2^\lambda r^b + 1) \],

\[ \bigcup_{\lambda = 0} (l-1, r^{n/4} - 1, 2^\lambda r^b + 1) \].

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We are now ready to state our main theorem.

**Theorem.** (i) $SK_1(Z[G_1]) = 0$.

(ii) $SK_1(Z[G_2]) \cong Z_2^{t(2)}$ if $d$ is an odd integer,

(iii) $SK_1(Z[G_3]) \cong Z_2^{t'(2)}$ if $d$ is an even integer.

(iv) $SK_1(Z[G_4]) \cong Z_2^{t(3)}$.

Example 1.1. When $d = 3$, we have

(i) $SK_1(Z[G_3]) = Z_2^{#D(n,3)} \cdot #D(m) - 1$,
Example 1.2. For $G_2$, when $m = 35$, $n = 72$, $r = 4$, $k = 55$, $l = 29$, we have $d = 6$ and then,

$$SK_1(\mathbb{Z}[G_2]) \cong \mathbb{Z}_2^8.$$
Lemma 2.1 ([10, Theorem 6.1.11]). \( \text{Syl}_2(G_2) \cong \langle R, B^v \rangle \cong Q_{2^{u+1}} \)
\( \text{Syl}_2(G_3) = \langle P, Q \rangle = Q_8 \), and \( \text{Syl}_2(G_4) \cong \langle P, Q, R \rangle = \langle PR, P \rangle \cong Q_{16} \),
where \( Q_{2^N} \) denotes the generalized quaternionic group of order \( 2^N \).

When \( H \) is a subgroup of \( G \), \( C_G(H) \) denotes the centralizer of \( H \) in \( G \) and \( N_G(H) \) denotes the normalizer of \( H \) in \( G \).

Lemma 2.2 ([8, Example 14.4]). Let \( G \) be a finite group whose 2-Sylow subgroups are dihedral, quaternionic, or semidihedral. Then

\[ SK_1(\mathbb{Z}[G])_2 \cong \mathbb{Z}_2^t, \]

where \( t \) is the number of conjugacy classes of cyclic subgroups \( \sigma \subset G \) such that (a) \( |\sigma| \) is odd, (b) \( C_G(\sigma) \) has a non-abelian 2-Sylow subgroup, and (c) there is no \( g \in N_G(\sigma) \) with \( gxg^{-1} = x^{-1} \) for all \( x \in \sigma \).

By Lemma 2.1, \( G_2, G_3 \) and \( G_4 \) satisfy the assertion in Lemma 2.2. We now prepare the next lemmas for the calculation of \( SK_1(\mathbb{Z}[G_i]) \) \( (i = 2, 3, 4) \), whose proof will be given in the last section. For integers \( \alpha \) and \( \beta \), we put
\[ D(\alpha) = \{ x \in \mathbb{N} \mid x \text{ is a divisor of } \alpha \}, M_\beta = (r^\beta - 1, m). \] Then
Lemma 2.3. For any \( \beta \in D(n) \), we have 
\[
\left( \frac{r^n - 1}{r^\beta - 1}, M_\beta \right) = 1.
\]

Lemma 2.4. For any integer \( \alpha \), we have 
\[
(m, r^\beta - 1, \alpha_{\frac{r^n - 1}{r^\beta - 1}}) = (\alpha, M_\beta).
\]

Lemma 2.5. Let \( \langle A^\mu B^\nu \rangle \) be the cyclic group which is generated by the element of the form \( A^\mu B^\nu \). We put \( \beta = (n, \nu) \). Then, there exists an integer \( \alpha \) such that \( \langle A^\mu B^\nu \rangle = \langle A^\alpha B^\beta \rangle \).

Proposition 2.6. Let \( \alpha \) be an integer, and \( \beta \) an element in \( D(n) \). Put \( n' = n/\beta \). Then we have 
\[
|\langle A^\alpha B^\beta \rangle| = \frac{M_\beta \cdot n'}{(M_\beta, \alpha)}.
\]

**Proof.** It is clear that \( |\langle A^\alpha B^\beta \rangle| \) is divisible by \( n' \). We have \( (A^\alpha B^\beta)^{n'} = A^{\alpha(r^n - 1)/(r^\beta - 1)} \). Put \( r^n - 1 = m \cdot s' \), \( r^\beta - 1 = M_\beta \cdot s \), and \( m = M_\beta \cdot t \), then we have \( (r^n - 1)/(r^\beta - 1) = t \cdot s'/s \). Set \( M_\beta = \alpha_1 \cdots \alpha_{\xi} \), \( t = \beta_1 \cdots \beta_{\eta} \), and \( s = \gamma_1 \cdots \gamma_t \), where \( \alpha_i \), \( \beta_i \) and \( \gamma_i \) are prime numbers, and \( e_i \), \( f_i \), \( g_i \) are positive integers. By the fact \( (t, s) = 1 \) and Lemma 2.3, we have \( s' = \beta_1 \cdots \beta_{\eta} \gamma_1 \cdots \gamma_t \delta_1 \cdots \delta_\omega \) for some...
prime numbers \( \delta_1, \ldots, \delta_\kappa \), non-negative integers \( f_1, \ldots, f_\eta \) and positive integers \( g_1, \ldots, g_i, h_1, \ldots, h_\kappa \), with \( g_i' \geq g_i \) (\( i = 1, \ldots, \eta \)). Since

\[
\frac{r^{n-1}}{r-1} \equiv M' S = \beta_1^{m_1+1} \cdots \beta_{\kappa}^{m_{\kappa}+1} g_1^{g_1-g_1'} \cdots h_1^{h_1-h_1'} \gamma_1 \cdots \gamma_\eta \delta_1 \cdots \delta_\kappa,
\]

the smallest positive integer \( x \) satisfying that

\[
\alpha \frac{r^{n-1}}{r-1} x \equiv 0 \pmod{m'} \text{ is } \frac{M'}{(\alpha, M')}.
\]

Hence we have \( |\langle A' B \rangle| = \frac{M'}{(M', \alpha)} \).

**Proposition 2.7.** Let \( \alpha \) and \( \alpha' \) be integers, and \( \beta \) and \( \beta' \) elements in \( D(n) \). \( \langle A' B \rangle \) is conjugate to \( \langle A' B' \rangle \) in \( G_2, G_3 \) and \( G_4 \) if and only if \( |\langle A' B \rangle| = |\langle A' B' \rangle| \).

**Proof.** Suppose that \( |\langle A' B \rangle| = |\langle A' B' \rangle| \). By using Proposition 2.6, we obtain that \( \beta = \beta' \). Since

\[
A' (A' B) B' A^{-1} = A^{-\alpha} (1 - r^\beta) B' \quad \text{and} \quad (A' B')^{C_n+1} = A^{\alpha (1 + c \frac{r^{n-1}}{r-1})} B^{\alpha (1 + c \frac{r^{n-1}}{r-1})}
\]

for any integers \( a \) and \( c \), by Lemma 2.4, two cyclic subgroups whose orders are same are conjugate. The converse is clear.

As an immediate consequence of Lemma 2.5 and Proposition 2.7, we have:

**Proposition 2.8.** Let \( \mu \) and \( \nu \) be integers. Put \( \beta = (\nu, n) \), then there exists an element \( \alpha \in D(M_{\beta}) \) such that
\[ \langle A^\mu B^\nu \rangle \text{ is conjugate to } \langle A^\alpha B^\beta \rangle. \]

3. Proof of (ii) of Theorem

Every element in \( G_2 \) is represented by the form \( A^\mu B^\nu e \) for some integers \( \mu \) and \( \nu \), where \( e \) is either 0 or 1. We see that \( |\langle A^\mu B^\nu e \rangle| \) is even, and that a generator of a cyclic subgroup of odd order is represented by the element of the form \( A^\alpha B^\nu \) for an integer \( \nu \). Put \( \beta = (\nu, \nu') \). By Proposition 2.8, there exists an integer \( \alpha \in D(M_{\nu}) \) such that \( \langle A^\nu B^\nu \beta \rangle \) is conjugate to \( \langle A^\nu B^\nu \beta \rangle \). Thus, from now on, we will consider the cyclic subgroups generated by the element of the form \( A^\alpha B^\nu \beta \) for any \( \beta \in D(\nu) \) and any \( \alpha \in D(M_{\nu}) \).

At first, we state some observations on \( G_2 \).

Observation 3.1. \( 2^\nu \) is divisible by \( d \).

Proof. Since \( r^n \equiv r^{k-1} \equiv 1 \pmod{m} \), \( d \) is a common divisor of \( n \) and \( k-1 \). Since \( k+1 \equiv 0 \pmod{2^u} \), \( (k+1, k-1) = 2 \), and \( u \geq 2 \), \( k-1 \) is divisible by 2, but not divisible by 4. Since \( n = 2^u \nu \), \( d \) is a divisor of \( 2^\nu \). \( \square \)

When \( d \) is an even integer, we put \( d' = d/2 \). Then we have:
Observation 3.2. For any integer $a$,

$$\langle A_{\mathbb{Z}^4}^{(1-r^4)}, A_{\mathbb{Z}^4}^{(1-s)} \rangle \cong \mathbb{Q}_8.$$ 

If $d$ is an even integer, then for any integer $a$,

$$\langle A_{\mathbb{Z}^4}^{(1-r^4)}, A_{\mathbb{Z}^4}^{(1-r^d)} \rangle \cong \mathbb{Q}_8.$$ 

Lemma 3.3. In the case that $d$ is an odd integer, for any $\beta \in D(v)$ and any $\alpha \in D(M_{u, v})$, $C_{G}(A_{\mathbb{Z}^4}^{2u_{\beta}})$ has a subgroup $H$ which is isomorphic to $\mathbb{Q}_8$ if and only if $\beta(k - 1) \equiv 0 (v)$ and 

$$(\alpha + a(r^2u_{\beta} - 1))(l - 1, r^{n/4} - 1) \equiv 0 (m)$$

for some integer $a$.

In the case that $d$ is an even integer, for any $\beta \in D(v)$ and any $\alpha \in D(M_{u, v})$, $C_{G}(A_{\mathbb{Z}^4}^{2u_{\beta}})$ has a subgroup $H$ which is isomorphic to $\mathbb{Q}_8$ if and only if $\beta(k - 1) \equiv 0 (v)$ and 

$$\beta(k - 1) \equiv 0 (v) \text{ and } (\alpha + a(r^2u_{\beta} - 1))(l - 1, r^{n/4} - 1) \equiv 0 (m)$$

or $\beta(k - 1) \not\equiv 0 (v)$ and 

$$\beta(k - 1) \not\equiv 0 (v) \text{ and } (\alpha + a(r^2u_{\beta} - 1))(l - 1, r^{n/4} - 1) \equiv 0 (m)$$

for some integer $a$.

Proof. In the case that $\beta(k - 1) \equiv 0 (v)$ and 

$$(\alpha + a(r^2u_{\beta} - 1))(l - 1, r^{n/4} - 1) \equiv 0 (m)$$

for some integer $a$, 

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we see that $C_G(<AA'B^2U^n\beta>) \supset <A^{a(1-r^{n/4})}B^{n/4}, A^{a(1-\ell)}R>$. In the case that $\beta(k-1) \equiv 0 \pmod{v}$ and $(\alpha + a(r^2\beta - 1))(2r^d' - 1, r^{n/4} - 1) \equiv 0 \pmod{m}$ for some integer $a$, we see that $C_G(<AA'B^2U^n\beta>) \supset <A^{a(1-r^{n/4})}B^{n/4}, A^{a(1-\ell r^d')}B^d'R>$. Conversely, assume that $C_G(<AA'B^2U^n\beta>)$ has a subgroup $H$ which is isomorphic to $Q_8$.

Since $K = <B^v, R>$ is one of the 2-Sylow subgroups of $G$ and $H$ is a 2-group of $G$, we have $g^{-1}Hg \subset K$ for some $g \in G$. Now we consider the quotient group of $K/<B^v>$ and the projection $p : K \longrightarrow K/<B^v>$. Since $\ker p = <B^v> \text{ and } g^{-1}Hg \cong Q_8$, we have $\ker(p|g^{-1}Hg) = <B^{n/4}>$. Hence, $g^{-1}Hg = <B^{n/4}, B^\tau R>$ for some integer $\tau$ which is divisible by $v$. Now put $g = A^{a}B^{b}R^{c}$ where $a$ and $b$ are some integers, and $c$ is either 0 or 1.

Then,

$$H = g<B^4, B^\tau R>g^{-1}$$

$$= A^{a}B^{b}R^{c}<B^4, B^\tau R>R^cB^{-b}A^{-a}$$

$$= A^{a}B^{b}<B^4, B^\tau'R>^bB^{-a} \quad (\text{for some integer } \tau')$$

$$= A^{a}<B^4, B^\tau''R>A^{-a} \quad (\text{for some integer } \tau'')$$

$$= <A^{a(1-r^{n/4})}B^{n/4}, A^{a(1-\ell r^\tau'')}B^\tau''R>.$$
\[(r^4 - 1)(a(r^2 u^2 - 1) + \alpha) \equiv 0 \pmod{m} .\]

On the other hand, since \(A^{(1 - \ell r^\tau''')} B r^\tau'' R \in C_G(\langle A\beta^u \rangle),\) we have

\[
\begin{cases}
(\ell r^\tau'' - 1)(\alpha + a(r^2 u^2 - 1)) \equiv 0 \pmod{m} \\
\beta(k - 1) \equiv 0 \pmod{v}
\end{cases}
\]

Now, since \(\tau'' = \tau k + b(1 - k)\) if \(c = 1,\) and \(\tau'' = \tau + b(1 - k)\) if \(c = 0,\) we have \(r^\tau'' = r^\tau.\) Moreover, \(r^\tau = 1\) or \(r^{d'}\) because \(\tau\) is divisible by \(v\) and \(d\) is a divisor of \(2v.\)

Thus the lemma was proved. \(\Box\)

As an immediate consequence of Lemma 3.3, we have:

**Corollary 3.4.** In the case that \(d\) is an odd integer, for any \(\beta \in D(v)\) and any \(\alpha \in D(M_u),\) \(C_G(\langle A\beta^u \rangle)\) has a subgroup \(H\) which is isomorphic to \(Q_8\) if and only if \(\beta(k - 1) \equiv 0 \pmod{v}\) and \((\alpha + aM_u)(l - 1, r^{n/4} - 1) \equiv 0 \pmod{m}\) for some integer \(a\) with \(0 \leq a < m/M_u .\)

In the case that \(d\) is an even integer, for any \(\beta \in D(v)\) and any \(\alpha \in D(M_u),\) \(C_G(\langle A\beta^u \rangle)\) has a subgroup \(H\) which is
isomorphic to $Q_8$ if and only if $\beta (k - 1) \equiv 0 (v)$ and
$$(\alpha + a M) (l - 1, r^n/4 - 1) \equiv 0 (m)$$
or
$$(\alpha + a M) (l - 1, r^n/4 - 1) \equiv 0 (m)$$
for some integer $a$
with $0 \leq a < m/M_{2 \beta}$. \hfill \Box

It is clear that $C_G(\langle A^\alpha B^\beta \rangle)$ has a nonabelian 2-Sylow
subgroup if and only if $C_G(\langle A^\alpha B^\beta \rangle)$ has a subgroup $H$ which
is isomorphic to $Q_8$. Let $\langle A^\alpha B^\beta \rangle$ be a cyclic subgroup of $G_2$
satisfying the conditions (a) and (b). Assume that it does not
satisfy the condition (c). In the case that
$$(A^a B^b)(A^\alpha B^\beta)(A^a B^b)^{-1} = (A^\alpha B^\beta)^{-1}$$
for some integers $a$ and
$b$, we have

$$\begin{cases} 
\alpha (r^b + 1) \equiv 0 (m) \\
\beta \equiv 0 (v) 
\end{cases}$$

On the other hand, in the case that
$$(A^a B^b R)(A^\alpha B^\beta)(A^a B^b R)^{-1} = (A^\alpha B^\beta)^{-1}$$
for some integers $a$ and
$b$, we have

$$\begin{cases} 
a + \alpha \beta r^b - \alpha r^2 \beta k + \alpha r^2 \beta \equiv 0 (m) \\
\beta (k + 1) \equiv 0 (v) 
\end{cases}$$
Since it follows from Corollary 3.4 that \( \beta(k - 1) \equiv 0 \pmod{v} \), in this case we have

\[
\begin{align*}
\alpha(\ell r^b + 1) & \equiv 0 \pmod{m} \\
\beta & \equiv 0 \pmod{v}
\end{align*}
\]

Hence for \( \alpha \in D(m) \) satisfying that \( \alpha(\ell^\lambda r^b + 1) \equiv 0 \pmod{m} \) \((\lambda = 0, 1)\), \( \langle A^\alpha \rangle \) does not satisfy the condition (c). This completes the proof of (ii) of Theorem.

4. Proof of (iii) of Theorem

Lemma 4.1. Let \( \sigma \in G_3 \) be a cyclic subgroup of odd order. Then, there exist \( \beta \in D(n) \) and \( \alpha \in D(M_\beta) \) such that \( \sigma \) is conjugate to \( \langle A^\alpha B^\beta \rangle \).

Proof. Every element in \( G_3 \) can be represented by the form \( XA^\mu B^v \) for some \( X \in \langle P, Q \rangle \) and some integers \( \mu \) and \( v \). We see that \( \langle A^\mu B^v \rangle \) has odd order. In the case that \( v \equiv 0 \pmod{3} \), we see that \( \langle XA^\mu B^v \rangle \) has even order. In the other cases, we see that \( \langle XA^\mu B^v \rangle \) has even order or is conjugate to \( \langle A^\mu B^v \rangle \). The conclusion now follows from Proposition 2.8.

Hence from now on we will consider the cyclic subgroups generated by the element of the form \( A^\alpha B^\beta \) for \( \beta \in D(v) \) and \( \alpha \in D(M_\beta) \). Since \( \langle P, Q \rangle \) is a normal subgroup of \( G_3 \), \( C_{G_3} (\langle A^\alpha B^\beta \rangle) \)
has a non-abelian 2-Sylow subgroup if and only if $C_{G_3}(\langle A^\alpha B^\beta \rangle)$ includes $\langle P, Q \rangle$. And it is easy to show that $C_{G_3}(\langle A^\alpha B^\beta \rangle)$ includes $\langle P, Q \rangle$ if and only if $\beta$ is an element of $D(n, 3)$. Let $\langle A^\alpha B^\beta \rangle$ be a cyclic subgroup of $G_3$ satisfying the conditions (a) and (b). Assume that $(A^a B^b)(A^\alpha B^\beta)(A^a B^b)^{-1} = (A^\alpha B^\beta)^{-1}$ for some integers $a$ and $b$. Since $n$ is an odd integer, we have

$$\begin{align*}
\alpha(1 + r^b) &\equiv 0 \pmod{m} \\
\beta &\equiv 0 \pmod{n}
\end{align*}$$

Since $(1 + r^b, m) = 1$ for any $b \in \mathbb{Z}$ when $n$ is odd, we have $\langle A^\alpha B^\beta \rangle = 1$. This completes the proof of (iii) of Theorem.

5. Proof of (iv) of Theorem

Lemma 5.1. Let $\sigma \in G_4$ be a cyclic subgroup of odd order. Then, there exist $\beta \in D(n)$ and $\alpha \in D(M_\beta)$ such that $\sigma$ is conjugate to $\langle A^\alpha B^\beta \rangle$.

Proof. Every element in $G_4$ can be represented by the form $XA^{\mu}B^{\nu}$ for some $X \in \langle P, Q, R \rangle$ and some integers $\mu$ and $\nu$. We see that $\langle A^{\mu}B^{\nu} \rangle$ has odd order. And it is shown that $|\langle XA^{\mu}B^{\nu} \rangle|$ is even or $\langle XA^{\mu}B^{\nu} \rangle$ is conjugate to $\langle A^{\mu}B^{\nu} \rangle$. The conclusion now follows from Proposition 2.8.

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Hence from now on we will consider the cyclic subgroups generated by the element of the form $\alpha^\beta B$ for $\beta \in D(v)$ and $\alpha \in D(M_\beta)$.

**Lemma 5.2.** If $C_{G_4}(\langle A^\alpha B^\beta \rangle)$ has a non-abelian 2-Sylow subgroup, then $C_{G_4}(\langle A^\alpha B^\beta \rangle)$ includes $\langle P \rangle$, $\langle Q \rangle$ or $\langle PQ \rangle$.

**Proof.** We put $K = \langle P, Q, R \rangle = \langle PR, P \rangle$. If $C_{G_4}(\langle A^\alpha B^\beta \rangle)$ has a non-abelian 2-Sylow subgroup, if and only if $C_{G_4}(\langle A^\alpha B^\beta \rangle)$ has a subgroup $H$ which is isomorphic to $Q_8$. Since $H$ is a 2-group of $G$, we have $g^{-1}Hg \subset K$ for some $g \in G$. We note that $\langle PR \rangle$ is a cyclic subgroup of $K$ whose order is 8. Now we consider the quotient group $K/\langle PR \rangle$, and the projection $p : K \longrightarrow K/\langle PR \rangle$. Since $\ker p = \langle PR \rangle$ and $g^{-1}Hg \cong Q_8$, we have that $\ker (p|g^{-1}Hg)$ is a cyclic subgroup of $\langle PR \rangle$ whose order is 4. Hence we have $\ker (p|g^{-1}Hg) = \langle PR \rangle^2 = \langle Q \rangle$. Thus, we have $g^{-1}Hg = \langle Q, (PR)^\lambda P \rangle$ for some $\lambda \in \mathbb{Z}$. We note that if $\lambda$ is an odd integer, then $g^{-1}Hg = \langle Q, R \rangle$, and that if $\lambda$ is an even integer, then $g^{-1}Hg = \langle P, Q \rangle$. Thus, we obtain:

- $H = \langle P, Q \rangle$ or $\langle RA^a(b-1)^b(k-1), Q \rangle$ if $b \equiv 0 \pmod{3}$,
- $H = \langle P, Q, RA^a(b-1)^b(k-1), PQ \rangle, \langle RA^a(b-1)^b(k-1), P \rangle$
- or $\langle QRA^a(b-1)^b(k-1), P \rangle$ if $b \equiv 1 \pmod{3}$,
- $H = \langle P, Q, RA^a(b-1)^b(k-1), PQ \rangle, \langle RA^a(b-1)^b(k-1), PQ \rangle$
- or $\langle RPA^a(b-1)^b(k-1), PQ \rangle$ if $b \equiv 2 \pmod{3}$,
where $a$ and $b$ are integers. Hence $H$ includes $\langle P \rangle$, $\langle Q \rangle$ or $\langle PQ \rangle$. 

\[ \text{Lemma 5.3.} \quad C_{G_4}(\langle \alpha \beta \rangle) \text{ has a non-abelian } 2\text{-Sylow subgroup if and only if } \beta \equiv 0 \text{ (3)}. \]

\[ \text{Proof.} \quad \text{If } C_{G_4}(\langle \alpha \beta \rangle) \text{ has a non-abelian } 2\text{-Sylow subgroup, by Lemma 5.2, we have } P, Q \text{ or } PQ \text{ are elements of } C_{G_4}(\langle \alpha \beta \rangle). \text{ In the case that } P \text{ or } Q \text{ are elements of } C_{G_4}(\langle \alpha \beta \rangle), \text{ we have } \beta \equiv 0 \text{ (3) as in proof of (iii) of Theorem. On the other hand it is easy to show that if } PQ \text{ is an element of } C_{G_4}(\langle \alpha \beta \rangle), \text{ then } \beta \equiv 0 \text{ (3). Conversely, if } \beta \equiv 0 \text{ (3), it follows from proof of (iii) of Theorem that } C_{G_4}(\langle \alpha \beta \rangle) \text{ includes } \langle P, Q \rangle, \text{ that is a non-abelian } 2\text{-group. This completes the proof.} \]

Now for $\beta \in D(n, 3)$ and $\alpha \in D(M_\beta)$, we assume that $\langle \alpha \beta \rangle$ doesn't satisfy the condition (c). If $(\alpha \beta)(\alpha \beta)(\alpha \beta)^{-1} = (\alpha \beta)^{-1}$, then we have $\alpha \beta = 1$. If $(\alpha \beta)(\alpha \beta)(\alpha \beta)^{-1} = (\alpha \beta)^{-1}$, then we have

\[
\begin{align*}
\ell(\alpha r^b + a(1 - r^b)) + a \alpha r^b & \equiv 0 \text{ (m)} \\
\beta(k + 1) & \equiv 0 \text{ (n)}
\end{align*}
\]
Since \( d \) is a common divisor of \( n \) and \( k - 1 \), we have 

\((k + 1, d) = 1\), and so \( \beta \) must be divisible by \( d \). Hence we have 

\( \alpha(1 + kr^b) \equiv 0 \pmod{m} \). Since \( k^2 \equiv 1 \pmod{m} \), we have 

\( \alpha(l + r^b) \equiv 0 \pmod{m} \). By these equations, we have 

\( \alpha(l + 1)(r^b + 1) \equiv 0 \pmod{m} \). Since \( (r^b + 1, m) = 1 \), we have 

\( \alpha(l + 1) \equiv 0 \pmod{m} \).

Conversely, under the conditions \( \beta(k + 1) \equiv 0 \pmod{m} \) and 

\( \alpha(l + 1) \equiv 0 \pmod{m} \), we see that 

\[ R(A^\alpha B^\beta)R^{-1} = (A^\alpha B^\beta)^{-1} \]

then \( \langle A^\alpha B^\beta \rangle \) doesn't satisfy the condition (c). This completes the proof (iv) of Theorem.

6. Proof of Lemmas in Section 2

Proof of Lemma 2.3. Put \( n' = n/\beta \) and \( r^\beta - 1 = M_\beta \cdot s \).

Then we have

\[
\frac{r^{n-1}}{r^\beta - 1} = \sum_{i=0}^{n'-1} r^{\beta i} = \sum_{i=0}^{n'-1} (M_\beta \cdot s + 1)^i = n' (M_\beta). 
\]

Now since \((n', M_\beta) = 1\), we have \((r^n - 1)/(r^\beta - 1), M_\beta) = 1\). □

Lemma 2.4 is an immediate consequence of Lemma 2.3.

Proof of Lemma 2.5. Since \( \beta = (n, v) \), there exists an integer \( x \) such that \( vx \equiv \beta(n) \). Put \( n' = n/\beta \), then we see
that \((x, n') = 1\). We note that the order of \(<A^\mu B^\nu>\) is a divisor of \(mn'\). If \((x, m) = 1\), we have \((A^\mu B^\nu)^x = A^\alpha B^\beta\) for some integer \(\alpha\) and \(<(A^\mu B^\nu)^x> = <A^\mu B^\nu>\). If \((x, m) \neq 1\), since there exists an integer \(c\) such that \((x + cn', n'm) = 1\), we have \((A^\mu B^\nu)^x+cn' = A^\alpha B^\beta\) for some integer \(\alpha\) and \(<(A^\mu B^\nu)^x+cn'> = <A^\mu B^\nu>\). This completes the proof.

7. Appendix ([10, Theorem 6.1.11])

Let \(G\) be a finite solvable group. Then \(G\) has a fixed point free complex representation if and only if \(G\) is of type I, II, III or IV below, with the additional condition: if \(d\) is the order of \(r\) in the multiplicative group of residues modulo \(m\), of integers prime to \(m\), then \(n/d\) is divisible by every prime divisor of \(d\).

**Type I.** A group of order \(mn\) that is generated by the elements of the form \(A\) and \(B\), and that has relations:

\[A^m = B^n = 1, \quad BAB^{-1} = A^r,\]

where \(m, n\) and \(r\) satisfy the following conditions:

\[m \geq 1, \quad n \geq 1, \quad (n(r - 1), m) = 1, \quad r^n \equiv 1 \pmod{m}.

**Type II.** A group of order \(2mn\) that is generated by the elements of the form \(A, B\) and \(R\), and that has relations:
\[ R^2 = B^2, \quad RAM^{-1} = A^\ell, \quad RBM^{-1} = B^k \]

in addition to the relations in 1, where \( m, n, r, \ell \) and \( k \) satisfy the following conditions:

\[ \ell^2 \equiv r^{k-1} \equiv 1 \pmod{m}, \quad k \equiv -1 \pmod{2^u}, \]
\[ n = 2^u v \quad (u \geq 2, \quad (v, 2) = 1), \quad k^2 \equiv 1 \pmod{n} \]

in addition to the conditions in 1.

**Type III.** A group of order \( 8mn \) that is generated by the elements of the form \( A, B, P \) and \( Q \), and that has relations:

\[ P^2 = Q^2 = (PQ)^2, \quad AP = PA, \quad AQ = QA, \]
\[ BPB^{-1} = Q, \quad BQB^{-1} = PQ \]

in addition to the relations in 1, where \( m, n \) and \( r \) satisfy the following conditions:

\[ n \equiv 1 \pmod{2}, \quad n \equiv 0 \pmod{3} \]

in addition to the conditions in 1.

**Type IV.** A group of order \( 16mn \) that is generated by the elements of the form \( A, B, P, Q \) and \( R \), and that has relations:
\[ R^2 = P^2, \quad RPR^{-1} = QP, \quad RQR^{-1} = Q^{-1}, \]
\[ RAR^{-1} = A^l, \quad RBR^{-1} = B^k \]

in addition to the relations in III, where \( m, n, r, k \) and \( l \)
satisfy the following conditions:

\[ k^2 \equiv 1 \quad (n), \quad k \equiv -1 \quad (3), \quad r^{k-1} \equiv l^2 \equiv 1 \quad (m) \]

in addition to the conditions in III.

References.


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